From Exponential to Matrix-Exponential Distributions

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How to get more flexibility without losing tractability!

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C.A. O' Cinneide. *charecterization of Phase-type distributions*. Comm. Statist. Stohastic Models, 6:1-57, 1990.

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Dense, in the metric of weak convergence of distributions, in all distributions on $[0, \infty)$.

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PH distributions act as the computational vehicle for many applied probability models since they constitute a very versatile class of distributions defined on the non negative real line that lead to models which are algorithmically tractable.

Their formulation allow us to retain the Markov structure of Stochastic Models while being act as a reasonable approximation to a general distribution.

Continuous Phase type distributions

Let $\{X_t : t \geq 0\}$ be a CTMC on the finite state space $E = \{1, 2, ..., p, p+1\}$, where the states 1,2,....,p are transient (i.e given that we start in any one of these states, there is a non-zero probability that we will never return to it) and the state p+1 is absorbing.

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Then $\{X_t : t \geq 0\}$ has an intensity matrix of the form

$$\Lambda = \left[\begin{array}{cc} T & t \\ \mathbf{0} & 0 \end{array} \right]$$

where T is a $p \times p$ matrix (satisfying $T_{ii} < 0$ and $T_{ij} \geq 0$, for $i \neq j$), and t is a p-dimensional column vector satisfying $Te + t = 0$.

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$\therefore P_{ij}^s = P(X_s = j \mid X_0 = i) = exp(Ts)_{ij}$ for i,j = 1,2,...,p.

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Laplace Transform

$$E(e^{-s\tau}) = \pi(sI - T)^{-1} t = \pi(s(-T)^{-1} + I)^{-1} e.$$

Properties of Phase type distributions

- The family of PH distributions is closed under finite number of convolutions.
  In particular if $X \sim PH_m(\alpha, T)$ and $Y \sim PH_n(\beta, S)$, both being independent , then $Z = X + Y \sim PH(\gamma, L)$ where $\gamma = (\alpha, \alpha_{m+1}\beta)$,

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  Example(Generalized Erlang k distribution)
  Let $Z = \sum_{i=1}^{k} X_i$ with $X_i \sim exp(\lambda_i)$, then

  $$\gamma = (1, 0, 0, ..., 0), L = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & -\lambda_{k-1} & \lambda_{k-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_k \end{bmatrix}$$

- Any finite convex mixture of PH variates is a PH variate.
  Let $X_i \sim PH(\alpha_i, T_i), i = 1, 2, ..., k$ such that $Z = X_i$ with probability p_i. Then $Z \sim PH(\gamma, L)$ where

$$\gamma = (p_1\alpha_1, p_2\alpha_2, ..., p_k\alpha_k) \;\; and \;\; L = \begin{bmatrix} T_1 & 0 & \cdot & \cdots & 0 \\ 0 & T_2 & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & T_k \end{bmatrix}.$$

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  Example(Hyper-exponential distribution)
  Let $X_i \sim exp(\lambda_i)$ and Z takes the value of X_i with probability p_i. Then Z has the representation (γ, L) where

$$\gamma = (p_1, p_2, ..., p_k), \quad and \quad L = \begin{bmatrix} -\lambda_1 & 0 & \cdot & \cdots & 0 \\ 0 & -\lambda_2 & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -\lambda_k \end{bmatrix}.$$

- If $Y_1 \sim PH_k(\beta, S)$ and $Y_2 \sim PH_m(\pi, T)$, then $min(Y_1, Y_2) \sim PH(\gamma, L)$, where $\gamma = \beta \otimes \pi$ and $L = S \oplus T$.

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  Proof:
  Let $\overline{F}_{Y_i}$, i= 1,2 denote the respective survival functions. Since $\overline{F}(u) = P(Y_1 > u, Y_2 > u) = \overline{F}_{Y_1}(u)\overline{F}_{Y_2}(u)$, the density function of the minimum, f is given by

$$f(x) \qquad = -f_{Y_1}(x)\overline{F}_{Y_2}(x) - \overline{F}_{Y_1}(x)f_{Y_2}(x)$$

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$$U = \begin{bmatrix} T \otimes I_m + I_k \otimes S & I_k \otimes s & t \otimes I_m \\ 0 & T & 0 \\ 0 & 0 & S \end{bmatrix}$$

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More closure properties of PH distributions can be seen in

- G.Latouche and V. Ramaswami, *Introduction to Matrix Analytic Methods in Stochastic Modeling*, SIAM,1999.
- R.S. Maier and C.A.O' Cinneide, *A Closure characterization of Phase type distributions*, J. App. Prob., 29:92-103, 1992.

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Joint likelihood function of the sample:

$$L = \prod_{k=1}^{m} \beta_k^{B_k} \prod_{i=1}^{m} \prod_{j\neq i}^{m} S_{ij}^{N_{ij}} e^{-S_{ij} Z_i} \prod_{l=1}^{m} (s_l^0)^{N_l} e^{-s_l^0 Z_l}$$

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$$1 \leq i, j \leq m, i \neq j$$

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$$\hat{S}_{ii} = -\sum_{j \neq i} \hat{S}_{ij} - \hat{s}_i^0.$$

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Conditional expections given y , of the sufficient statistics for the parameters are

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$$M(y,\beta,S) = \int_0^y e^{S(y-u)} s^0 \beta e^{Su} du = \sum_{r=0}^{\infty} e^{-cy} \frac{(cy)^{r+1}}{(r+1)!} \sum_{m=0}^{r} P^m p \beta P^{r-m}$$

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$$c = max(-S_{ii}), P = (1/c)S + I, p = (1/c)s^0.$$

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- $Y \sim \sum_{i \in S_+} \sum_{n=1}^{N_i^*} Y_{i,n}$
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$$[Q_+^*]_{ij} = \begin{cases} \frac{q_{ii}(1-[P_+^*]_{ii})}{r(i)} & if \quad i = j \\ \frac{-q_{ii}[P_+^*]_{ij}}{r(i)} & if \quad i \quad \neq j \end{cases}$$

Matrix-exponential distributions

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$L_Y(s) = \frac{f_1 s^{m-1} + f_2 s^{m-2} + \ldots + f_m}{s^m + g_1 s^{m-1} + \ldots. + f_m}$.

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Another characterization of the class of matrix-exponential distributions is given by Asmussen and Bladt.

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A random variable Y is matrix-exponentially distributed if and only if there exists a triplet (β, D, d) such that the density f of Y can be expressed as $f(y) = \beta e^{Dy} d$, and we write $Y \sim ME(\beta, D, d)$.

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This distribution has following three distinct representation

$(\alpha, T, t), (\beta, S, s), (\gamma, R, r)$

$$\alpha = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}, T = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}, t = \begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix}$$

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Mark William Fackrell, *Characterization of matrix-exponential distributions*, Doctoral Thesis, 2003.

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- Moments: $M_i = E(Y^i) = i!\beta(-D)^{-(i+1)} d$,i=0,1,2,...

Theorem:

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The coefficients c_i can be calculated in terms of the Hankel determinants

$$\phi_n = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ . & . & \cdots & . \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \end{bmatrix} \; for \quad n = 1, 2, ...$$

$$\psi_n = \begin{bmatrix} \mu_2 & \mu_3 & \cdots & \mu_n \\ \mu_3 & \mu_4 & \cdots & \mu_{n+1} \\ . & . & \cdots & . \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-2} \end{bmatrix} \textit{for} \quad n = 2, 3, ...$$

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Also, refer Esparza, L. J. R., Nielsen, B. F., and Bladt, M. *Maximum likelihood estimation of phase-type distributions.*, 2011. Kgs. Lyngby, Denmark: Technical University of Denmark (DTU). (IMM-PHD-2010-245).

Properties of matrix-exponential distributions

- If $Y_1 \sim ME(\beta, S, s)$ and $Y_2 \sim (\pi, T, t)$, then $Y_1 + Y_2$ is matrix-exponential with representation

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- Let $p \in (0,1)$. Then the mixture $f = pf_{Y_1} + (1-p)f_{Y_2}$ is again a matrix- exponential density with representation

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Matrix-exponential representations

Theorem: If the LT of a ME distribution is expressed as
$\phi(\lambda) = \frac{a_p\lambda^{p-1}+a_{p-1}\lambda^{p-2}+\ldots+a_1}{\lambda^p+b_p\lambda^{p-1}+b_{p-1}\lambda^{p-2}+\ldots+b_1} + \alpha_0$
where $p \geq 1, a_1, a_2, ..., a_p, b_1, b_2, ..., b_p$ are all real and $0 \leq \alpha_0 < 1$, then the ME distribution has a minimal representation (α, T, t) where
$\alpha = (a_1, a_2, ..., a_p)$

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -b_1 & -b_2 & -b_3 & \cdots & -b_{p-1} & -b_p \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix} = e_p.$$

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form a minimal representation for ME distribution.

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How can we know whether a given LT corresponds to an ME distribution or not? SEE.....

Theorem:

Let $a = (a_1, a_2, ..., a_p)$, $b = (b_1, b_2, ..., b_p)$,
$b(\lambda) = \lambda^p + b_p\lambda^{p-1} + b_{p-1}\lambda^{p-2} + ... + b_1$ and

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -b_1 & -b_2 & -b_3 & \cdots & -b_{p-1} & -b_p \end{bmatrix}.$$

Then a and b correspond to a non trivial ME distribution if and only if

- $f(u) = ae^{Bu}e \geq 0$ for $u > 0$
- $0 < \frac{a_1}{b_1} \leq 1$
- there exists a zero of $b(\lambda)$ of maximal real part that is both real and negative.

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For the proof and more details on matrix exponential distributions, see

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For the proof and more details on matrix exponential distributions, see Mark William Fackrell, *Characterization of matrix-exponential distributions*, Doctoral Thesis, 2003.


Thank You!