Lecture Notes

Relativity - Special Theory

(part of **Classical Mechanics (II) PH33003/PH43017**)

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Contents

Chapter 1

The Issue

In this chapter we review the relevant aspects of classical mechanics, and the issues that led to the Special Theory of Relativity.

1.1 Galelian transformations and the Galeian group

Consider two observers in two reference frames, S and S', designated by coordinates (x, y, z) and time t, and (x', y', z') and time t', respectively. Let the two reference frames be in uniform relative motion with respect to each other with constant velocity $\mathbf{V} = \{v_x, v_y, v_z\}$. Also, let the two reference frames differ in their orientation (which remains a constant), and differ in their origin in space and time. Then the two space and time coordinates are in general related by

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \left(\begin{array}{ccc|c} & & & v_x \\ & \mathcal{R}_{3\times3} & & v_y \\ & & & v_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix}. \tag{1.1}$$

Here, $\mathcal{R}_{3\times3}$ is a constant 3-parameter rotation (orthogonal 3×3) matrix representing the orientation of the S' frame with respect to S frame, and (x_0, y_0, z_0) and t_0 are the difference in their spacial and temporal origins. In all, there are 10 parameters that relate the two space and time coordinates (3 for constant spacial rotations in $\mathcal{R}_{3\times3}$, 3 *boost* parameters -(v_x, v_y, v_z), 3 spacial displacements -(x_0, y_0, z_0), and one temporal displacement-t_0).

Eq. (1.1) constitutes the most general *Galelian* transformation connecting two coordinate systems. Any *non-accelerating* reference frame is defined as an *inertial* reference frame. Any reference frame related to any other *inertial* reference frame by a *Galelian* transformation will also be *inertial*. Indeed, the significance of such transformations arises from the fact - Forces (or accelerations) seen in one inertial reference frame is the same in all inertial reference frames. More strictly, the form of the equations of mechanics are invariant with respect to such transformations. I.e.,

$$m_i\ddot{\mathbf{r}}_i = m_i\ddot{\mathbf{r}}'_i. \tag{1.2}$$

Further,

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i([\mathbf{r}], t) \tag{1.3}$$

will go over to

$$m_i\ddot{\mathbf{r}}'_i = \mathbf{F}_i([\mathbf{r}(\mathbf{r}')], t(t')) = \mathbf{F}'_i([\mathbf{r}'], t'). \tag{1.4}$$

Some facts about the transformation in Eq. (1.1) can be immediately seen: i) Identity transformation is just a particular case of Eq. (1.1). ii) Inverse of Eq. (1.1) relating S' to S is also of the same form. iii) If we consider two such relations relating S to S' and S' to S'', respectively, say

$$\begin{pmatrix} x'' \\ y'' \\ z'' \\ t'' \end{pmatrix} = \left(\begin{array}{ccc|c} & & & v'_x \\ & \mathcal{R}'_{3\times3} & & v'_y \\ & & & v'_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} + \begin{pmatrix} x'_0 \\ y'_0 \\ z'_0 \\ t'_0 \end{pmatrix}. \tag{1.5}$$

It can be worked out that the relation between S and S'' is again of the same form. I.e.,

$$\begin{pmatrix} x'' \\ y'' \\ z'' \\ t'' \end{pmatrix} = \left(\begin{array}{ccc|c} & & & v''_x \\ & \mathcal{R}''_{3\times3} & & v''_y \\ & & & v''_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \begin{pmatrix} x''_0 \\ y''_0 \\ z''_0 \\ t''_0 \end{pmatrix}, \tag{1.6}$$

for suitable constant $\mathcal{R}''_{3\times3}$, (v''_x, v''_y, v''_z), (x''_0, y''_0, z''_0) and t''_0. Thus, if one considers the set of all possible transformations of the type Eq. (1.1), it forms a group under composition - the *10-parameter Galelian group*.

Moral: Equations of mechanics preserve their form under Galelian transformations. Or simply, mechanics pocesses Galelian group symmetry, or is Galelian invariant.

Practically this means that we can write and follow the same equations, using one's own lab coordinates, irrespective of our reference frame.

1.2 Maxwell's equations do not agree

The results of the previous section constitute what is referred to as Galelian/Newtonian/Classical relativity, and consequently a relativity principle - a belief that *laws of physics must be same in all inertial reference frames*, wherein the reference frames are related by transformations of the type Eq. (1.1).

However, Maxwell's equations of electrodynamics do not show this invariance. Under the transformation $x' = x - v_0 t$, $y' = y$, $z' = z$ andes $t' = t$, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \tag{1.7}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \tag{1.8}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \tag{1.9}$$
$$\frac{\partial}{\partial t} = -v_0 \frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}. \tag{1.10}$$

Consider the wave equations in vacuum for the electric and magnetic fields, propagating in the $\hat{\mathbf{x}}$ direction

$$\frac{\partial^2 \mathbf{E_i}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E_i}}{\partial t^2} \tag{1.11}$$
$$\frac{\partial^2 \mathbf{B_i}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B_i}}{\partial t^2}. \tag{1.12}$$

The solution is a Electro-Magnetic wave with velocity of propagation $c = 1/\sqrt{(\mu_0\epsilon_0)}$. However, under a transformation $x' = x - v_0 t, t' = t$, the equation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{1.13}$$

$$\frac{\partial^2\phi}{\partial x'^2} = \frac{1}{(c-v_0)^2}\frac{\partial^2\phi}{\partial t'^2} \tag{1.14}$$

I.e., a E-M wave of velocity $c - v_0$, as one would expect from classical relativity. However, several experiments have shown that the speed of light in vacuum is the same, c, *irrespective of the velocity of the source* or *velocity of the reference frame.*

Practically this implies that if Maxwell's equations are valid in on inertial reference frame, then it should not be valid for an observer in any other *non-trivial* inertial reference frame. I.e., electrodynamics, as expressed in the form of Maxwell's equations, *must by valid only in some specific special reference frame.* However, any attempt to locate this *special* frame was a failure[1]

1.3 Possible resolutions

There are three possible resolutions to the issue:

1. *Mechanics is fine and Galelian relativity is valid. But Maxwell's equations are wrong.* If this is so then first of all we must be able to show this experimentally, and hopefully correct the equations to the ones that show the symmetry.

2. *Mechanics is fine and Maxwell's equations are fine. But the relativity principle does not exist for electrodynamics.* If this is true then doing physics will be tough. For, we may have to write up different equations for each geographical location or even each day. However, if this is indeed true we must be able to confirm this experimentally.

   Every effort, running over several decades, along these two lines showed a negative result. Maxwell's equations were valid to the best of accuracy. The *special* reference frame could not be found. Curiously enough, as was mentioned earlier, the speed of light was found to be the same *irrespective of the velocity of the source, or the observer*[2]

3. *Maxwell's equations are fine and a relativity principle exists. But, the expressions as in Eq. (1.1) are wrong.*

The third possibility was the one investigated by Einstein, and the one that resolved all issues.

[1] Look up any standard text in relativity for a discussion on this period in history (do this atleast once), and some experiments, particularly the ones by Michelson-Morley, Kennedy-Thorndike. *Relativity* by R. Resnick and *Feynman Lectures, Vol. 1* are good enough.

[2] One historically important remedy suggested was by Lorentz-Fitzgerald, that objects in motion shrink by a certain factor along the direction of motion depending on their relative velocity. Though this theory could not completely resolve all issues, it is well acknowledged as the prime motivator for Einstein's theory. Again, see Feynman or Resnick for a discussion.

Chapter 2

The Special Theory

We shall deliberately ignore the chronological development of the theory [1], and start directly from the answer.

2.1 The Postulates

The resolution came in the form of two (just two!!) postulates.

PI. The principle of relativity: *Laws of physics must be the same in all inertial reference frames.*

Though this assertion may sound nothing new, it has to be appreciated that, first of all, this is a postulate. Besides, the change is in its privilege, now as an *apriori* assertion.

The second postulate brings in some fundamental changes in our notion of space and time. While the following sections in this chapter are devoted to a more detailed discussion on these aspects, we shall briefly define the bare minimum first, just enough material to state the postulate.

The special theory forces us to look upon *space* and *time* not independently, but as a *space-time* continuum. Just as we speak of a point in 3-space given by 3-coordinates, we have *Events*(noun) designated by 4 coordinates - 3 spacial and 1 temporal. Thus we have a 4-dimensional space-time, and every space-time point is defined as an 'Event'. For a start it may be convenent to think of these Events as usual events(verb).

Consider two Events in space-time, say $(t_i; x_i, y_i, z_i), i = 1, 2$ (say two firecrackers bursting in the sky at two different points at different times). Contrary to our usual notion that time intervals $\Delta t = (t_2 - t_1)$, and lengths $\Delta l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$, are *independently* invariant in any reference frame, we have

PII. *The space time interval between two Events, defined as*

$$\Delta s^2 = c^2 \Delta t^2 - \Delta l^2, \tag{2.1}$$

is an invariant in any inertial reference frame, where $'c'$ is a universal constant whose value is roughly $3 \times 10^8 m/s$.

[1] Just as we do not care anymore about the chronological development of Newton's laws, which after all were not written overnight.

I.e., if the same two events are in another inertial reference frame designated by the space-time coordinates $(t'_i; x'_i, y'_i, z'_i)$, $i = 1, 2$, then

$$c^2(t_2-t_1)^2-[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2] = c^2(t'_2-t'_1)^2-[(x'_2-x'_1)^2+(y'_2-y'_1)^2+(z'_2-z'_1)^2]. \quad (2.2)$$

The speed of light *happens to be* $'c'$.

The remaining part of the course will be confined to convincing ourselves that the special theory does indeed resolve the issues discussed in the previous chapter, and make the relevant changes in Newtonian mechanics. However, as mentioned earlier, PII changes our notion of space and time fundamentally. This chapter will detail these changes.

2.2 Space-time, Events and World Lines

In classical relativity, the instantaneous position of a particle, or an object, is specified by its space coordinates, and its time evolution is illustrated by a trajectory it traces out in space. But, PII puts space and time in equal footing, thus prohibiting looking upon the two independently. Consequently, The trajectory of, say, a particle in space becomes a trajectory connecting space-time 'Events'. Such a trajectory shall be called a 'World Line'. The four coordinates do not have the same dimension, however. This is taken care of by scaling the time axis with the universal constant $'c'$.

A few examples are shown in Figures 2.1 and 2.2.

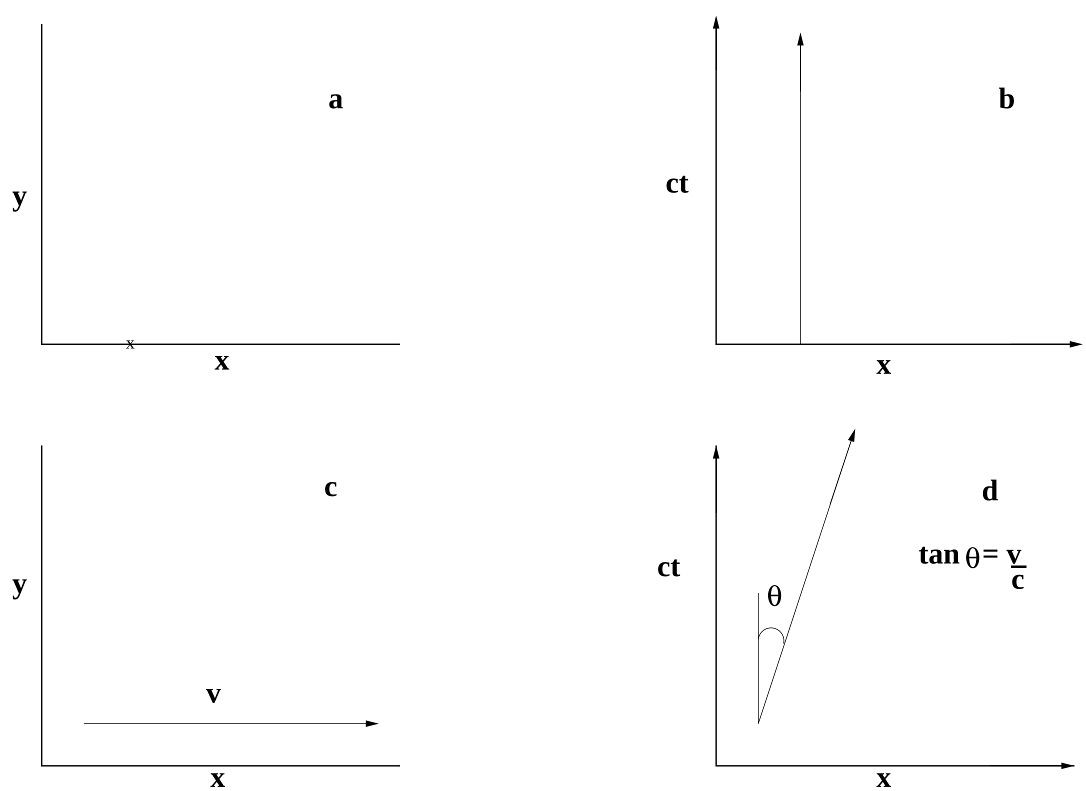

Figure 2.1: Two simple situations and their world line view: a) an object at rest, and c) a object in uniform motion along x direction in space, and their respective world lines c) and d). World lines of objects in uniform motion are straight lines. y and z directions are not shown in b) and d).

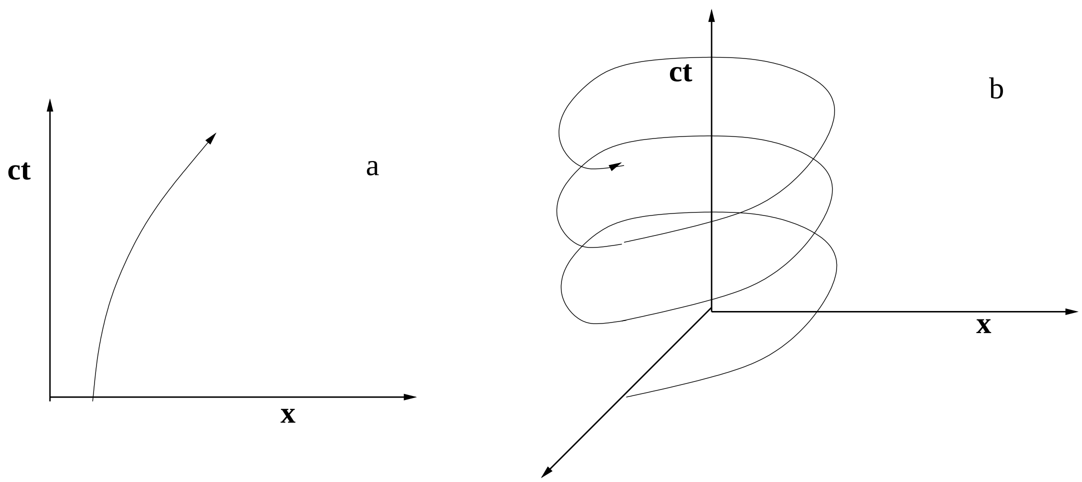

Figure 2.2: a) The world line of a accelerating object, moving in the x direction. The curvature of the world line is a measure of its acceleration. The other two spacial directions are suppressed for simplicity. b) The world line of a planet orbiting in the $x-y$ plane. The world line for the planet is a helix (i.e, if the orbit is circular).

Thus even an object at rest is represented by a line. In some sense, even an object at rest is moving in space time. Two observations are immediate: a) Objects in uniform motion in a *inertial* (henceforth a default unless specified explicitly) reference frame are represented by *straight* world lines in the space-time diagram drawn in any reference frame. b) Objects that are accelerating will have curved world lines. More specifically, the acceleration at a point (ct, x, y, z) will be given by the curvature of the world line at that point.

2.3 Two Events in two different inertial frames

In comparing two points in space, Galelian relativity was wound around the *postulate* that the spacial separation between the two points will be the same for any two observers. And so was the case for temporal separation between two events.

2.3.1 Correspondence

Consider two events A and B, with coordinates $(ct_1; x_1)$ and $(ct_2; x_2)$ (y and z coordinates being same for simplicity) in a reference frame S. Say an object starts at A and reaches B with uniform velocity v in the $\hat{\mathbf{x}}$ direction as in Figure 2. Then

$$\Delta s^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 = (c^2 - v^2)(t_2 - t_1)^2. \tag{2.3}$$

Let S' be another reference frame moving with respect to S with some velocity such that the same two events in S' are $(ct'_1; x'_1)$ and $(ct'_2; x'_2)$, and the velocity of the object v'. From PII we have

$$\Delta s^2 = (c^2 - v^2)(t_2 - t_1)^2 = (c^2 - v'^2)(t'_2 - t'_1)^2 = \Delta s'^2. \tag{2.4}$$

When velocities v_1 and v_2 are small compared to c, Eq. (2.4) reduces to

$$(t_2 - t_1)^2 = (t'_2 - t'_1)^2, \tag{2.5}$$

or simply, one gets back Galelian relativity. Indeed, practically relativistic effects are significant only at velocities of the order $0.5c$ and more.

2.3.2 Objects with velocity c are special

Suppose the object in the previous section is moving at speed $v = c$ in S. Then evidently,

$$\Delta s^2 = 0. \tag{2.6}$$

Consequently $\Delta s'^2 = 0$ and $v' = c$. I.e., if an object is moving at speed c in some reference then it moves with the same speed c in all reference frames. Objects traveling with speed c are treated by space-time with a little added privilege. Hence, the velocity of light in vacuum being c, will be the same as seen from any reference frame, just as several experiments had suggested.[2] Indeed, from Eq. (2.4) it follows that an object moving with a speed less than (greater than) c in one reference will be seen to be moving with a speed less than (greater than) c in any reference frame. In fact, the invariance of the space-time intervals allows us to classify them into three types based on their sign:

2.3.3 Time like, space like and light like intervals: Causality

Two events are said to be time like, space like or light like separated if their space-time intervals are such that $\Delta s^2 > 0$, $\Delta s^2 < 0$ or $\Delta s^2 = 0$ respectively. The primary significance being that they maintain this classification in any reference frame.

We wish to see how the spacetime separation between events look like in different reference frames. Of two events A and B, let A coincide with the origin in two reference frames S and S' (i.e, the two reference frames coincide at their origin). Given $B = (ct; x, y, z)$ in frame S, we wish to see where $B = (ct', x', y', z')$ can possibly be in S'. Since the space time coordinates of A being the origin, we have,

$$\Delta s^2 = c^2t^2 - x^2 - y^2 - z^2. \tag{2.7}$$

If the two events are time like,space like and light like separated, then we have $\Delta s^2 = a^2$, $\Delta s^2 = -a^2$ and $\Delta s^2 = 0$, respectively, leading to

$$\begin{aligned} c^2t'^2 - x'^2 - y'^2 - z'^2 &= a^2 \; ; \quad \textit{time like} \\ c^2t'^2 - x'^2 - y'^2 - z'^2 &= -a^2 \; ; \quad \textit{space like} \\ c^2t'^2 - x'^2 - y'^2 - z'^2 &= 0 \; ; \quad \textit{light like.} \end{aligned} \tag{2.8}$$

The equations may be recognized as *paraboloid*, *hyperboloid* and a *hyper cone* separating the two. This is illustrated in the $\{ct, x, y\}$ space in the Figure 2.3. If an event B lies in the paraboloid in one reference frame S, then it lies in the same paraboloid in any other reference frame S' (if S and S' have the same origin). The same holds true for the light cone and the hyperboloid (See Figure 2.4)

[2] 'The speed of light in vacuum is same in all reference frame' is in fact the version of PII originally given by Einstein, and is equivalent to the version we have used. Einstein's version is deliberately avoided for the following reasons: a) the postulate PII as we have stated is functionally more useful, b) it shows the precise difference between Galelian and Special relativity and c) on a naive reading Einstein's version is likely to give an impression that we are talking about some special property of light. Whereas, it should be realized that what we have encountered is a property of space-time. After all, it is not just light, but any object moving with speed c that has the same speed in all reference frames.

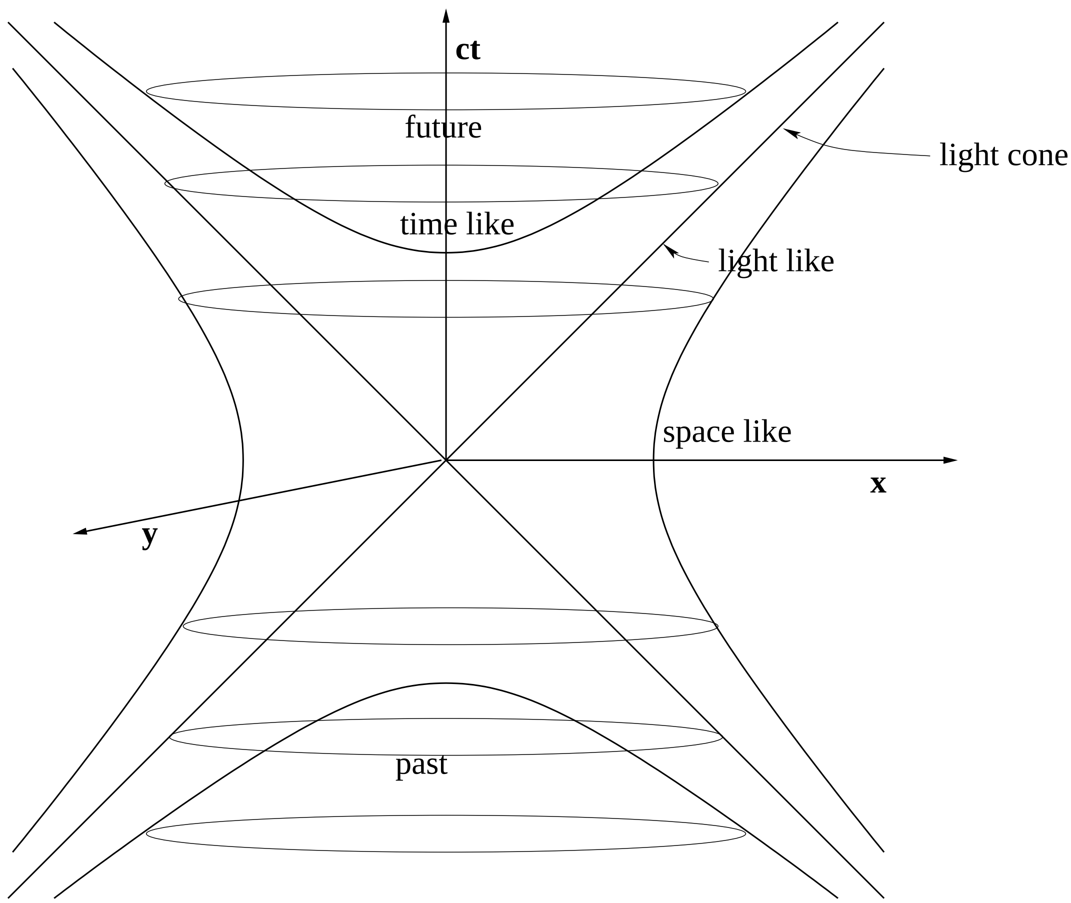

Figure 2.3: The light cone attached to an event in space time (origin). The events in the *future* fall inside the cone in the $+ve$ t direction. The events outside the cone are space like seperated from the origin and are not accessible. If a future event falls on the paraboloid inside the cone, then in any other reference frame (with the same origin) the event will be some other point in the same paraboloid. The actual structure is 4 dimensional. z direction is suppressed for simplicity.

An important observation with respect to time like and space like separated events concerns their time ordering. Suppose B and A are time like separated, and say $t > 0$, i.e, A precedes B in frame S. Then in any frame the time ordering is preserved. *Causality is preserved if the events are time like separated*. However, if A and B are space like separated, then it is possible that A precedes B in frame S, while B precedes A in S' or some other reference frame. Event B could be at $B1$(see figure) in one reference frame and $B2$ in another. Notice the difference between Newtonian causality and Einstein's causality. In Newtonian causality, A and $B1$ are causally connected. However, in Special relativity they are not, as in some other reference frame $B1$ could be $B2$.

2.4 Time dilation: Proper time

The most important facet of special relativity is that it identifies *space* and *time* not as separate entities but part of a single *space-time* continuum. It forces us a rethink on our basic notion of time and space measurements in the elementary of situations.

Consider a clock at rest in a reference frame S. The world line of the clock is shown in Figure .

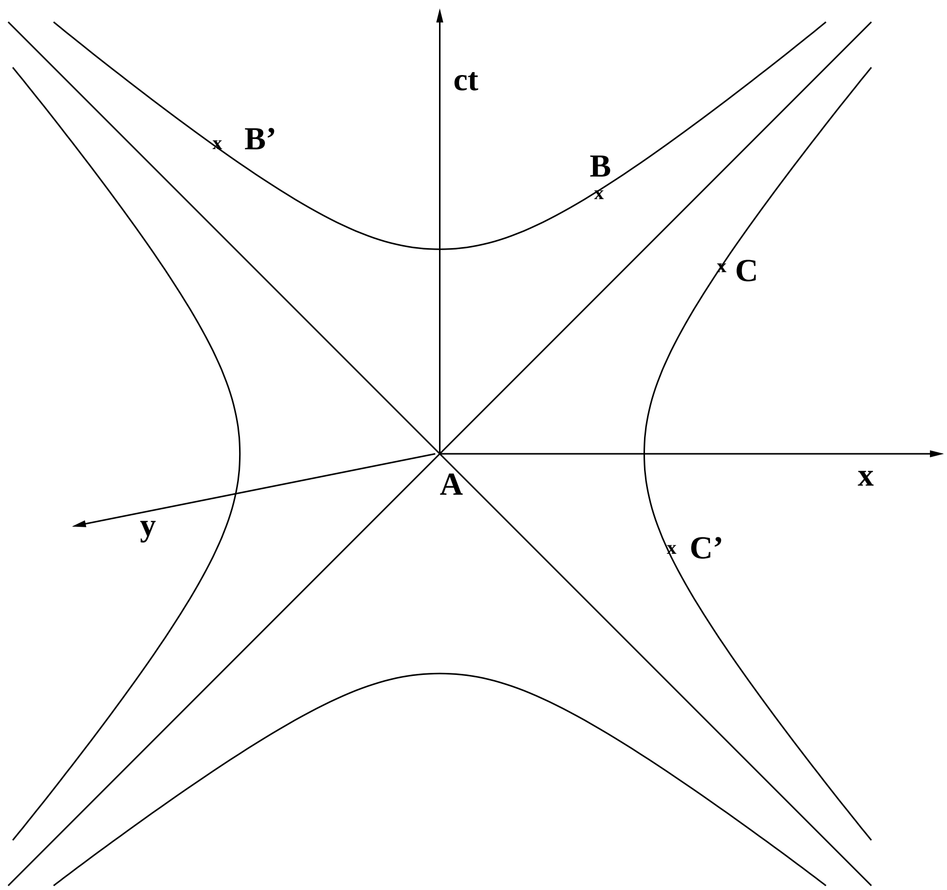

Figure 2.4: Event B might transform to B' in some other reference frame S'. However, the time ordering between events A and B is preserved ($A' = A$). This is true for any two events that are time like separated. But the same is not true for events C and A, which are space like separated. C transforms to C' in S'. Since the time ordering is reversed, A precedes C in S, but is reversed in S'.

The ticking of the clock at intervals $t_1, t_2, ..$ are the events denoted by the dots along the time axis. The space time interval between the first two 'ticks' is given by

$$\Delta s^2 = c^2(t_2 - t_1)^2. \tag{2.9}$$

Let S' be a reference frame moving with speed v in the x direction with respect to S. The two ticks of the clock in this reference frame do not happen at the same point in space. Let the two events in this frame be (ct'_1, x'_1) and (ct'_2, x'_2) (we do not bother about the y', z' coordinates since they are same as the unprimed ones, the motion being in the x direction). The space time interval in this frame is given by

$$\Delta s'^2 = c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 \tag{2.10}$$

$$= c^2(t'_2 - t'_1)^2 - v^2(t'_2 - t'_1)^2. \tag{2.11}$$

Since the two intervals are same by PII, it follows that

$$(t_2 - t_1) = \sqrt{(1 - \frac{v^2}{c^2})} \cdot (t'_2 - t'_1) \tag{2.12}$$

or

$$(t'_2 - t'_1). = \gamma(t_2 - t_1), \tag{2.13}$$

where $\gamma = 1/\sqrt{(1 - v^2/c^2)}$. So, the times shown by a moving clock will be different from the one it shows when it is at rest. It can be seen that $1 \leq \gamma \leq \infty$. $(t'_2 - t'_1)$ is the time interval between two 'ticks' in a frame in which the clock is moving. From Eq. (2.13) it follows that $(t'_2 - t'_1) \geq (t_2 - t_1)$. *Moving clocks run slower by a factor* γ.

Definition: The **Proper time** between two events is defined as the time interval between the events in a frame in which the two events happen at the same place. The time shown by a clock in a reference frame in which the clock is at rest is the **Proper time shown by the clock**. Evidently, proper time is not defined between two events that are *space like* separated.

2.5 Length contraction: Proper length

Having seen that time interval measurements in two reference frames are different, it is natural to expect the same about length measurements too. The definition of proper length goes along the same line as that of proper time.

Definition: The **Proper length** of an object is its length measured in a frame in which the object is at rest.

Let us compare the length of an object as measured in two reference frames. Frame S in which the object is at rest with length L, say along the x direction, and another frame S' in motion with respect to S along the x direction with speed v. The world lines of the object in the two frames are shown below.

Let A and B be the two events at which the two ends of the object cross the observer. In S

$$A = (ct_A; x_A) \ ; \quad B = (ct_B; x_B), \tag{2.14}$$

and in S'

$$A = (ct'_A; 0) \ ; \quad B = (ct'_B; 0). \tag{2.15}$$

Then the length of the object in S' is given by

$$L' = v(t'_B - t'_A). \tag{2.16}$$

The space time interval between A and B is the same in the two reference frames. This gives

$$\begin{aligned} c^2(t'_B - t'_A)^2 &= c^2(t_B - t_A)^2 - (x_B - x_A)^2 \\ \Rightarrow c^2\frac{L'^2}{v^2} &= \frac{c^2}{v^2}L^2 - L^2. \end{aligned} \tag{2.17}$$

Or,

$$L' = L\sqrt{(}1 - \frac{v^2}{c^2}) = \frac{L}{\gamma}. \tag{2.18}$$

Since $\gamma \geq 1$, $L' \leq L$, i.e objects in motion appear smaller along the direction of motion.

PS: We have considered only the contraction along the direction of motion. However, it is straight forward to show (using detailed space time diagrams) that in the other two directions, i.e., the direction perpendicular to motion, there is no contraction.

Chapter 3

Four Vectors and the $*$ product

As we saw in the last chapter, special relativity identifies space and time not as independent entities but as two facets of a space-time continuum. Consequently, a point in space time is identified 4 coordinates, one time and three space. Thus the position vector of an object in space time is given by[1]

$$\underline{\mathbf{r}} = (ct; x, y, z) = (ct, \mathbf{r}). \tag{3.1}$$

3.1 The four-velocity vector

Every observer makes measurements in her own lab frame in her own clock - *her own proper time*. Evidently, thats one time on which everybody will agree. It seems natural therefore to define the four velocity of an object as the *proper time derivative* of its position 4-vector. However, between two observers in relative motion we know they do not measure the same time interval between two given events. Indeed, the proper time interval $\Delta\tau$ is related to the time interval observed in any other frame Δt by

$$\Delta\tau = \frac{\Delta t}{\gamma}. \tag{3.2}$$

The time derivatives in the two reference frames are related by

$$\frac{d}{d\tau} = \gamma\frac{d}{dt}. \tag{3.3}$$

The four velocity of a moving object is defined thus:

$$\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{d\tau} = \gamma\frac{d\underline{\mathbf{r}}}{dt} = \gamma(c; \mathbf{v}). \tag{3.4}$$

Let's look at each of the quantities in Eq. (3.4) carefully. The situation is the following: In a given frame (say, the rest frame) an object, whose position vector is $\mathbf{r}$, is observed to be moving with velocity $\mathbf{v}$. τ is the time as measured in the object's own reference frame - *its proper time*. The four velocity of the object in the object's own reference frame will be (and so will be the four velocity of anyone or anything in its own reference frame)

$$\underline{\mathbf{v}} = (c; 0). \tag{3.5}$$

Notice that the four velocities in Eq. (3.4) and Eq. (3.5) are one and the same, but just observed from two different reference frames.

[1]The semicolon separating time and space coordinates need not be taken seriously. Henceforth 3-vectors will be denoted by **bold** faces, and 4-vectors by underlined **bold** faces.

3.2 The $*$ product

Analogous to the $\cdot$ product of three-vectors, we define the $*$ product of two four-vectors $\underline{\mathbf{A}} = (a_t; a_x, a_y, a_z)$ and $\underline{\mathbf{B}} = (b_t; b_x, b_y, b_z)$ as

$$\underline{\mathbf{A}} * \underline{\mathbf{B}} = a_t b_t - a_x b_x - a_y b_y - a_z b_z. \quad (3.6)$$

With this definition, we notice

$$\underline{\mathbf{v}} * \underline{\mathbf{v}} = c^2. \quad (3.7)$$

A four vector connects two space time points. Consequently, PII can be rewritten as

- **PII:** *The $*$ product of any two four-vectors defined in space time is an invariant in any reference frame.*

3.3 The four-accelaration

The four-accelaration is defined as

$$\underline{\mathbf{a}} = \frac{d\underline{\mathbf{v}}}{dt} = \gamma \frac{d}{d\tau}(\gamma c; \gamma \vec{\mathbf{v}}). \quad (3.8)$$

Differentiating Eq. (3.7), we get

$$\underline{\mathbf{v}} * \underline{\mathbf{a}} = 0. \quad (3.9)$$

Thus, $\underline{\mathbf{a}}$ and $\underline{\mathbf{v}}$ are always *orthogonal* (Note that the notion of orthogonality here is purely abstract, and should not be confused with vectors being perpendicular, as in 3-D).

3.3.1 The case of uniform accelaration

We shall look in detail an object that is uniformly accelarated, i.e., an object that feels uniformly accelarated, with accelaration say 'a' in the x direction. From the object's own frame it is possible to measure its accelaration (but not speed), without any reference to an external reference point. The question we adress here is, if a ground observer S were to measure the acceleration of the object, will it be the same as 'a'.

It is necessary to understand here quantitatively what we mean by a. Let the velocity of the object at some instant be v in the x direction. This speed is changing as the object accelerates. The acceleration a is the change in velocity as seen by the object itself. Thus if we consider a reference frame S' at a speed close to v (in the x direction), with respect to the ground observer at the same instant, then the acceleration of the object seen by S' at that instant is roughly a. If v' is the velocity of the object as seen by S', then the four-velocity of the object at that instant is $(c; v', 0, 0)$ (taking $\gamma \sim 1$). The velocity being small with respect to S', the four-accelaration is

$$\underline{\mathbf{a}} \sim \frac{d}{dt}(c; v', 0, 0) = (0; a, 0, 0). \quad (3.10)$$

Eq. (3.10) gives the four-acceleration as seen by S' around the time when the object is at speed v, and it is assumed that $\gamma = 1$. From Eq. (3.10)

$$\underline{\mathbf{a}} * \underline{\mathbf{a}} = -a^2. \quad (3.11)$$

But, the star product being an invariant, it must be same in the ground reference frame S also.

Given the special relation satisfied by v, Eq. (3.7), we choose the following parameterization:

$$\underline{\mathbf{v}} = (v_t; v_x) = c(\cosh(\kappa\tau); \sinh(\kappa\tau)), \tag{3.12}$$

a choice that is valid always (y and z components are zero, and hence not shown in the above). Upon integration with respect to τ we get

$$\underline{\mathbf{r}} = \frac{c}{\kappa}(\sinh(\kappa\tau); \cosh(\kappa\tau) - 1). \tag{3.13}$$

The constant of integration -1 in the space part in Eq. (3.13) has been introduced so that $x = 0$ at $\tau = 0$. Differentiating $\underline{\mathbf{v}}$ with respect to τ we get $\underline{\mathbf{a}}$ as

$$\underline{\mathbf{a}} = c\kappa(\sinh(\kappa\tau); \cosh(\kappa\tau)). \tag{3.14}$$

Evidently $\underline{\mathbf{v}} * \underline{\mathbf{a}} = 0$. The condition

$$\underline{\mathbf{a}} * \underline{\mathbf{a}} = -a^2 \tag{3.15}$$

gives

$$c^2\kappa^2 = a^2. \tag{3.16}$$

Thus we have the final solution,

$$\underline{\mathbf{r}} = \frac{c^2}{a}(\sinh(a\tau/c); \cosh(a\tau/c) - 1)\ ;\ \underline{\mathbf{v}} = c(\cosh(a\tau/c); \sinh(a\tau/c))\ ;\ \ \underline{\mathbf{a}} = a(\sinh(a\tau/c); \cosh(a\tau/c)). \tag{3.17}$$

The velocity as seen from the ground frame is given by

$$\frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \frac{1}{\gamma}\frac{dx}{d\tau} \tag{3.18}$$

From the expression for $\underline{\mathbf{v}}$ we find $\gamma = \cosh(a\tau/c)$ and $dx/d\tau = c\sinh(a\tau/c)$. Thus the velocity as seen by the ground observer is $c\tanh(a\tau/c)$. So, inspite of continued uniform acceleration, the object never reaches c in finite time.

It is important to realize here that when we say uniform acceleration, we are referring to the acceleration as seen by the accelerating object itself, in its own reference frame.

Chapter 4

Velocity addition and Lorentz transformations

We have noted already that in going from one reference frame to another the simple velocity addition rule does not hold. In this chapter we shall find the correct relativistic expression for adding velocities, the relations connecting space-time coordinates in two reference frames in relative motion - the Lorentz transformation relations.

4.1 Velocity addition

Let two observers 1 and 2 in relative motion observe an object 3 in motion. Say 1 is the ground reference frame, 2 moving with respect to 1 along their common x direction with velocity v_{21}, and let 3 also move in the x direction with speed v_{32} with respect to 2. Evidently the velocity of 1 with respect to 2 is $v_{12} = -v_{21}$. The task is to find v_{31}, the velocity of 3 with respect to 1.

Let us write the corresponding four velocities:

$$\underline{\mathbf{v}}_{21} = \gamma_{21}(c; v_{21}, 0, 0), \tag{4.1}$$

$$\underline{\mathbf{v}}_{32} = \gamma_{32}(c; v_{32}, 0, 0), \tag{4.2}$$

$$\underline{\mathbf{v}}_{12} = \gamma_{12}(c; v_{12}, 0, 0) = \gamma_{21}(c; -v_{21}, 0, 0) \tag{4.3}$$

$$\underline{\mathbf{v}}_{31} = \gamma_{31}(c; v_{31}, 0, 0) =? \tag{4.4}$$

where, as usual, $\gamma_{ij} = 1/\sqrt{1 - v_{ij}^2/c^2}$.

To determine v_{31} we shall use the fact that the $*$ product of two four-vectors is invariant in any reference frame, and that the four-velocity of any object in its own reference frame is $(c; 0)$, since in its own reference frame it is at rest. Thus, the speed of 1 in its own reference frame is zero, and consequently the four-vector

$$\underline{\mathbf{v}}_{11} = (c; 0). \tag{4.5}$$

From the invariance of the $*$ product it follows that

$$\underline{\mathbf{v}}_{32} * \underline{\mathbf{v}}_{12} = \underline{\mathbf{v}}_{31} * \underline{\mathbf{v}}_{11}. \tag{4.6}$$

Substituting for these four vectors from Eqs. (4.1)-(4.5), we have,

$$\gamma_{32}\gamma_{12}(c^2 - v_{32}v_{12}) = \gamma_{31}(c^2), \tag{4.7}$$

or,

$$\gamma_{31} = \gamma_{32}\gamma_{12}(1 - \beta_{32}\beta_{12}) = \gamma_{32}\gamma_{21}(1 + \beta_{32}\beta_{21}). \tag{4.8}$$

After some simple algebra we find

$$\beta_{31} = \sqrt{1 - \frac{1}{\gamma_{31}^2}} = \frac{\beta_{32} + \beta_{21}}{1 + \beta_{32}\beta_{21}} \tag{4.9}$$

$$v_{31} = \frac{v_{32} + v_{21}}{1 + v_{32}v_{21}/c^2}. \tag{4.10}$$

Notice that the expression for v_{31} reduces to the usual Galelian expression $v_{31} = v_{32} + v_{21}$ when the velocities are small compared to c.

Summary: If an object is moving with velocity v_{32} with respect to frame 2, and the frame 2 itself is moving with respect another frame 1 with velocity v_{21}, then the velocity of the object with respect to 1, v_{31}, is **not** the simple Galelian velocity addition law, but the expression given by Eq. (4.10), which reduces to the Galelian form for small velocities.

Not that in the above demo it is assumed that all velocivites is are in the same common x direction. If the velocities are different the expressions are a little more complicated.

4.2 Lorentz transformations

Two observers in two different reference frames look at the the same events differently. Their coordinates may be different, and so could be their directions in space-time. Thus the same four vector $\underline{\mathbf{A}}$ in space-time may be designated differently by the two observers in frames S and S' as

$$\underline{\mathbf{v}} = a_t\underline{\mathbf{t}} + a_x\underline{\mathbf{x}} + a_y\underline{\mathbf{y}} + a_z\underline{\mathbf{z}} = a_{t'}\underline{\mathbf{t}}' + a_{x'}\underline{\mathbf{x}}' + a_{y'}\underline{\mathbf{y}}' + a_{z'}\underline{\mathbf{z}}' \tag{4.11}$$

We intend to find the relations between the two sets of unit four-vectors, and the relations between the two sets of coordinates.

To do this we start with a known four vector. Recall that the four-veolcity of any object *in its own reference frame* is given by $(c; 0, 0, 0)$. Thus, let us look at the four velocity of the frame S' in the two frames S and S'. Let v be the velocity (say, along the x direction for simplicity) of S' with respect to S.

$$\underline{\mathbf{v}} = \gamma c\underline{\mathbf{t}} + \gamma v\underline{\mathbf{x}} = c\underline{\mathbf{t}}'. \tag{4.12}$$

The two vectors are the same, as they refer to the same four-vector in space time. Eq. (4.12) gives a relation between the time unit vector $\underline{\mathbf{t}}'$ in frame S' and that of the $\underline{\mathbf{x}}$ and $\underline{\mathbf{t}}$ in S. Now we have to determine $\underline{\mathbf{x}}'$. Let

$$\underline{\mathbf{x}}' = a\underline{\mathbf{t}} + b\underline{\mathbf{x}}. \tag{4.13}$$

To determine a and b we make use of the ortho normality conditions.

$$\underline{\mathbf{x}}' * \underline{\mathbf{x}}' = -1 \;\; ; \;\; \underline{\mathbf{t}}' * \underline{\mathbf{x}}' = 0 \tag{4.14}$$

yeild

$$a^2 - b^2 = -1 \; ; \;\; ac - bv = 0. \tag{4.15}$$

Substituting $a = b\beta$ in the first expression in Eq. (4.15), we have

$$b = \gamma \; ; \;\; a = \gamma\beta. \tag{4.16}$$

Thus for the simple choice that the velocity is along the common x direction of the two frames, we find the relations between the unit vectors as

$$\underline{\mathbf{t}}' = \gamma \underline{\mathbf{t}} + \gamma\beta \underline{\mathbf{x}}, \tag{4.17}$$
$$\underline{\mathbf{x}}' = \gamma\beta \underline{\mathbf{t}} + \gamma \underline{\mathbf{x}}, \tag{4.18}$$
$$\underline{\mathbf{y}}' = \underline{\mathbf{y}}, \tag{4.19}$$
$$\underline{\mathbf{z}}' = \underline{\mathbf{z}}. \tag{4.20}$$
$$\tag{4.21}$$

The last two vectors are the same since the relative motion is along the x direction. It is now straight forward to relate the coordinates in the two frames. In Eq. (4.11) we take $*$ product with $\underline{\mathbf{t}}'$ on both sides. This gives,

$$\gamma a_t - \gamma\beta a_x = a_{t'}. \tag{4.22}$$

Repeating the procedure for $\underline{\mathbf{x}}'$, $\underline{\mathbf{y}}'$ and $\underline{\mathbf{z}}'$ (and adjusting the signs), we get

$$-\gamma\beta a_t + \gamma a_x = a_{x'}, \tag{4.23}$$
$$a_y = a_{y'} \tag{4.24}$$
$$a_z = a_{z'} \tag{4.25}$$
$$\tag{4.26}$$

In particular, the relation between the coordinates are

$$t' = \gamma t - \gamma\beta x \tag{4.27}$$
$$x' = \gamma\beta t - \gamma x \tag{4.28}$$
$$y' = y \tag{4.29}$$
$$z' = z. \tag{4.30}$$

As earlier, the coordinates y and z are unchanged in the two frames as the motion is only along the x direction. In general the expression connecting the two sets are

They constitute the Lorentz transformation relations connecting two frames, whose axes are aligned, and their origins coincide. Such a transformation is called a *pure Lorentz* transformation. The transformation matrix $\mathbf{L}$ in such a case is symmetric.

In general frame S' can be rotated through some constant angle with respect to S, and their origins need not coincide.

Chapter 5

Relativistic Mechanics

The purposefulness of energy and momentum come from their conservation criterions. For an isolated system the total momentum is conserved, and so is the energy. But, will the momentum conservation hold if we were to look at the same system from any reference frames. Given that velocity addition is not the simple Galelian relation, but the expression in Eq. (4.10) (in fact more complicated for a general case), momentum conservation is tricky business as viewed from different reference frames. In this chapter we analyze this question, and that of energy, and look at the modifications Special relativity makes to mechanics.

5.1 Momentum, as we know it, of an isolated system of particles is not conserved if we look at it from different reference frames

Consider an isolated system of 2 particles that collide *head on* and *perfectly elastically*. The motion is essentially one dimensional. Let v_1 and v_2 be their velocities before collision and v_1' and v_2' after the collision. If m_1 and m_2 are their respective velocities, then the momentum conservation demands

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'. \tag{5.1}$$

Or, dividing through out by c,

$$m_1 \beta_1 + m_2 \beta_2 = m_1 \beta_1' + m_2 \beta_2'. \tag{5.2}$$

Let us consider another reference frame S' moving at velocity v_0, with respect to the lab frame in the same direction of motion of the two particles. Let the velocities in S' frame be $\tilde{v}_i$ and $\tilde{v}_i'$, $i = 1, 2$, or again dividing by c, $\tilde{\beta}_i$ and $\tilde{\beta}_i'$. If momentum conservation holds then it must hold in all reference frames, and hence in S' we demand

$$m_1 \tilde{\beta}_1 + m_2 \tilde{\beta}_2 = m_1 \tilde{\beta}_1' + m_2 \tilde{\beta}_2'. \tag{5.3}$$

But from the velocity addition rule in Eq. (4.9), we should have

$$\tilde{\beta}_i = \frac{\beta_0 + \beta_i}{1 + \beta_0 \beta_i}, \tag{5.4}$$

and similarly for $\tilde{\beta}'_i$. Substituting in Eq. (5.3), we have

$$m_1\frac{\beta_0+\beta_1}{1+\beta_0\beta_1}+m_2\frac{\beta_0+\beta_2}{1+\beta_0\beta_2}=m_1\frac{\beta_0+\beta'_1}{1+\beta_0\beta'_1}+m_2\frac{\beta_0+\beta'_2}{1+\beta_0\beta'_2}. \tag{5.5}$$

Given Eq. (5.1), and that β_0 is arbitrary in the range $[0,1]$, there is no way Eq. (5.5) will be satisfied for all β_0. The momentum, *as we know it*, of an isolated system is not a constant!! It is easily to check along similar lines that the kinetic energy of the two particle system is not conserved either!

5.2 The four-momentum

Momentum and energy of an isolated system not being a constant as seen from different reference frames poses a serious problem for mechanics. It means that what is an isolated system for one observer might appear to be acted upon by forces from some other *inertial* reference frame - questioning Newton's laws. The resolution is that the expression for momentum and kinetic energy that we have been using is only an approximate one. We have to find a correct expression for momentum and energy, that are conserved in different reference frames, and reduce to the Newtonian expressions for low velocities. The answer comes in the form of the four-momentum.

Having already familiarized with four-velocity (see Eq. (3.4)), we define the four-momentum of an object as

$$\underline{\mathbf{p}}=m\gamma(c;\vec{\mathbf{v}}), \tag{5.6}$$

where m is the mass of the object measured when it is at rest, or its *rest mass*. It is straight forward to check that the four-momentum for the system of two colliding particles is conserved in S and S'. If four-momentum is conserved in S, then

$$m_1\gamma_1(c;\beta_1)+m_2\gamma_2(c;\beta_2)=m_1\gamma'_1(c;\beta'_1)+m_2\gamma'_2(c;\beta'_2), \tag{5.7}$$

where we have not bothered to write the y and z components, as all motion is along the spacial x direction. Being a vectorial relation, Eq. (5.7) breaks into two equations

$$m_1\gamma_1+m_2\gamma_2=m_1\gamma'_1+m_2\gamma'_2 \tag{5.8}$$

$$m_1\gamma_1\beta_1+m_2\gamma_2\beta_2=m_1\gamma'_1\beta'_1+m_2\gamma'_2\beta'_2. \tag{5.9}$$

We demand, in reference frame S'

$$m_1\tilde{\gamma}_1(c;\tilde{\beta}_1)+m_2\tilde{\gamma}_2(c;\tilde{\beta}_2)=m_1\tilde{\gamma}'_1(c;\tilde{\beta}'_1)+m_2\tilde{\gamma}'_2(c;\tilde{\beta}'_2). \tag{5.10}$$

Given Eq. (5.4), the relation between $\tilde{\beta}_i$ and β_i, it is easily seen that (Eq. (4.8) for instance)

$$\tilde{\gamma}_i=\gamma_0\gamma_i(1+\beta_0\beta_i). \tag{5.11}$$

Substituting from Eqs. (5.4) and (5.11) in Eq. (5.10), we have the temporal and spacial components as

$$m_1\gamma_0\gamma_1(1+\beta_0\beta_1)+m_2\gamma_0\gamma_2(1+\beta_0\beta_2)=m_1\gamma_0\gamma'_1(1+\beta_0\beta'_1)+m_2\gamma_0\gamma'_2(1+\beta_0\beta'_2) \tag{5.12}$$

$$\begin{aligned}m_1\gamma_0\gamma_1(1+\beta_0\beta_1)\frac{\beta_0+\beta_1}{(1+\beta_0\beta_1)}+m_2\gamma_0\gamma_2(1+\beta_0\beta_2)\frac{\beta_0+\beta_2}{(1+\beta_0\beta_2)}&=m_1\gamma_0\gamma'_1(1+\beta_0\beta'_1)\frac{\beta_0+\beta'_1}{(1+\beta_0\beta'_1)}\\&+m_2\gamma_0\gamma'_2(1+\beta_0\beta'_2)\frac{\beta_0+\beta'_2}{(1+\beta_0\beta'_2)}\end{aligned} \tag{5.13}$$

Evidently, Eqs. (5.12) and (5.13) follow from Eqs. (5.8) and (5.9). Hence, we have shown that if the four-momentum is conserved in S, so will it be in S'. And we do know that the four momentum is conserved in a reference frame where the velocities are small: in which case Eq. (5.8) is the sum of masses after and before collision, and Eq. (5.9) is the sum of Newtonian momentum.

Summary: It is not the Newtonian momentum and kinetic energy that is conserved in collision of an isolated system of two particles, but the four-momentum.

5.3 Four-momentum: Interpretation

5.3.1 Momentum, mass and force

The spacial part of the four-momentum, $\gamma m\vec{\mathbf{v}}$ reduces to the usual Newtonian momentum for small velocities. We identify this as the relativistic momentum of an object, or the correct expression for momentum. Thus the momentum of a object of *rest mass* m and moving with velocity $\vec{\mathbf{v}}$ is

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}}. \tag{5.14}$$

Mass is defined as momentum upon velocity, and hence we identify that the mass of an object is γm, where m is its rest mass. With this correction, the statement of Newton's law remains the same, except that the momentum is as defined in Eq. (5.14)

$$Force: \ \vec{\mathbf{F}} = \frac{d\tilde{\mathbf{p}}}{d\tau} = \gamma\frac{d(\gamma m\tilde{\mathbf{v}})}{dt}. \tag{5.15}$$

With this definition for momentum, Eq. (5.14), it is clear that an object can never reach the velocity c. Upon acceleration its momentum will change, consistent with Newton's law, but it starts reflecting in the mass than the velocity.

5.3.2 Energy

The most important feature of the time part of four-momentum $p_t = \gamma mc$ in Eq. (5.6) is that it is conserved for an isolated system as seen from any reference frame. To get more clarity on this quantity, let us look at the same in familiar territory, i.e., at low velocities.

At $v = 0$, $p_t = mc$, which doesn't tell us much. For small velocities, γ can be expanded as

$$\gamma \sim 1 + \frac{1}{2}\beta^2. \tag{5.16}$$

Thus we find

$$p_t \sim \frac{1}{c}(mc^2 + \frac{1}{2}mv^2). \tag{5.17}$$

Also,

$$\frac{dp_t}{d\tau} = \gamma\frac{d\gamma}{dt}mc = \gamma^4 mc\beta\frac{d\beta}{dt}. \tag{5.18}$$

But from Eq. (5.15), for the simple case of v along x say, we have

$$\begin{aligned} F &= \gamma\frac{d\gamma}{dt}mv + \gamma^2 mc\frac{d\beta}{dt} \\ &\Rightarrow \gamma^4\beta\frac{d\beta}{dt}mc\beta + \gamma^2 mc\frac{d\beta}{dt}. \end{aligned} \tag{5.19}$$

This gives

$$F\beta = (\gamma^2\beta^2 + 1)\gamma^2 mc\frac{d\beta}{dt}\beta = \gamma^4 mc\frac{d\beta}{dt}\beta. \tag{5.20}$$

Comparing Eq. (5.20) and Eq. (5.18), we have

$$\gamma\frac{d(cp_t)}{dt} = Fv. \tag{5.21}$$

But Eq. (5.21) is the work energy theorem in the differential form. Consequently we define the *Relativistic energy* as

$$E = cp_t = \gamma mc^2, \tag{5.22}$$

following the three facts we have collected: a) It is conserved for an isolated system, in any reference frame, b) It goes like the kinetic energy plus a *rest energy* mc^2 for small velocities, and c) it obeys the work energy theorem.

Thus the four momentum

$$\underline{\mathbf{P}} = (\frac{E}{c}; \vec{\mathbf{p}}) \tag{5.23}$$

incorporates in it both the conservation of momentum and energy. Note that the conservation of either of Eq. (5.12) and Eq. (5.13) require both Eq. (5.8) and Eq. (5.9). Thus the four dimensional nature is indispensable. Form Eq. (5.6) it follows that

$$\underline{\mathbf{P}} * \underline{\mathbf{P}} = m^2c^2. \tag{5.24}$$

Thus

$$E^2c^2 - p^2 = m^2c^2. \tag{5.25}$$

The quantity $E - mc^2$ will be defined as kinetic energy K. Thus

$$K = E - mc^2 = (\gamma - 1)mc^2. \tag{5.26}$$

5.4 Decay process

We shall analyze a simple case of a particle of rest mass M decaying, or splitting spontaneously into two particles of rest mass m_1 and m_2. We shall look at the problem from the reference frame in which the particle was initially at rest. Given the rest masses, the kinetic energy of each particle after the decay can be found using the conservation of four-momentum. The four momentum of the particle before decay is (since it is at rest)

$$\underline{\mathbf{p}} = (Mc; 0). \tag{5.27}$$

The momentum of the two particles are

$$\underline{\mathbf{p}}_1 = (\gamma_1 m_1 c; \gamma_1 m_1 v_1, 0, 0) \ ; \quad \underline{\mathbf{p}}_2 = (\gamma_2 m_2 c; \gamma_2 m_2 v_2, 0, 0). \tag{5.28}$$

In the above, the direction of motion of the particles after splitting is taken as x direction, hence the momentum components along the y and z directions are zero. From four momentum conservation we have

$$\underline{\mathbf{p}} = \underline{\mathbf{p}}_1 + \underline{\mathbf{p}}_2. \tag{5.29}$$

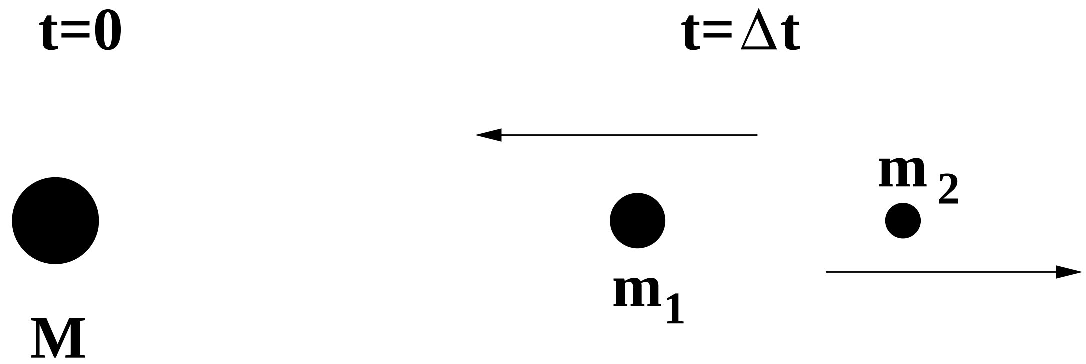

Figure 5.1: A decay process, showing a mass M initially at rest decaying into two particles of rest masses m_1 and m_2. Since the two particles have kinetic energy, $M > m_1 + m_2$.

Breaking into time and space part, we have

$$\begin{aligned} M &= \gamma_1 m_1 + \gamma_2 m_2 \\ \gamma_1 m_1 v_1 &= \gamma_2 m_2 v_2. \end{aligned} \tag{5.30}$$

From Eq. (5.30) we notice that $M \geq m_1 + m_2$, with the '=' holding only when $\gamma_1 = \gamma_2 = 0$, i.e., both particles are at rest after decay. The difference in mass reflects in the form of kinetic energy of the particles. After some jugglery it can be shown that the kinetic energies of the particles are given by

$$\begin{aligned} K_1 &= \frac{(Mc^2 - m_1c^2)^2 - (m_2c^2)^2}{2Mc^2} \\ K_2 &= \frac{(Mc^2 - m_2c^2)^2 - (m_1c^2)^2}{2Mc^2} \end{aligned} \tag{5.31}$$

5.5 Compton scattering

The momentum four-vector of light can be written in consonance with the interpretation of the time part as energy, and the space part as the linear momentum. Thus invoking quantum mechanics, we have the momentum four-vector for light

$$\underline{\mathbf{k}}_{Light} = (\frac{h\nu}{c}; \hbar\mathbf{k}). \tag{5.32}$$

Scattering of a massive particle by light is referred to as Compton scattering. A simple scenario is shown in Figure 5.2, in which a photon of energy $E = h\nu$ collides with a electron of mass m_e. After collision the rest mass of the electron stays the same, although it has acquired some kinetic energy. The wavelength of the scattered photon is related to the angle of scattering. It is this relation that we try to find here. For convenience we have chosen a reference frame in which the particle is initially at rest. Besides, the scattering the assumed to happen in the $x - y$ plane. The momentum four vectors for light and the particle, before and after the collision are, respectively,

Before:

$$(\frac{h\nu}{c}; \frac{h\nu}{c}, 0) \ ; \quad (m_e c; 0, 0), \tag{5.33}$$

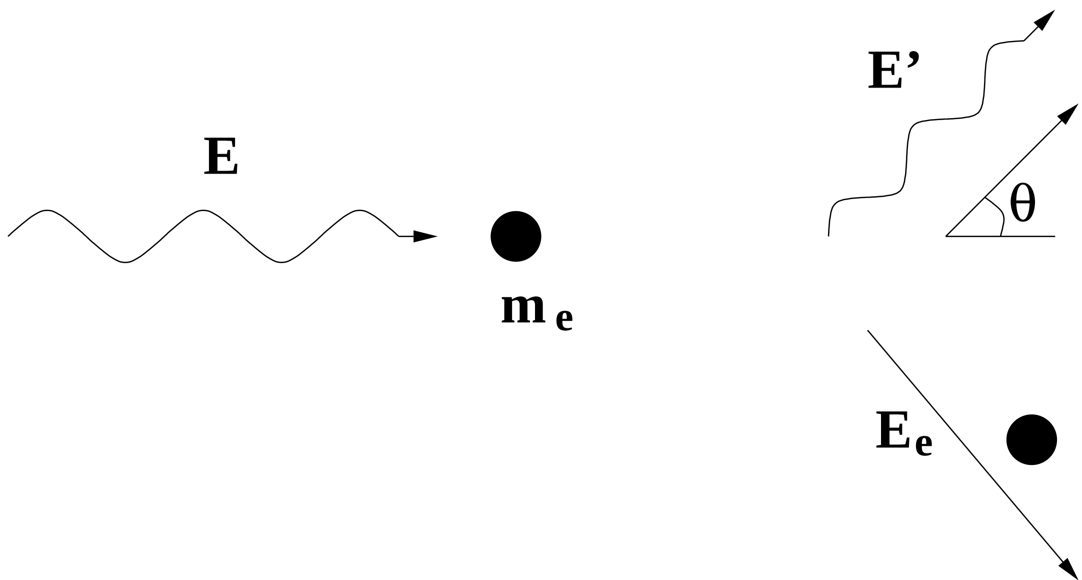

Figure 5.2: Scattering of a electron, initially at rest, by a photon of energy $E = h\nu$. The change in frequency (or wavelength) of the photon depends on the angle of scattering θ.

After:

$$(\frac{h\nu'}{c}; \frac{h\nu'}{c}\cos\theta, \frac{h\nu'}{c}\sin\theta)\ ;\quad (\frac{E_e}{c}; P_{ex}, P_{ey}), \tag{5.34}$$

where

$$E_e^2 - (P_{ex}^2 + P_{ey}^2)c^2 = m_e^2 c^4, \tag{5.35}$$

and ν' is the frequency of the scattered photon. Conservation of four-momentum gives

$$h\nu + m_e c^2 = h\nu' + E_e, \tag{5.36}$$

$$h\nu = h\nu'\cos\theta + P_{ex}c, \tag{5.37}$$

$$h\nu'\sin\theta = P_{ey}c. \tag{5.38}$$

Substituting Eqs. (5.36)-(5.38) in Eq. (5.35), we have

$$m_e^2 c^4 = (h\nu + m_e c^2 - h\nu')^2 - (h\nu - h\nu'\cos\theta)^2 - h\nu'^2\sin^2\theta. \tag{5.39}$$

Ultimately, this reduces to

$$0 = -(\nu - \nu')m_e c^2 + h\nu\nu' - h\nu\nu'\cos\theta. \tag{5.40}$$

Substituting $\nu = c/\lambda$ and $\nu' = c/\lambda'$ in Eq. (5.40), we get

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta). \tag{5.41}$$

Eq. (5.41) gives the desired relation between change in wavelength and the angle of scattering. The quantity $h/(m_e c)$ is the Compton wavelength of electron and is roughly equal to $2.43 \times 10^{-12} m$. The angle of scattering θ may vary from zero (no scattering, or forward scattering), to 180^o (back scattering).

Chapter 6

Waves

A wave is a propagating disturbance of some field. A sound wave is a fluctuation of the density field of the medium in which it is propagating. Light is a periodic propagating electromagnetic field. A wave is characterized by its wave vector $\mathbf{k}$, giving the direction of propagation and related to the wavelength as $|\mathbf{k}| = 2\pi/\lambda$, the angular frequency ω_k, and amplitude. A simple plane wave is given by

$$\Phi = \Phi_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega_k t) \tag{6.1}$$

The phase velocity of the wave is $v_k = \omega_k/|\mathbf{k}|$. The subscript k has been used to indicate that waves of different frequencies have different velocities in a medium. In this chapter we shall relate the observation of these quantities as seen by two observers in relative motion.

If the wave is observed with wave vector $\mathbf{k}$ and angular frequency $\omega_{\mathbf{k}}$ in frame S, we intend to find the corresponding quantities $\mathbf{k}'$ and $\omega'_{k'}$ as seen from the reference frame S' in relative motion with respect to S.

A simple plane wave is shown in Figure 6.1. A wave can be seen as a sequence of events, say the crests of the field, that happen at regular intervals. Two observers looking at the same wave will, though they may not agree on the frequency and wavelength, have no ambiguity in which are the crests. The same goes for the troughs, and consequently for any phase of the wave. That is to say, their observations will be such that

$$\mathbf{k} \cdot \mathbf{r} - \omega_k t = \mathbf{k}' \cdot \mathbf{r}' - \omega'_{k'} t'. \tag{6.2}$$

Recalling that $(ct; \mathbf{r})$ and $(ct'; \mathbf{r}')$ are the four vectors in the two frames, Eq. (6.2) is merely the statement that

$$\underline{\mathbf{k}} * \underline{\mathbf{r}} = \underline{\mathbf{k}}' * \underline{\mathbf{r}}', \tag{6.3}$$

if we define

$$\underline{\mathbf{k}} = (\frac{\omega_k}{c}; \mathbf{k}) \ ; \ \ \underline{\mathbf{k}} = (\frac{\omega'_{k'}}{c}; \mathbf{k}'). \tag{6.4}$$

It may be noted that this definition of $\underline{\mathbf{k}}$ is consistent with the definition of the momentum four vector for light in Eq. (5.32).

It must be noted that a four vector cannot be constructed arbitrarily. For instance, multiplying a four-vector by γ will change its character. Owing to PII in the form Eq. (2.1), the four-position as defined in Eq. (3.1) immediately qualified as a four-vector, such that the $*$ product is invariant. Later on, all other four-vectors were carefully defined, using the derivative with respect to the *invariant* τ. In Eq. (6.4), using the phase matching argument, we have *identified* the wave four-vector relevant for special relativity. With this identification what is left is to study the consequence of PII, that $*$ product of two four-vectors is invariant.

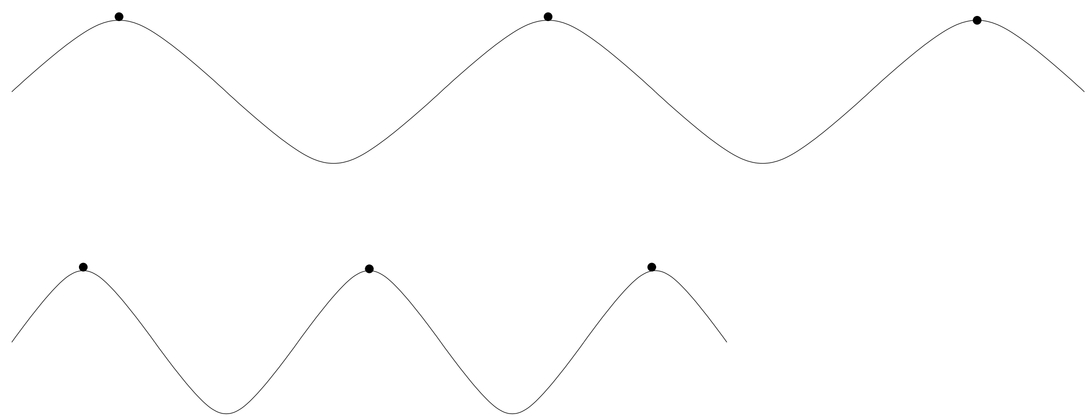

Figure 6.1: A wave as seen in two different reference frames in relative motion. The two observers may not agree on the wave length and frequency, but if the wave is seen as a sequence of events, say the maxima of the field, they agree on the events, that are the peaks. I.e., the phases as seen in the two frames are the same.

Also Eq. (6.4) defines the wave four-vector associated with any field, sound, light, or what ever. Specifically, in the case of light in vacuum, $\omega_k = c|\mathbf{k}|$, hence for light

$$\underline{\mathbf{k}} = (|\mathbf{k}|; \mathbf{k}), \tag{6.5}$$

and is evidently a null vector.

6.1 Doppler effect

The shift in frequency of a wave due to relative motion between the source and observer is called the Doppler effect.

6.1.1 Longitudinal Doppler effect

Let $\mathbf{v}_0$ be the relative velocity between the observer and source as shown in the figure (Figure 6.2). If $\underline{\mathbf{t}}, \underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}$ and $\underline{\mathbf{t}}', \underline{\mathbf{x}}', \underline{\mathbf{y}}', \underline{\mathbf{z}}'$ are the unit four-vectors in the source and observer frames, respectively, then the four-vector of the wave in the two frames are given by

$$\underline{\mathbf{k}} = \frac{\omega_k}{c}\underline{\mathbf{t}} + k_x\underline{\mathbf{x}} + k_y\underline{\mathbf{y}} + k_z\underline{\mathbf{z}} = \frac{\omega'_{k'}}{c}\underline{\mathbf{t}}' + k'\underline{\mathbf{x}}' + k'_y\underline{\mathbf{y}}' + k'_z\underline{\mathbf{z}}'. \tag{6.6}$$

The four-velocity of the observer in the two frames are

$$\underline{\mathbf{v}} = c\gamma_0\underline{\mathbf{t}} + \gamma_0 v_{0x}\underline{\mathbf{x}} + \gamma_0 v_{0y}\underline{\mathbf{y}} + \gamma_0 v_{0z}\underline{\mathbf{x}} = c\underline{\mathbf{t}}', \tag{6.7}$$

where $\gamma_0 = (1 - v_0^2/c^2)^{-1/2} = (1 - \beta_0^2)^{-1/2}$, as usual. Invoking PII, we have $\underline{\mathbf{k}} * \underline{\mathbf{v}}$ is a invariant. Thus,

$$\gamma_0(\omega_k - \mathbf{k} \cdot \mathbf{v}_0) = \omega'_{k'}. \tag{6.8}$$

It must be remembered that the expression for $\omega'_{k'}$ in Eq. (6.8) is true for any wave. Particularly for light in vacuum, we have,

$$\omega_k = c|\mathbf{k}|. \tag{6.9}$$

Writing $\mathbf{k} = k\hat{\mathbf{k}}$, we have

$$\omega'_{k'} = \gamma_0 \omega_k (1 - \hat{\mathbf{k}} \cdot \vec{\beta}_0), \tag{6.10}$$

where $\vec{\beta}_0 = \mathbf{v}_0/c$. If both the wave vector and the observer are moving same direction, i.e., if the observer is moving away from the source, we have

$$\omega'_{k'} = \omega_k \frac{(1-\beta_0)}{\sqrt{1-\beta_0^2}} = \omega_k \sqrt{\frac{1-\beta_0}{1+\beta_0}}. \tag{6.11}$$

Hence $\omega'_{k'} < \omega_k$, or the frequency is *red shifted*. Similarly, when the observer moves towards the source, we have

$$\omega'_{k'} = \omega_k \sqrt{\frac{1+\beta_0}{1-\beta_0}}. \tag{6.12}$$

I.E., the frequency is *blue shifted*.

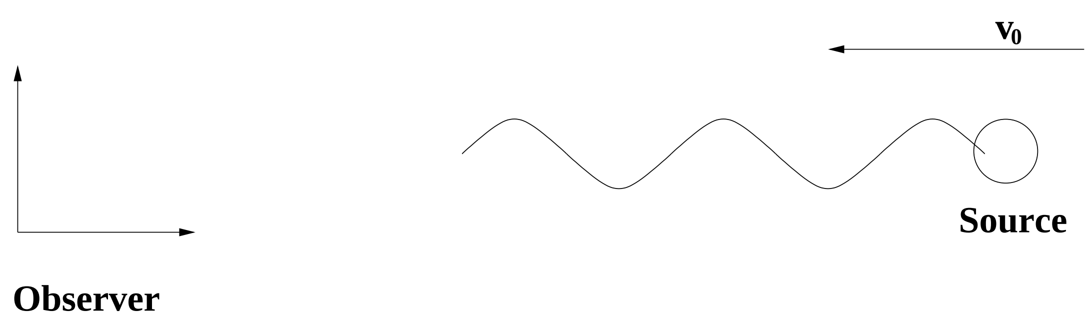

Figure 6.2: Longitudinal Doppler effect: A source and observer in relative motion see different frequencies for the wave. Here the source is moving towards the observer. The frequency as seen by the observer will be higher, or *blue shifted*, than the frequency seen in the reference frame in which the source is at rest.

6.1.2 Transverse Doppler effect

A situation of transverse doppler effect is shown in Figure 6.3. Here again we have the observer and source in relative motion, but the wave vector is transverse to the direction of motion of the source, as seen by the observer. We shall again employ the same trick, after writing the four-vectors in the two reference frames. Taking the direction of motion of the source to be the x direction, and the direction of the wave vector to be the y direction, we have the wave vector in the source frame and observer frame as

$$\underline{\mathbf{k}} * \underline{\mathbf{v}} = (\frac{\omega_k}{c}; \mathbf{k}) * (c; 0) = (\frac{\omega'_{k'}}{c}; \mathbf{k}') * (\gamma_0 c; \gamma_0 \mathbf{v}_0). \tag{6.13}$$

Hence

$$\omega_k = \gamma_0(\omega'_{k'} - \mathbf{k}' \cdot \mathbf{v}_0) = \gamma_0 \omega'_{k'}, \tag{6.14}$$

since $\mathbf{k}' \cdot \mathbf{v}_0 = 0$. Thus $\omega'_{k'} = \omega_k/\gamma_0$ is always less than ω_k.

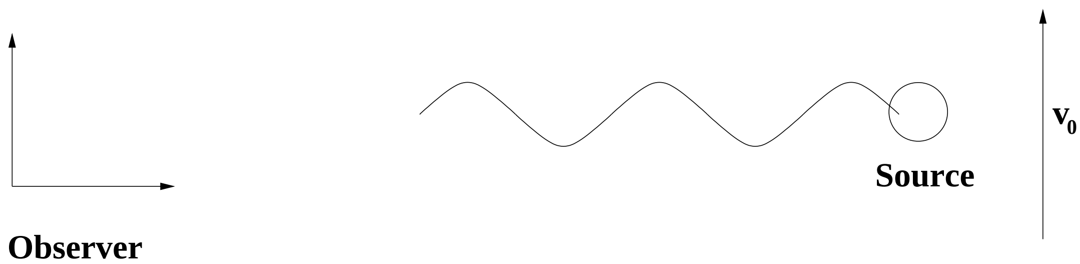

Figure 6.3: Transverse Doppler effect: A source in relative motion with respect to the observer. But the wave vector is now transverse to the motion of the source.

6.2 Aberration

When an observer is in relative motion with respect to the source of a wave, in some arbitrary direction, the angle of incidence of the wave is different as seen by two observers in relative motion. Aberration refers to this difference in the angle between angle of incidence of the wave (say the angle between line of motion of the observer and wave vector in the source reference frame), as seen in the two reference frames.

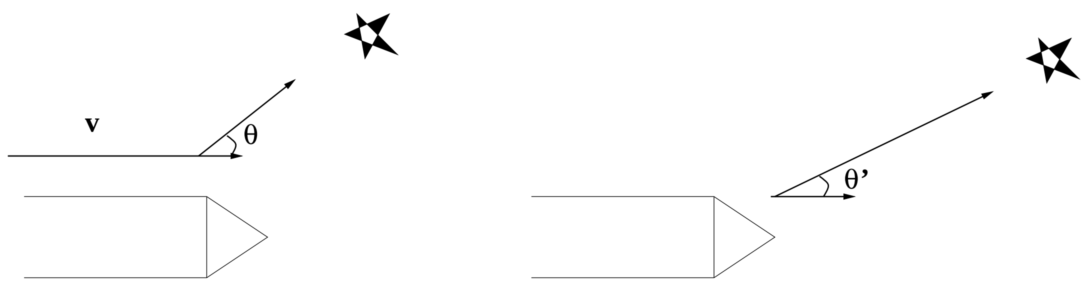

Figure 6.4: A spacecraft in motion along the x direction with speed v as seen from earth. The angle at which the observer in the spacecraft sees a distant star is different from the angle between the star and the spacecraft as seen from earth.

The case of stellar aberration is shown in the figure, from the two reference frames. A spacecraft moves along the common x direction with velocity v as seen in the earth reference frame. A star (assumed stationary) is at an angle θ as seen from this reference frame. In the spacecraft reference frame this angle is θ'. If the wave four-vector as seen the earth reference frame is (for the geometry in Figure 6.4)

$$\underline{(k)} = k\underline{\mathbf{t}} - k\cos\theta\underline{\mathbf{x}} - k\sin\theta\underline{\mathbf{y}}, \tag{6.15}$$

then the components in the spacecraft reference frame are, by Lorentz transformation

$$\begin{aligned} k' &= \gamma k + \gamma\beta k\cos\theta \\ -k'\cos\theta' &= -\gamma\beta k - \gamma k\cos\theta \\ -k'\sin\theta' &= -k\sin\theta. \end{aligned} \tag{6.16}$$

Therefore,

$$\sin\theta' = \frac{k}{k'}\sin\theta = \frac{\sin\theta}{\gamma(1+\beta\cos\theta)}. \tag{6.17}$$

As seen from Eq. (6.17), the angle as seen by the spacecraft is smaller than the angle noticed from earth - $\theta' \leq \theta$ if the observer is moving towards the source. Similarly $\theta' \geq \theta$ if the observer is moving away from the source.

Update summary

2 Nov' 08

1. Some corrections made in definition of Force in Sec. 5.3.
2. Sections 5.4 on Decay process and 5.5 on Compton scattering added.
3. Chapter 6 on waves added.