Streamfunction-Vorticity Formulation

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1 Introduction

In any numerical computation it is important to demonstrate the grid convergence of the numerical solution. A common way to demonstrate the grid convergence is to repeat a computation on grid with half of the original grid spacing, and compare the two solutions.

If numerical solution is available on two different set of grids, a technique called *Richardson extrapolation* can be used to generate more accurate numerical solutions. It is built on the concept that by combining two separate discrete solutions, on two different grids, the leading order error term in the assumed error expansion can be eliminated.

The basic technique can be illustrated with the following example. The central difference formula for the first-order derivative is given by

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h} \underbrace{-\frac{h^2}{6} f'''(x_i) - \frac{h^4}{120} f^{(5)}(x_i) + \cdots}_{\text{Truncation error}} \tag{1}$$

This formula describes precisely how the truncation error behaves. Richardson extrapolation can be used whenever the truncation error has a predictable form and depends on a parameter such as grid-point spacing h without ever knowing f''', $f^{(5)}, \cdots$.

2 Extrapolation formula for first-order accurate approximation

Suppose that $M(x)$ is the exact solution and $N_1(x;h)$ is a numerical solution (approximate), we may write

$$N_1(h) \approx M$$

and the truncation error, $E_1(h) = M - N_1(h)$. The truncation error may be expanded as a power series in h. That is,

$$E_1 = M - N_1(h) = k_1 h + k_2 h^2 + k_3 h^3 + \cdots \tag{2}$$

for some unknown constants $k_1, k_2, k_3, \cdots$. Let us rewrite equation (2) in the following form:

$$M = N_1(h) + k_1 h + k_2 h^2 + k_3 h^3 + \cdots \tag{3}$$

We can construct a second such equation just by using a different step size, say $h/2$. That is,

$$M = N_1\left(\frac{h}{2}\right) + k_1\frac{h}{2} + k_2\frac{h^2}{4} + k_3\frac{h^3}{8} + \cdots \tag{4}$$

We now eliminate the term involving h from the above two equations. This is achieved by multiply the 4th equation by 2 and subtract the 3rd equation

$$\begin{aligned} M &= 2N_1\left(\frac{h}{2}\right) - N_1(h) + k_2\left[\frac{h^2}{2} - h^2\right] + k_3\left[\frac{h^3}{4} - h^3\right] + \cdots \\ &= N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right] - \frac{k_2}{2}h^2 - \frac{3k_3}{4}h^3 - \cdots \end{aligned} \tag{5}$$

Defining

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right] \tag{6}$$

we can write equation (5) as

$$M = N_2(h) - \frac{k_2}{2}h^2 - \frac{3k_3}{4}h^3 - \cdots \tag{7}$$

which has $O(h^2)$ truncation error. Thus we have combined the multiple numerical solutions of $O(h)$ to generate solution of $O(h^2)$.

Example 1

The first-order forward difference formula for first-order derivative is given by

$$f'(x_i) = \frac{f(x_i+h) - f(x_i)}{h} \underbrace{-\frac{h}{2}f''(x_i) - \frac{h^2}{6}f'''(x_i) + \cdots}_{\text{Truncation error}}$$

Consider the function

$$f(x) = xe^x$$

with $x_i = 2$. The first derivative of $f(x)$ is:

$$f'(x) = (x+1)e^x$$

therefore the exact value of $f'(2) = 22.16717$. Now the forward difference formula can be used to determine approximate values of $f'(2)$ with step size 0.1 and 0.05. So we have

h	$N_1(h)$	% error
0.1	23.70845	6.95298
0.05	22.92170	3.40382

As expected, the error decreases by a factor of approximately 2 when we halve the step size h, because the error in the forward difference formula is of $O(h)$.

Applying then Richardson extrapolation to obtain a more accurate result:

$$N_2(0.1) = N_1(0.05) + [N_1(0.05) - N_1(0.1)] = 22.13495$$

The absolute % error when Richardson extrapolation used is: $100 \times |f'(2) - N_2(0.1)|/f'(2) = 0.14535\%$. As expected, the error in the extrapolated value $N_2(0.1)$ is considerably small when compared to the errors in the original computed values $N_1(0.1)$ and $N_1(0.05)$.

Example 2

The second-order central difference formula for first-order derivative is given by

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h} \underbrace{- \frac{h^2}{6} f'''(x_i) - \frac{h^4}{120} f^{(5)}(x_i) + \cdots}_{\text{Truncation error}}$$

Consider the function

$$f(x) = xe^x$$

As we have seen in the last example, the exact value of $f'(2) = 22.16717$. The central difference formula can be used to determine approximate values of $f'(2)$ with step size 0.1 and 0.05. So we have

h	$N_1(h)$	% error
0.1	22.22879	0.27798
0.05	22.18256	0.06943

As expected, the error decreases by a factor of approximately 4 when we halve the step size h, because the error in the central difference formula is of $O(h^2)$.

Applying the Richardson extrapolation to obtain a more accurate approximation:

$$N_2(0.1) = N_1(0.05) + [N_1(0.05) - N_1(0.1)] = 22.13633$$

The absolute % error when Richardson extrapolation used is: $100 \times |f'(2) - N_2(0.1)|/f'(2) = 0.13912\%$. Surprisingly, the extrapolated value $N_2(0.1)$ in this case is found to be less accurate than the non-extrapolated value $N_1(0.05)$. This has happened because truncation error in the central difference formula is second order. However, the formula for $N_2(h)$ has been obtained by eliminating the first order truncation error which is nonexistent in the central difference formula.

3 Extrapolation formula for second-order accurate approximation

The procedure to derive Richardson extrapolation formula for second-order accurate approximation can be used to obtain Richardson extrapolation formula for higher-order approximations. Suppose that $M(x)$ is the exact solution and a second-order approximate solution, $N_2(x;h)$; then we can write

$$M = N_2(h) + k_2h^2 + k_3h^3 + k_4h^4 + \cdots \tag{8}$$

For a step size $h/2$, we have

$$M = N_2\left(\frac{h}{2}\right) + k_2\frac{h^2}{4} + k_3\frac{h^3}{8} + k_4\frac{h^4}{16} + \cdots \tag{9}$$

We now eliminate the term involving h^2 from the above two equations by multiplying the 8th equation by 4 and subtract the 7th equation to yield

$$3M = \left[4N_2\left(\frac{h}{2}\right) - N_2(h)\right] + k_3\left[\frac{h^3}{2} - h^3\right] + k_4\left[\frac{h^4}{4} - h^4\right] + \cdots$$

If we divide this equation by 3 we obtain

$$M = \frac{1}{3}\left[4N_2\left(\frac{h}{2}\right) - N_2(h)\right] - \frac{k_3}{6}h^3 - \frac{k_4}{4}h^4 + \cdots \tag{10}$$

Defining

$$N_3(h) = \frac{1}{3}\left[4N_2\left(\frac{h}{2}\right) - N_2(h)\right] = N_2\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right] \tag{11}$$

we can write equation (10) as

$$M = N_3(h) - \frac{k_3}{6}h^3 - \frac{k_4}{4}h^4 + \cdots \tag{12}$$

which has $O(h^3)$ truncation error. Thus we have combined the multiple numerical solutions of $O(h^2)$ to generate solution of $O(h^3)$.

Example 3

We again consider the function

$$f(x) = xe^x$$

and its second-order central difference approximation. We have already computed the approximate solution of $f'(2)$ using central difference formula with step size 0.1 and 0.05 as given in the table below.

h	$N_1(h)$	% error
0.1	22.22879	0.27798
0.05	22.18256	0.06943

Applying the Richardson extrapolation using the formula (11) to obtain a more accurate approximation:

$$N_3(0.1) = N_2(0.05) + \frac{1}{3}[N_2(0.05) - N_2(0.1)] = 22.16715$$

The absolute % error when Richardson extrapolation used is: $100 \times |f'(2) - N_3(0.1)|/f'(2) = 9.022 \times 10^{-5}\%$. As expected, the error in the extrapolated value $N_3(0.1)$ is considerably small when compared to the errors in the original computed values $N_2(0.1)$ and $N_2(0.05)$.

4 Extrapolation formula for third-order accurate approximation

Let us obtain the Richardson extrapolation formula for third-order accurate approximations. Suppose that $M(x)$ is the exact solution and a third-order approximate solution, $N_3(x;h)$; then we can write

$$M = N_3(h) + k_3h^3 + k_4h^4 + k_5h^5 + \cdots \tag{13}$$

For a step size $h/2$, we have

$$M = N_3\left(\frac{h}{2}\right) + k_3\frac{h^3}{8} + k_4\frac{h^4}{16} + k_5\frac{h^5}{32} + \cdots \tag{14}$$

We now eliminate the term involving h^3 from the above two equations by multiplying the 14th equation by 8 and subtract the 13th equation to yield

$$7M = \left[8N_3\left(\frac{h}{2}\right) - N_3(h)\right] + k_4\left[\frac{h^4}{2} - h^4\right] + k_5\left[\frac{h^5}{4} - h^5\right] + \cdots$$

If we divide this equation by 7 we obtain

$$M = \frac{1}{7}\left[8N_3\left(\frac{h}{2}\right) - N_3(h)\right] - \frac{k_4}{14}h^4 - \frac{3k_5}{28}h^5 + \cdots \tag{15}$$

Defining

$$N_4(h) = \frac{1}{7}\left[8N_3\left(\frac{h}{2}\right) - N_3(h)\right] = N_3\left(\frac{h}{2}\right) + \frac{1}{7}\left[N_3\left(\frac{h}{2}\right) - N_3(h)\right] \tag{16}$$

we can write equation (15) as

$$M = N_4(h) - \frac{k_4}{14}h^4 - \frac{3k_5}{28}h^5 + \cdots \tag{17}$$

which has $O(h^4)$ truncation error. Thus we have combined the multiple numerical solutions of $O(h^3)$ to generate solution of $O(h^4)$.

Example 4

The third-order backward biased difference formula for first-order derivative is given by

$$f'(x_i) = \frac{2f(x_i+h)+3f(x_i)-6f(x_i-h)+f(x_i-2h)}{6h} + O(h^3)$$

Consider the now familiar function

$$f(x) = xe^x$$

with $x_i = 2$. The exact value of $f'(2) = 22.16717$. Now the forward difference formula can be used to determine approximate values of $f'(2)$ with step size 0.1 and 0.05. So we have

h	$N_1(h)$	% error
0.1	22.17070	0.015924
0.05	22.16762	0.002030

Applying the Richardson extrapolation using the formula (16) to obtain a more accurate approximation:

$$N_4(0.1) = N_3(0.05) + \frac{1}{7}[N_3(0.05) - N_3(0.1)] = 22.16718$$

The absolute % error when Richardson extrapolation used is: $100 \times |f'(2) - N_4(0.1)|/f'(2) = 4.511 \times 10^{-5}\%$. As expected, the error in the extrapolated value $N_4(0.1)$ is considerably small when compared to the errors in the original computed values $N_3(0.1)$ and $N_3(0.05)$.

5 Extrapolation formula for jth-order accurate approximation

If the error representation is of the form

$$E = k_1 h + k_2 h^2 + k_3 h^3 + k_4 h^4 + \cdots$$

the following general formula for Richardson extrapolation with truncation error $O(h^j)$ can be obtained for $j = 2,3,4,\ldots$

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{1}{2^{j-1}-1}\left[N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)\right] \tag{18}$$

Thus, for $j = 2,3,4$, and 5, we have the following formula:

$$\begin{aligned}
N_2 &= N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right] \\
N_3 &= N_2\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right] \\
N_4 &= N_3\left(\frac{h}{2}\right) + \frac{1}{7}\left[N_3\left(\frac{h}{2}\right) - N_3(h)\right] \\
N_5 &= N_4\left(\frac{h}{2}\right) + \frac{1}{15}\left[N_4\left(\frac{h}{2}\right) - N_4(h)\right]
\end{aligned}$$

If the error representation contains only even power terms, i.e.,

$$E = k_2 h^2 + k_4 h^4 + k_6 h^6 + k_8 h^8 + \cdots$$

we could obtain a general formula for Richardson extrapolation with truncation error $O(h^j)$ for $j = 4,6,8,\ldots$

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{1}{2^{j-2}-1}\left[N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)\right] \tag{19}$$

Thus, for $j = 4,6,8$, and 10, we have the following formula:

$$\begin{aligned}
N_4 &= N_2\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right] \\
N_6 &= N_4\left(\frac{h}{2}\right) + \frac{1}{15}\left[N_4\left(\frac{h}{2}\right) - N_4(h)\right] \\
N_8 &= N_6\left(\frac{h}{2}\right) + \frac{1}{63}\left[N_6\left(\frac{h}{2}\right) - N_6(h)\right] \\
N_{10} &= N_8\left(\frac{h}{2}\right) + \frac{1}{255}\left[N_8\left(\frac{h}{2}\right) - N_8(h)\right]
\end{aligned}$$

It must be noted that, Richardson extrapolation has limitations. First, if it is applied to primitive variables of fluid dynamics such as velocity, its implication regarding momentum conservation may be inaccurate. Second, the method implicitly assumes that the solution is a smooth function and has derivatives to all orders. Hence, its results are not valid in the vicinity of discontinuities in the solution.