Conservation Equations of Fluid Dynamics

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– February 2011 –

This is a summary of conservation equations (continuity, Navier–Stokes, and energy) that govern the flow of a Newtonian fluid. Equations in various forms, including vector, indicial, Cartesian coordinates, and cylindrical coordinates are provided. The nomenclature is listed at the end.

I Equations in vector form

• Compressible flow:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overline{V}) = 0 \tag{1}$$

$$\rho \frac{D\overline{V}}{Dt} = \rho \overline{g} - \nabla p - \frac{2}{3}\nabla \left(\mu \nabla \cdot \overline{V}\right) + \nabla \cdot \left[\mu \left(\nabla \overline{V} + \left(\nabla \overline{V}\right)^T\right)\right] \tag{2}$$

$$\rho c_p \frac{DT}{Dt} = \rho \dot{q}_g + \nabla \cdot (k \nabla T) + \beta T \frac{Dp}{Dt} + \Phi \tag{3}$$

where the viscous dissipation rate Φ is

$$\Phi = \overline{\overline{\tau}} : \nabla \overline{V} = \left(-\frac{2}{3}\mu \nabla \cdot \overline{V}\,\overline{\overline{I}} + \mu \left[\nabla \overline{V} + \left(\nabla \overline{V}\right)^T\right]\right) : \nabla \overline{V}$$

The foregoing equations (1), (2), and (3) represent the continuity, Navier–Stokes, and energy respectively.

• Incompressible flow with constant fluid properties:

$$\nabla \cdot \overline{V} = 0 \tag{4}$$

$$\rho \frac{D\overline{V}}{Dt} = \rho \overline{g} - \nabla p + \mu \nabla^2 \overline{V} \tag{5}$$

$$\rho c_p \frac{DT}{Dt} = \rho \dot{q}_g + k \nabla^2 T + \Phi \tag{6}$$

where the viscous dissipation rate Φ is

$$\Phi = \mu \left[\nabla \overline{V} + \left(\nabla \overline{V}\right)^T\right] : \nabla \overline{V}$$

The foregoing equations (4), (5), and (6) represent the continuity, Navier–Stokes, and energy respectively.

II Equations in indicial form

• **Compressible flow:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \tag{7}$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho g_i - \frac{\partial p}{\partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu \frac{\partial v_j}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] \tag{8}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i}\right) = \rho \dot{q}_g + \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \beta T\left(\frac{\partial p}{\partial t} + v_i \frac{\partial p}{\partial x_i}\right) + \Phi \tag{9}$$

where the viscous dissipation rate Φ is

$$\Phi = \tau_{ij}\frac{\partial v_i}{\partial x_j} = \left[-\frac{2}{3}\mu \frac{\partial v_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]\frac{\partial v_i}{\partial x_j}$$

The foregoing equations (7), (8), and (9) represent the continuity, Navier–Stokes, and energy respectively.

• **Incompressible flow with constant fluid properties:**

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{10}$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2} \tag{11}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i}\right) = \rho \dot{q}_g + k \frac{\partial^2 T}{\partial x_i^2} + \Phi \tag{12}$$

where the viscous dissipation rate Φ is

$$\Phi = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial v_i}{\partial x_j}$$

The foregoing equations (10), (11), and (12) represent the continuity, Navier–Stokes, and energy respectively.

III Equations in Cartesian coordinates

• **Compressible flow:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{13}$$

$$
\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) &= \rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(-\frac{2}{3}\nabla\cdot\overline{V}+2\frac{\partial u}{\partial x}\right)\right]\\
&+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) &= \rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left[\mu\left(-\frac{2}{3}\nabla\cdot\overline{V}+2\frac{\partial v}{\partial y}\right)\right]\\
&+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) &= \rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\mu\left(-\frac{2}{3}\nabla\cdot\overline{V}+2\frac{\partial w}{\partial z}\right)\right]\\
&+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right) &= \rho\dot{q}_g+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\
&+\beta T\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+\Phi
\end{aligned}
\tag{15}
$$

where the viscous dissipation rate Φ is

$$
\begin{aligned}
\Phi &= 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]\\
&+\mu\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2
\end{aligned}
$$

The foregoing equations (13), (14), and (15) represent the continuity, Navier–Stokes, and energy respectively.

• Incompressible flow with constant fluid properties:

$$
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{16}
$$

$$
\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) &= \rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) &= \rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) &= \rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}
\tag{17}
$$

$$
\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\rho\dot{q}_g+k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi \tag{18}
$$

where the viscous dissipation rate Φ is

$$\Phi = 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \mu \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right]$$

The foregoing equations (16), (17), and (18) represent the continuity, Navier–Stokes, and energy respectively.

IV Equations in cylindrical coordinates

• Compressible flow:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho u_\theta)}{\partial \theta} + \frac{\partial (\rho u_z)}{\partial z} = 0 \tag{19}$$

$$\begin{aligned}
\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \right) &= \rho g_r - \frac{\partial p}{\partial r} \\
&+ \frac{\partial}{\partial r}\left[\mu \left(-\frac{2}{3}\nabla \cdot \overline{V} + 2\frac{\partial u_r}{\partial r} \right) \right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right) \right] \\
&+ \frac{\partial}{\partial z}\left[\mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \right] + \frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r} \right) \\
\rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) &= \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \\
&+ \frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu \left(-\frac{2}{3}\nabla \cdot \overline{V} + \frac{2}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{2u_r}{r} \right) \right] + \frac{\partial}{\partial z}\left[\mu \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \right) \right] \\
&+ \frac{\partial}{\partial r}\left[\mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right) \right] + \frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \\
\rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) &= \rho g_z - \frac{\partial p}{\partial z} \\
&+ \frac{\partial}{\partial z}\left[\mu \left(-\frac{2}{3}\nabla \cdot \overline{V} + 2\frac{\partial u_z}{\partial z} \right) \right] + \frac{\partial}{\partial r}\left[\mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_z}{\partial z} \right) \right] \\
&+ \frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \right] + \frac{\mu}{r}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)
\end{aligned} \tag{20}$$

$$\begin{aligned}
\rho c_p \left(\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z \frac{\partial T}{\partial z} \right) &= \rho \dot{q}_g + \frac{1}{r}\frac{\partial}{\partial r}\left(kr \frac{\partial T}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{k}{r}\frac{\partial T}{\partial \theta} \right) \\
&+ \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) + \beta T \left(\frac{\partial p}{\partial t} + u_r \frac{\partial p}{\partial r} + \frac{u_\theta}{r}\frac{\partial p}{\partial \theta} + u_z \frac{\partial p}{\partial z} \right) + \Phi
\end{aligned} \tag{21}$$

where the viscous dissipation rate is

$$\Phi = 2\mu \left[\left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 \right]$$
$$+ \mu \left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)^2 + \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \right)^2 + \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right)^2 \right]$$
$$- \frac{2}{3}\mu \left(\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right)^2$$

The foregoing equations (19), (20), and (21) represent the continuity, Navier–Stokes, and energy respectively.

• Incompressible flow with constant fluid properties:

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{22}$$

$$\begin{aligned}
\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \right) &= \rho g_r - \frac{\partial p}{\partial r} \\
&+ \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right] \\
\rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) &= \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \\
&+ \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] \\
\rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) &= \rho g_z - \frac{\partial p}{\partial z} \\
&+ \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right]
\end{aligned} \tag{23}$$

$$\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z \frac{\partial T}{\partial z} = \frac{\dot{q}_g}{c_p} + \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{\Phi}{\rho c_p} \tag{24}$$

where the viscous dissipation rate is

$$\Phi = 2\mu \left[\left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 \right]$$
$$+ \mu \left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)^2 + \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \right)^2 + \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right)^2 \right]$$

The foregoing equations (22), (23), and (24) represent the continuity, Navier–Stokes, and energy respectively.

V Navier–Stokes equations in stress form

It is sometimes convenient to write the Navier–Stokes equations in terms of stresses. Below we give the stress form of the Navier–Stokes equations in both Cartesian and cylindrical coordinates.

- **Cartesian coordinates:**

$$\begin{aligned} \rho\frac{Du}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \rho\frac{Dv}{Dt} &= \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \rho\frac{Dw}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{aligned} \tag{25}$$

where the deviatoric stress components are given by Stokes' law of viscosity

$$\begin{aligned} \tau_{xx} &= -\frac{2}{3}\mu\nabla\cdot\overline{V} + 2\mu\frac{\partial u}{\partial x} \\ \tau_{yy} &= -\frac{2}{3}\mu\nabla\cdot\overline{V} + 2\mu\frac{\partial v}{\partial y} \\ \tau_{zz} &= -\frac{2}{3}\mu\nabla\cdot\overline{V} + 2\mu\frac{\partial w}{\partial z} \\ \tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \tau_{zx} &= \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned}$$

- **Cylindrical coordinates:**

$$\begin{aligned} \rho\left(\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right) &= \rho g_r - \frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r} \\ \rho\left(\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r}\right) &= \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} \\ \rho\frac{Du_z}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \end{aligned} \tag{26}$$

where the deviatoric stress components are given by Stokes' law of viscosity

$$\begin{aligned} \tau_{rr} &= -\frac{2}{3}\mu\nabla\cdot\overline{V} + 2\mu\frac{\partial u_r}{\partial r} \\ \tau_{\theta\theta} &= -\frac{2}{3}\mu\nabla\cdot\overline{V} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \\ \tau_{zz} &= -\frac{2}{3}\mu\nabla\cdot\overline{V} + 2\mu\frac{\partial u_z}{\partial z} \end{aligned}$$

$$
\begin{aligned}
\tau_{r\theta} &= \tau_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
\tau_{\theta z} &= \tau_{z\theta} = \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
\tau_{zr} &= \tau_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)
\end{aligned}
$$

Nomenclature

α $\rightarrow$ thermal diffusivity

β $\rightarrow$ thermal expansion coefficient

μ $\rightarrow$ dynamic viscosity

ν $\rightarrow$ kinematic viscosity

ρ $\rightarrow$ density

Φ $\rightarrow$ viscous dissipation rate

c_p $\rightarrow$ specific heat at constant pressure

k $\rightarrow$ thermal conductivity

p $\rightarrow$ pressure

$\dot{q}_g$ $\rightarrow$ rate of heat generation per unit volume

r, θ, z $\rightarrow$ cylindrical coordinate variables

t $\rightarrow$ time

T $\rightarrow$ temperature

u, v, w $\rightarrow$ cartesian velocity components

u_r, u_θ, u_z $\rightarrow$ cylindrical velocity components

x, y, z $\rightarrow$ cartesian coordinate variables

δ_{ij} $\rightarrow$ Kronecker delta

τ_{ij} $\rightarrow$ $(ij)^{\text{th}}$component of stress tensor

g_i $\rightarrow$ i^{th}component of gravitational acceleration

v_i $\rightarrow$ i^{th}component of velocity vector

x_i $\rightarrow$ i^{th}cartesian coordinate variable

$\overline{g}$ $\rightarrow$ gravitational acceleration

$\overline{V}$ $\rightarrow$ velocity vector

$\overline{\overline{\tau}}$ $\rightarrow$ deviatoric stress tensor

$\overline{\overline{I}}$ $\rightarrow$ unit tensor

References

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