Inviscid Burgers' Equation

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– 31 December 2018 –

1 Nonlinear Wave Equation

We first consider the nonlinear one-way wave equation of the form:

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0 \tag{1}$$

with the initial wave profile

$$u(x,0) = F(x) \tag{2}$$

where $c(u)$ is the wave speed. We define the characteristic curves of (65) by the differential equation

$$\frac{dx}{dt} = g(u) \tag{3}$$

Then, along a particular such curve $x = x(t)$ we have

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + g(u)\frac{\partial u}{\partial x} = 0$$

Therefore u is constant along the characteristics, and the characteristics are straight lines since

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dg(u)}{dt} = g'\frac{du}{dt} = 0$$

In the nonlinear case, however, the speed of the characteristics as defined by (3) depends on the value u of the solution at a given point. To find the equation of the characteristic Γ through (x,t) we note that its speed is

$$\frac{dx}{dt} = g[u(x,t)] = g[u(\xi,0)] = g[F(\xi)] \tag{4}$$

This results from applying (3) at $(\xi,0)$. Equation (4) shows that the characteristics are straight lines emanating from $(\xi,0)$ with speed $g(F(\xi))$. Direct integration of (4) gives the equation of characteristic curve Γ

$$x = g[F(\xi)]t + \xi \tag{5}$$

where ξ is the x-intercept of the characteristic curve. This represents a family of straight lines whose slopes are not the same, but depends on ξ (see Fig. 1). Equation (5) defines $\xi = \xi(x,t)$

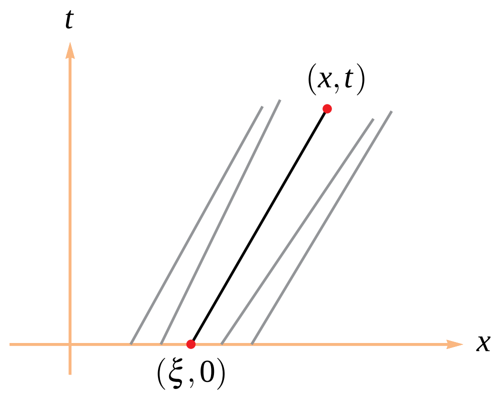

Figure 1: Characteristic curves for nonlinear wave equation.

implicitly as a function of x and t. The solution $u(x,t)$ of the initial value problem (1) and (2) is given by

$$u(x,t) = u(\xi,0) = F(\xi) \tag{6}$$

for $t \geq 0$, where ξ is implicitly defined by (5).

We next obtain the necessary condition that (6) represents the solution of (1). Putting $G(\xi) = g[F(\xi)]$, equation (5) can be rewritten as

$$x = G(\xi)t + \xi$$

Differentiating with respect to x and t, we obtain

$$1 = \left[1 + tG'(\xi)\right] \frac{\partial \xi}{\partial x} \qquad \text{and} \qquad 0 = G(\xi) + \left[1 + tG'(\xi)\right] \frac{\partial \xi}{\partial t}$$

Again, differentiating (6) with respect to x and t, we obtain

$$\frac{\partial u}{\partial x} = F'(\xi) \frac{\partial \xi}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial t} = F'(\xi) \frac{\partial \xi}{\partial t}$$

Eliminating ξ_x and ξ_t from the above equations gives

$$\frac{\partial u}{\partial x} = \frac{F'(\xi)}{1 + tG'(\xi)} \qquad \text{and} \qquad \frac{\partial u}{\partial t} = -\frac{F'(\xi)G(\xi)}{1 + tG'(\xi)} \tag{7}$$

Clearly, equation (1) is satisfied only if $1 + tG'(\xi) \neq 0$. The solution (6) also satisfies the initial condition at $t = 0$, since $\xi = x$, and the solution (6) is unique.

In summary, we have the following statement: The nonlinear initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + g(u) \frac{\partial u}{\partial x} &= 0, \qquad \infty < x < \infty, \quad t > 0, \\ u(x,0) &= F(x), \qquad \infty < x < \infty \end{aligned}$$

has a unique solution provided $1+tG'(\xi)\neq 0$, F and g are C^1 functions where $G(\xi)=g(F(\xi))$. The solution is given in the parametric form:

$$\begin{aligned} u(x,t) &= F(\xi), \\ x &= g[F(\xi)]t+\xi. \end{aligned}$$

Consider the solution of the nonlinear wave equation

$$u(x,t) = u(G(\xi)t+\xi,t) = f(\xi).$$

It is easy to see that the point $(\xi, f(\xi))$ moves parallel to the x-axis in the positive direction through a distance $G(\xi)t = gt$, and the distance moved $(x=\xi+gt)$ depends on ξ. This is a typical nonlinear phenomenon. In the linear case, the curve moves parallel to the x-axis with constant velocity $g=c$, and the solution represents waves travelling without change of shape. Thus, there is a striking difference between the linear and the nonlinear solution.

The solution of the linear wave equation (**??**) can be obtained as a special case of the nonlinear wave equation (1). When $g(u)=c$, a constant, the characteristic curves are $x=ct+\xi$ and the solution u is given by

$$u(x,t) = F(\xi) = F(x-ct).$$

for $t\geq 0$.

1.0.1 Breaking time

We have seen that the solution (a differentiable function $u(x,t)$) of the nonlinear initial value problem exists provided

$$1+tG'(\xi)\neq 0. \tag{8}$$

However, for smooth initial data, this condition is always satisfied for sufficiently small time t. It follows from results (7) that both u_x and u_t tend to infinity as $1+tG'(\xi)\to 0$. This means that the solution develops a singularity (discontinuity) when $1+tG'(\xi)=0$. We consider a point $(x,t)=(\xi,0)$ so that this condition is satisfied on the characteristics through the point $(\xi,0)$ at a time t such that

$$t = -\frac{1}{G'(\xi)} \tag{9}$$

which is positive provided $G'(\xi)=c'(F)F'(\xi)<0$. If we assume $c'(F)>0$, the above inequality implies that $F'(\xi)<0$. Hence, the solution ceases to exist for all time if the initial data is such that $F'(\xi)<0$ for some value of ξ. The time t^* at which this happens for the first time is called the breaking time. We will see more about the breaking time with regard to the inviscid Burgers' equation which is discussed in the next section.

2 The Inviscid Burgers' Equation

Inviscid Burgers' equation is a special case of nonlinear wave equation where wave speed $g(u) = u$. The inial value problem in this case can be posed as

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0, \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= F(x) \end{aligned} \tag{10}$$

The characteristic curves are defined by the differential equation

$$\frac{dx}{dt} = u \tag{11}$$

Since u is constant along the characteristics, the equation of the characteristic Γ through (x,t) can be found:

$$\frac{dx}{dt} = u(x,t) = u(\xi,0) = F(\xi) \tag{12}$$

The slope of characteristics curves in (x,t) plane are constant and is given by $1/F(\xi)$. Direct integration of (12) gives the equation of characteristic curve Γ

$$x = F(\xi)t + \xi = ut + \xi \tag{13}$$

where ξ is the x-intercept of the characteristic curve. Equation (13) defines $\xi = \xi(x,t)$ implicitly as a function of x and t.

Figure 2 shows a typical initial waveform for the inviscid Burgers' equation and the corresponding characteristic curves. It can be seen from (11) that the characteristics are straight lines emanating from $(\xi,0)$ with speed $c = F(\xi)$. Observe that, the larger the $F(\xi)$ is, the flatter the characteristic line, and faster that part of the wave travels. Also, since the slope of the characteristics depend on $F(\xi)$, the slope can change from characteristics to characteristics; this leads to the possibility of intersecting the characteristics as shown in Fig. 2.

The solution $u(x,t)$ of the initial value problem (10) is given by

$$u(x,t) = u(\xi,0) = F(\xi) = F(x-ut) \tag{14a}$$

for $t \geq 0$, which may also be written as

$$\left.\begin{aligned} u(x,t) &= F(\xi) \\ \xi &= x - ut \end{aligned}\right\} \tag{14b}$$

Consider, for example, the initial profile given by

$$u(x,0) = F(x) = \alpha x + \beta \tag{15}$$

where α and β are constants. This initial profile is a straight line with slope α and u-intercept β. Thus, the solution (14) becomes

$$u(x,t) = F(x-ut) = \alpha(x-ut) + \beta \tag{16}$$

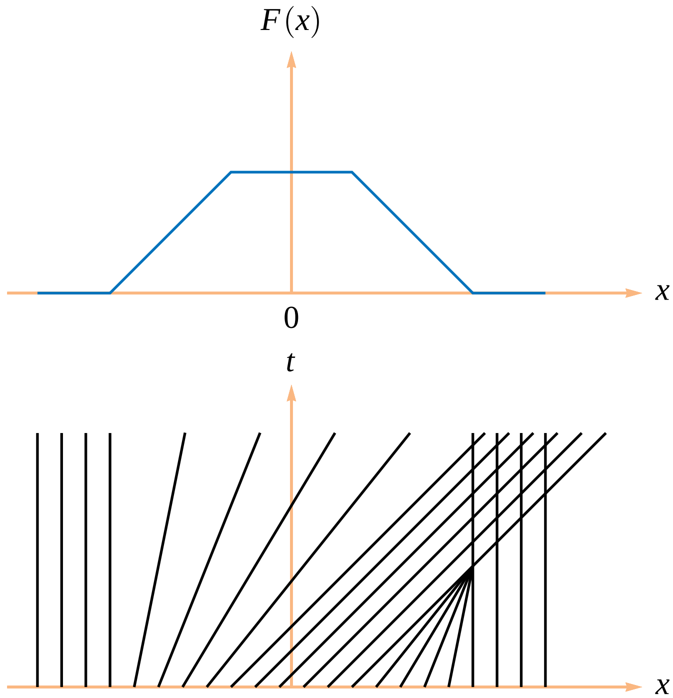

Figure 2: Initial profile and the characteristic curves for inviscid Burgers' equation.

This can be solved explicitly to yield the solution

$$u(x,t) = \frac{\alpha x + \beta}{\alpha t + 1} \tag{17}$$

for $t \geq 0$. It can be seen from equation (17) that, for each fixed time t, the solution represents a straight line with slope $\alpha/(1+\alpha t)$. If $\alpha > 0$, the slope of the straight line decreases as time increase and thus the solution flattens out with time. On the other hand, if $\alpha < 0$, the straight line rapidly steepens to vertical as t approaches to critical time called *breaking time* $t^* = -1/\alpha$ at which point the solution ceases to exist.

2.0.1 The case when $\alpha = -1$ and $\beta = 0$

The initial condition in this case is

$$u(x,0) = F(x) = -x$$

Initial profile in this case is a straight line passing through the origin with slope equal to -1. Equation (14) gives the implicit form of the solution as

$$u(x,t) = F(x-ut) = -(x-ut)$$

from which it follows that

$$u(x,t) = \frac{x}{t-1}$$

for $t \geq 0$. The solution shows that as t increases, the initial profile executes a clockwise rotation around the origin in the xu-plane. Since speed $u = 0$ at $x = 0$, the origin remains stationary (Fig. 3). Also, $|u|$ increases linearly with $|x|$, and points x farther away from the origin have linearly increasing velocity. At $t = 1$ the solution blows up as the waveform $u(x,t)$ coincide with the u-axis and thus becomes infinitely multivalued.

The breakdown of the solution and its multivaluedness at $t = 1$ may also be determined by considering the characteristic curves. The equation for characteristics can be directly obtained from (13) by noting that the initial profile, $F(\xi) = -\xi$. Thus,

$$t = -\frac{x}{\xi} + 1$$

The slopes of the characteristic lines are given by $-1/\xi$. Hence the slopes are negative for $\xi > 0$ and it is positive for $\xi < 0$. The magnitude of slopes of the characteristic lines decreases with distance from the origin as shown in Fig. 4. Since the t-intercepts of all the characteristic lines are 1, all of them converge at $t = 1$ and thus the solution becomes multivalued.

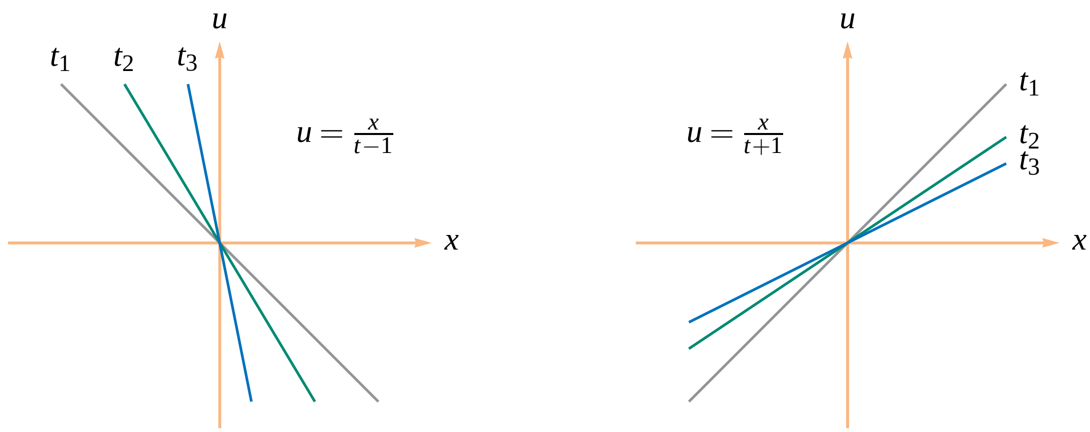

Figure 3: Solution of inviscid Burgers' equation at time $t_i = 0, 0.4, 0.8$ for initial conditions $F(x) = -x$ and x.

2.0.2 The case when $\alpha = 1$ and $\beta = 0$

If the initial condition is

$$u(x,0) = F(x) = x$$

the solution becomes

$$u(x,t) = \frac{x}{t+1}$$

for $t \geq 0$. This solution is defined for all $t > 0$ and hence never breaks. It shows that as t increases, the initial waveform executes a clockwise rotation around the origin in the xu-plane. Since $u = 0$ at $x = 0$, the origin remains stationary (Fig. 3). Also, $|u|$ increases linearly with $|x|$, and points x farther away from the origin have linearly increasing velocity. The equation for

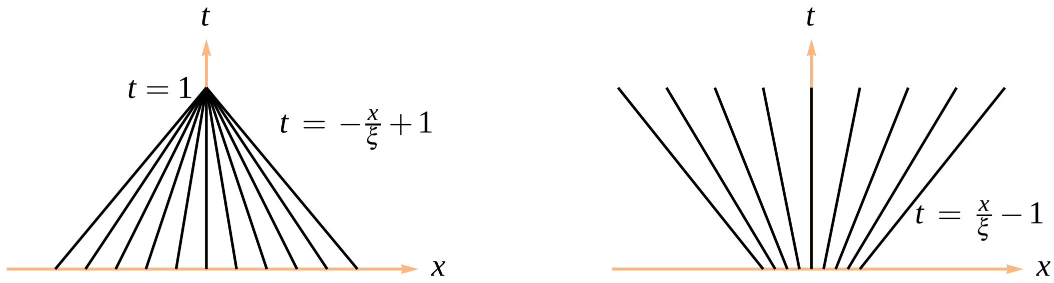

Figure 4: Characteristic curves of Burgers' equation for initial conditions $F(x) = -x$ and x.

characteristics in this case is given by

$$t = \frac{x}{\xi} - 1$$

The slopes of the characteristic lines are given by $1/\xi$. Hence the slopes are positive for $\xi > 0$ and it is negative for $\xi < 0$. The magnitude of slopes of the characteristic lines decreases with distance from the origin. Since the t-intercepts of all the characteristic lines are -1, none of them converges for $t > 0$ and hence they diverges from each other as shown in Fig. 4. No two characteristics intersect for $t > 0$.

It is instructive to compare the solution (14) of the quasilinear PDE in (10) with the solution $u(x,t) = F(x - ct)$ of the linear equation subject to the same initial condition $u(x,0) = F(x)$. In the case of linear equation the slope of the characteristic is $1/c =$ constant that the solution represents a steady translation of the initial wave profile along the x axis with speed c, and without change of shape or scale. In the (x,t) plane, where the solution represents a propagating wave, the function $u(x,t)$ is said to define the wave profile at time t. On the other hand, in the quasilinear case (inviscid Burgers' equation) the speed of translation of the wave depends on u, so different part of the wave will move with different speeds, causing it to distort as it propagates. It is this distortion that can lead to the nonuniqueness of solution in the quasilinear case. A physical example of this phenomena is found in the theory of shallow water waves, where the speed of propagation of a surface element of water is proportional to the square root of the depth. This has the effect that in shallow water the crest of the wave moves faster than the trough, leading to wave breaking close to the shore line.

2.1 Wave distortion

To illustrate the phenomenon of distortion of nonlinear waves as it propagates and the formation of the envelope of characteristics, let us consider the initial value problem governed by the quasilinear PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to the Cauchy data $u(x,0) = F(x) = \sin x$. The characteristic equations of the PDE are given by

$$\frac{dx}{dt} = u(x,t) \qquad \text{and} \qquad \frac{du}{dt} = 0$$

Thus, the slope of the characteristic curve is

$$\frac{dx}{dt} = u(x,t) = u(\xi,0) = F(\xi) = \sin(\xi)$$

which leads to

$$x = \sin(\xi)t + \xi = ut + \xi$$

The solution $u(x,t)$ of the initial value problem is given by (13)

$$u(x,t) = u(\xi,0) = F(\xi) = \sin(\xi) = \sin(x-ut)$$

The development of the wave profile as t increases is illustrated schematically in Fig.?? Points on the wave with larger values of u propagates faster and consequently overtakes parts of the wave propagating with smaller values of u. At time $t = 0$, the wave profile is pure sinusoid. At $t = 1$ the wave profile has steepened to the point where the tangent to the wave profile has become vertical at the point where the solution crosses the x axis, while at $t = 1.5$, shows that the wave is no longer single valued in periodic intervals of x. This illustrates how, as the time progresses, the wave profile distorts as it propagates eventually, after time $t = 1$, becoming a multiple-valued function with respect to x, thereby leading to the breakdown of the differentiability of the solution.

The characteristic through the arbitrary point $(\xi,0)$ on the x axis has the equation

$$x = ut + \xi = t\sin\xi + \xi$$

By defining

$$\psi(x,t,\xi) = x - t\sin\xi - \xi$$

the equation of this characteristic can be written in the parametric form $\psi(x,t,\xi) = 0$, with the ξ serving as the parameter. From elementary calculus it is known that, when it exists, the envelope formed by a family of curves $\psi(x,t,\xi) = 0$ with ξ as a parameter is found by eliminating ξ between the equations $\psi(x,t,\xi) = 0$ and $\partial\psi/\partial\xi = 0$. A simple calculation shows that in this case the envelope has the equation

$$x = \sqrt{t^2-1} + \pi - \cos^{-1}(1/t)$$

As the term $\sqrt{t^2-1}$ is real-valued for $t > 1$, this result confirms the uniqueness of the solution for $0 < t < 1$, because no envelope can form during this time interval. Defining the critical time as $t^* = 1$ it follows that a unique solution exists for $0 < t < t^*$, while the solution become nonunique for $t > t^*$.

3 Shock formation

Let us look at an example in which the solution develops discontinuity even if the initial waveform is continuous. The basic idea is simple: An initial value of u that is greater on the left side of a particular location x than on the right side of the same x will create waves that travel faster on the left side of x than on the right side. The fast waves will overtake the slow waves, causing a discontinuity. Let us consider the following initial waveform:

$$u(x,0) = F(x) = \begin{cases} u_l & \text{if } \; x < 0 \\ u_l - \dfrac{u_l - u_r}{x_r} x = u_l - \alpha x & \text{if } \; 0 \le x \le x_r \\ u_r & \text{if } \; x > x_r \end{cases} \tag{18}$$

where u_l and u_r are *positive* constants. This profile represents a ramp function with a negative slope of magnitude $\alpha = (u_l - u_r)/x_r$ as shown in Fig. 5. Using the initial condition (18), the

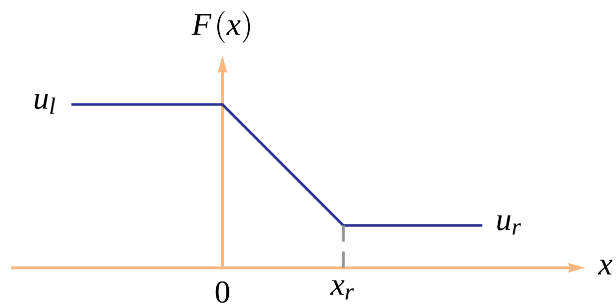

Figure 5: Initial profile with a ramp function.

equation of the characteristics (13) becomes

$$x = F(\xi)t + \xi = \begin{cases} u_l t + \xi & \text{if } \; \xi < 0 \\ (u_l - \alpha\xi)t + \xi & \text{if } \; 0 \le \xi \le x_r \\ u_r t + \xi & \text{if } \; \xi > x_r \end{cases} \tag{19}$$

for $t \ge 0$, where ξ is the x-intercept of the characteristics. We can also express the characteristic lines to give t as a function of x:

$$t = \begin{cases} \frac{1}{u_l}(x - \xi) & \text{if } \; \xi < 0 \\ \dfrac{x - \xi}{u_l - \alpha\xi} & \text{if } \; 0 \le \xi \le x_r \\ \frac{1}{u_r}(x - \xi) & \text{if } \; \xi > x_r \end{cases} \tag{20}$$

The plot of the equation for characteristic lines for the initial condition (18) is sketched in Fig. 6. The characteristics originating from region where $\xi < 0$ have slopes equal to $1/u_l$ and those originating from region where $\xi > x_r$ have slopes equal to $1/u_r$. Since $u_r < u_l$, the right characteristics are steeper than left characteristics. In the region where $0 < \xi < x_r$, the slope of characteristics increases continuously with ξ from $1/u_l$ at $\xi = 0$ to $1/u_r$ at $\xi = x_r$.

From (14), the solution of the Burgers' equation is given by

$$u(x,t) = F(x - ut) = \begin{cases} u_l & \text{if } \; x - ut < 0 \\ u_l - \alpha(x - ut) & \text{if } \; 0 \le x - ut \le x_r \\ u_r & \text{if } \; x - ut > x_r \end{cases}$$

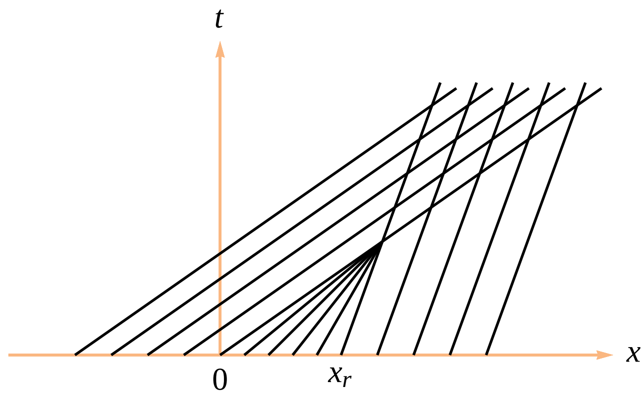

Figure 6: The characteristic curves for inviscid Burgers' equation.

which can also be written as

$$u(x,t) = \begin{cases} u_l & \text{if } \; x < u_l t \\ \dfrac{u_l - \alpha x}{1 - \alpha t} & \text{if } \; u_l t \le x \le x_r + u_r t \\ u_r & \text{if } \; x > x_r + u_r t \end{cases} \tag{21}$$

for $t \geq 0$. We will now see what happens to characteristics with advancement of time. Looking at Fig. 6 we see that, initially the characteristics do not cross, and the solution remains a well defined, single-valued function. However, since the characteristic lines are not parallel to each other they must cross each other in a finite time. The intersection points lie on two distinct characteristics with different slopes, and thus the solution u is no longer uniquely determined. The solution at intersection point becomes multivalued since the point can be traced back along either of the characteristics to an initial value of either u_l or u_r, given by the initial condition (18). This phenomena is called *shock* and the time t^* at which this happens for the first time is called the *breaking time*. Note that, if u represent a physical quantity, the multivalued solution associated with shock is not acceptable since physical quantity should have unique value at each point. The mathematical model has broken down, and fails to agree with the physical reality.

To fully appreciate what is going on, let us look at the solution (21). It is clear that the waves that issue from the region $\xi < 0$ move faster than the waves that issues from the region $\xi > x_r$. Because of this the ramp rapidly steepens to vertical as $(\alpha t - 1) \to 0$ or $t \to 1/\alpha$ and thus the solution becomes *discontinuous* at a finite time $t^* = 1/\alpha$.

3.1 Breaking time

A general expression for breaking time can be found as follows. As the time approaches breaking time t^*, the solution becomes vertical at $\xi = x^*$. Thus we have,

$$\frac{\partial u}{\partial x}(x^*, t) \to \infty \qquad \text{as} \qquad t \to t^*$$

The breaking time can be determined from the implicit solution formula (14). Differentiating (14) with respect to x yields

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}F(\xi) = \frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial x} = F'\frac{\partial}{\partial x}(x-ut) = F'\left(1-t\frac{\partial u}{\partial x}\right)$$

where $\xi = x-ut$ is the characteristic variable, which is constant along the characteristic lines. Rearranging the above equation to obtain

$$\frac{\partial u}{\partial x} = \frac{F'(\xi)}{1+tF'(\xi)} \tag{22}$$

This shows that if $F' > 0$, a *classical solution*, where the differential equation is satisfied at every point, exists for all time. On the other hand, if $F' < 0$ the classical solution exists only for a certain period of time as the solution blows up when $\partial u/\partial x \to \infty$. This happens when the denominator of equation (22) tends to zero. In other words

$$t \to -\frac{1}{F'(\xi)}$$

Hence if the initial profile has negative slope at position x, then the solution along the characteristic line issuing from the point $(x,0)$ will break down at the time $-1/F'(x)$. As a consequence, the earliest breaking time is

$$t^* = \min\left\{-\frac{1}{F'(x)}\right\} \qquad \text{when} \qquad F'(x) < 0 \tag{23}$$

In the present problem, $F'(x) = 0$ if $x \leq 0$ and $x \geq x_r$. However, in the interval $0 \leq x \leq x_r$, the value of the derivative $F'(x) = -\alpha$. Therefore the breaking time

$$t^* = 1/\alpha = \frac{x_r}{u_l - u_r}$$

The spatial location corresponding to breaking time can be easily obtained by using the equation of characteristics (19). Substituting t^* in this equation gives the breaking location x^*. Therefore, we have

$$x^* = u_l t^* = \frac{u_l x_r}{u_l - u_r} \qquad \text{or} \qquad x^* = u_r t^* + x_r = \frac{u_r x_r}{u_l - u_r} + x_r$$

3.1.1 Breaking time from characteristics

Expression for breaking time can also determined from the characteristics. Consider characteristics that emanate from the points ξ_1 and $\xi_2 = \xi_1 + \Delta\xi$. Since the breaking time corresponds to the first intersection of the characteristics, we have from equation (13)

$$F(\xi_1)t + \xi_1 = F(\xi_2)t + \xi_2$$

Solving for t to obtain

$$t = -\frac{\xi_2 - \xi_1}{F(\xi_2) - F(\xi_1)} = -\frac{\Delta\xi}{F(\xi_1 + \Delta\xi) - F(\xi_1)}$$

When $\Delta\xi \to 0$, the expression for the earliest breaking time becomes

$$t^* = \min\left\{\lim_{\Delta\xi\to 0} -\frac{\Delta\xi}{F(\xi_1+\Delta\xi) - F(\xi_1)}\right\} = \min\left\{-\frac{1}{F'(\xi)}\right\}$$

Since $x = \xi$ along $t = 0$ line, we have from the above equation

$$t^* = \min\left\{-\frac{1}{F'(x)}\right\}$$

which is same as the expression (23).

3.1.2 The special case of $u_l = 1$, $u_r = 0$, $x_r = 1$

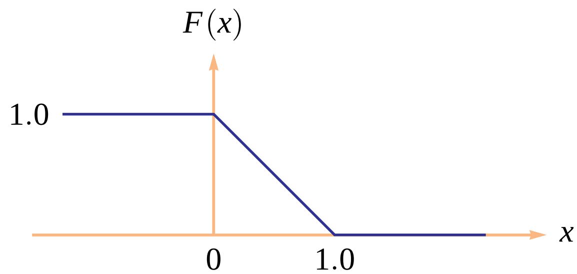

Figure 7: Initial profile with a ramp function.

The value of $\alpha = 1$. The initial condition is

$$u(x,0) = F(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{24}$$

which is sketched in Fig. 7. The equation of the characteristics becomes

$$x = F(\xi)t + \xi = \begin{cases} t+\xi & \text{if } \xi < 0 \\ (1-\xi)t+\xi & \text{if } 0 \le \xi \le 1 \\ \xi & \text{if } \xi > 1 \end{cases}$$

We can also express the characteristic lines to give t as a function of x:

$$t = \begin{cases} x-\xi & \text{if } \xi < 0 \\ \dfrac{x-\xi}{1-\xi} & \text{if } 0 \le \xi \le 1 \end{cases} \tag{25}$$

$$x = \xi \qquad \text{if } \xi > 1$$

The characteristics are plotted in Fig. 8. The solution of the Burgers' equation for the initial condition (24) is given by

$$u(x,t) = \begin{cases} 1 & \text{if } x-ut < 0 \\ 1-(x-ut) & \text{if } 0 \le x-ut \le 1 \\ 0 & \text{if } x-ut > 1 \end{cases}$$

which can also be written as

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ \dfrac{1-x}{1-t} & \text{if } t \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{26}$$

for $t \ge 0$. The characteristics lines are sketched in Fig. 8. For $x < 0$ the lines have speed 1; for $x > 1$ the lines have speed 0; for $0 < x < 1$ the lines have speed $1-x$. Since the breaking time $t^* = 1/\alpha$, for the present case we find that $t^* = 1$. So the solution cannot exist for $t > 1$, since the characteristics cross beyond that time and they carry different constant values of u. Figure 9 shows several wave profiles that indicate the steepening that is occurring. At $t = 1$ breaking of the wave occurs, which is the first instant when the solution becomes multiple valued. The breaking location is, $x^* = u_l t^* = 1$. Therefore, the characteristics first intersect at $(1,1)$. In general the nonlinear initial value problem (1) and (2) may have a solution only up to a finite breaking time t^*.

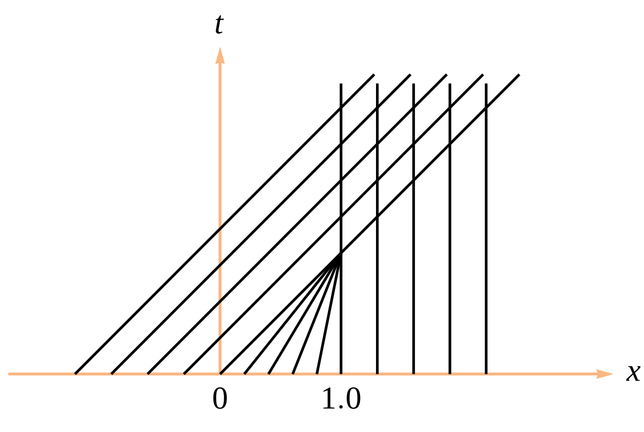

Figure 8: The characteristic curves for inviscid Burgers' equation.

3.1.3 Example 5

Consider the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

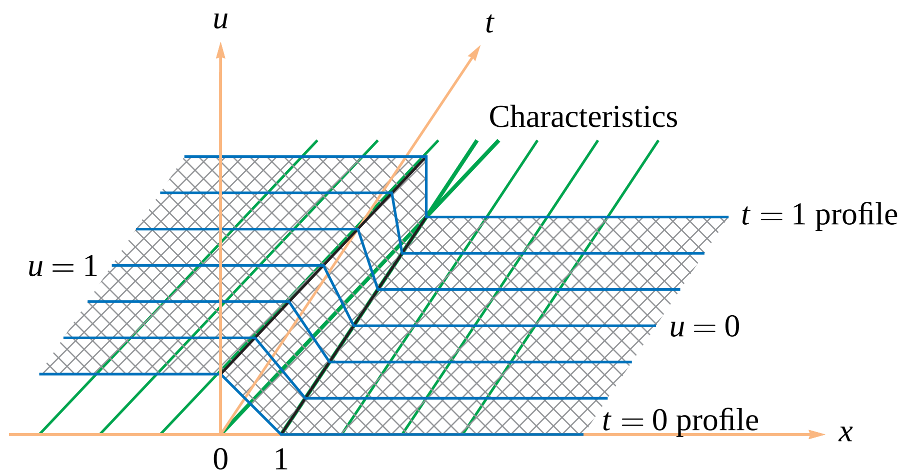

Figure 9: Solution surface of inviscid Burgers' equation.

subject to the Cauchy data

$$u(x,0) = F(x) = \begin{cases} 2 & \text{if } x < 0 \\ 2-x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Obtain and sketch the characteristic equation and the solution.

Solution Since the PDE is the inviscid Burgers' equation, the characteristics are straight lines emanating from $(\xi, 0)$ with speed $c = F(\xi)$. The equation of the characteristics becomes

$$x = \begin{cases} 2t + \xi & \text{if } \xi < 0 \\ (2-\xi)t + \xi & \text{if } 0 \le \xi \le 1 \\ t + \xi & \text{if } \xi > 1 \end{cases}$$

We can also express the characteristic lines to give t as a function of x:

$$t = \begin{cases} (x-\xi)/2 & \text{if } \xi < 0 \\ \dfrac{x-\xi}{2-\xi} & \text{if } 0 \le \xi \le 1 \\ x-\xi & \text{if } \xi > 1 \end{cases}$$

The characteristics are plotted in Fig. 10. The solution of the Burgers' equation for the given initial condition becomes

$$u(x,t) = \begin{cases} 2 & \text{if } x - ut < 0 \\ 2-(x-ut) & \text{if } 0 \le x-ut \le 1 \\ 1 & \text{if } x-ut > 1 \end{cases}$$

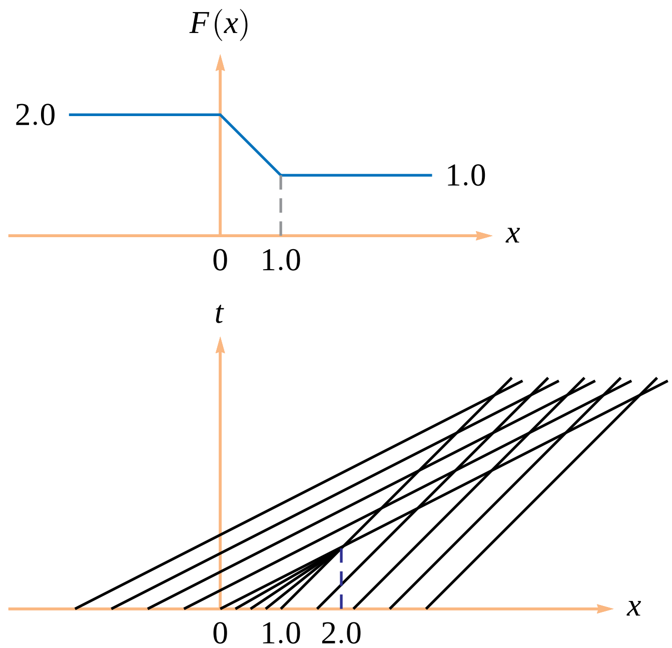

Figure 10: Initial profile and the characteristic curves for inviscid Burgers' equation.

which can also be written as

$$u(x,t) = \begin{cases} 2 & \text{if } x < 2t \\ \dfrac{2-x}{1-t} & \text{if } 2t \leq x \leq 1+t \\ 1 & \text{if } x > 1+t \end{cases} \tag{27}$$

for $t \geq 0$. The solution surface is sketched in Fig. 11. For $x < 0$ the lines have speed 2; for $x > 1$ the lines have speed 1; for $0 < x < 1$ the lines have speed $2 - x$. Since the breaking time $t^* = 1/\alpha$, for the present case we find that $t^* = 1$. So the solution cannot exist for $t > 1$, since the characteristics cross beyond that time and they carry different constant values of u. Figure 11 shows several wave profiles that indicate the steepening that is occurring. At $t = 1$ breaking of the wave occurs, which is the first instant when the solution becomes multiple valued. The breaking location is, $x^* = u_l t^* = 2$. Therefore, the characteristics first intersect at $(2,1)$.

3.1.4 The special case of jump discontinuity as initial waveform

An extreme case of wave breaking arises when the initial waveform has a jump discontinuity with the value of u behind the discontinuity greater than that ahead. If we have the initial waveform

$$F(x) = u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

with the condition that $u_l > u_r$, then breaking occurs immediately. This is illustrated in Fig. 12. The multivalued region starts right at the origin and is bounded by the characteristics $x = u_l t$

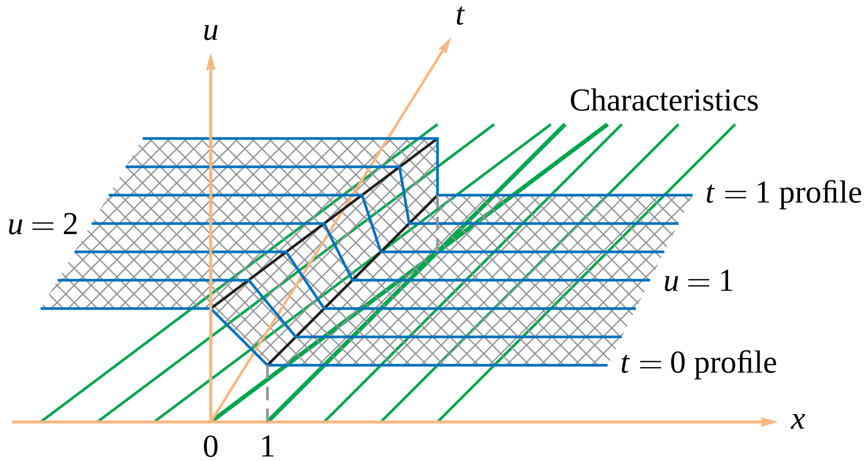

Figure 11: Solution surface of inviscid Burgers' equation.

and $x = u_r t$; the boundary is no longer a cusped envelope since $F(x)$ and its derivatives are not continuous. Nevertheless, the result may be considered as the limit of a series of smoothed-out steps, and the breaking point moves closer to the origin as the initial profile approaches the discontinuous step.

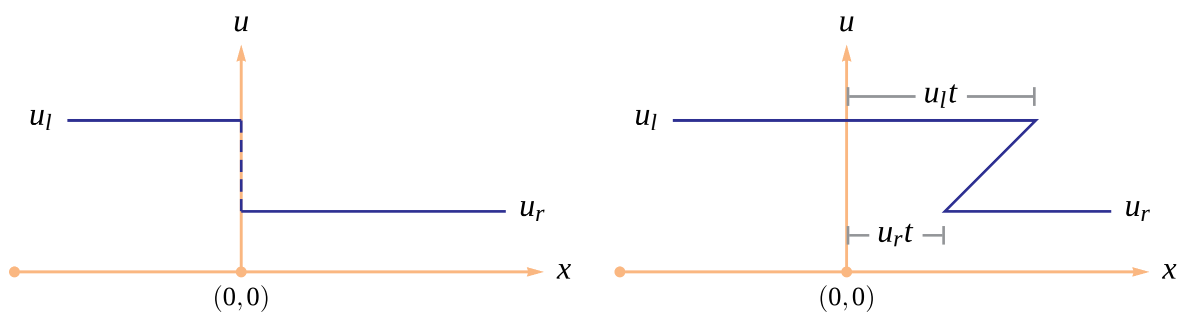

Figure 12: Initial waveform (jump discontinuity) and the solution at time t.

In most physical problems where (10) represents the mathematical model of a physical problem, $u(x,t)$ is just the property of some medium and is inherently single-valued. Therefore when breaking occurs (10) must cease to be valid as a description of the physical problem. Thus the situation is that some assumption or approximate relation in the formulation leading to (10) is no longer valid. In principle one must return to the physics of the problem, see what went wrong, and formulate an improved theory. However, it turns out, as we shall see, that the foregoing solution can be saved by allowing discontinuities into the solution; there is then a single-valued solution with a simple jump discontinuity to replace the multivalued continuous solution. This requires some mathematical extension of what we mean by a "solution" to (10), since strictly speaking the derivatives of u will not exist at a discontinuity. It can be done through the concept of a "weak solution". But it is important to appreciate that the real issue is not just a mathematical question of extending the solution of (10). The breakdown of the continuous solution is associated with the breakdown of some approximate relation in

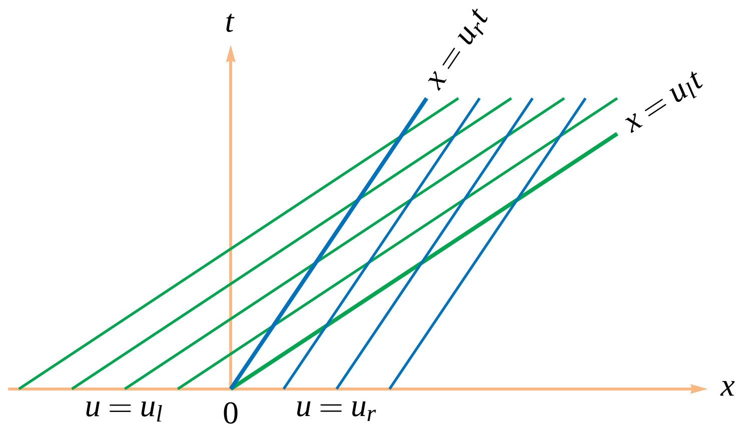

Figure 13: Characteristics of nonlinear wave equation with jump discontinuity as initial waveform.

the physics, and the two aspects must be considered together. It is found, for example, that there are several possible families of discontinuous solutions, all satisfactory mathematically; the nonuniqueness can be resolved only by appeal to the physics.

Clearly then, we cannot proceed further without discussion of some physical problems. The prototype is the nonlinear theory of waves in a gas and the formation of shock waves. When viscosity and heat conduction are ignored, the equations of gas dynamics have breaking solutions similar to the preceding ones. As the gradients become steep, just before breaking, the effects of viscosity and heat conduction are no longer negligible. These effects can be included to give an improved theory and waves no longer break in that theory. There is a thin region, a shock wave, in which viscosity and heat conduction are crucially important; outside the shock wave, viscosity and heat conduction may still be neglected. The flow variables change rapidly in the shock. This shock region is idealized into a discontinuity in the "extended" inviscid theory, and only shock conditions relating the jumps of the flow variables across the discontinuity need to be added to the inviscid theory.

4 Weak Solutions

Now the important question is that what happens to the solution after the breaking time at which the characteristics begin to cross? Beyond the breaking time there is no classical solution of the PDE, but can define a generalized solution called a *weak solution*. A solution is said to be the weak solution of a differential equation if it satisfies an integral formulation of the PDE and the corresponding conservation law. Thus any non-differentiable solution is called a weak solution of the differential form. However, a difficulty associated with the weak formulation of conservation laws is the possibility of more than one weak solution for the same initial waveform. This non-uniqueness can be resolved by eliminating the nonphysical solutions by imposing what is called *entropy conditions*.

4.1 Integral conservation law

We now derive the integral form of the general conservation law for a scalar variable. Let us consider a scalar quantity per unit volume u, defined as a flow related property. If ϕ is the same quantity per unit mass, then the relation between u and ϕ is given by

$$u = \rho \phi$$

We now consider an arbitrary control volume $\mathscr{V}$, bounded by a control surface S (see figure **??**). Our goal here is to obtain a mathematical expression for conservation law for a scalar quantity u over the domain $\mathscr{V}$. Let us first take the 'total amount of a quantity u inside a given domain' at a particular instant of time. If we consider the domain of volume $\mathscr{V}$, the total amount of u in $\mathscr{V}$ is given by

$$\int_{\mathscr{V}} u \, d\mathscr{V}$$

and the variation per unit time of the quantity u within the volume $\mathscr{V}$ is given by

$$\frac{d}{dt} \int_{\mathscr{V}} u \, d\mathscr{V}$$

Let $\overline{F}$ be the flux associated to the conserved variable u. The flux is defined as the amount of u crossing the unit surface per unit time. Thus, the amount of u crossing the surface element dS per unit of time is defined by the dot product of the flux and the local surface element,

$$\overline{F} \cdot \hat{n} \, dS$$

with the surface unit vector $\hat{n}$ pointing along the outward normal. The net amount of fluid crossing the total surface S is thus given by

$$-\oint_{\mathrm{S}} \overline{F} \cdot \hat{n} \, dS$$

The minus sign is introduced because we consider the flux contribution as positive when the flow enters the domain. Note that the dot product will be negative for an entering flux, as seen from Fig. **??**. Hence, in the absence of any sources, the general form of the conservation law for the quantity u is

$$\frac{d}{dt} \int_{\mathscr{V}} u \, d\mathscr{V} + \oint_{\mathrm{S}} \overline{F} \cdot \hat{n} \, dS = 0 \tag{28}$$

This is called the integral form of conservation law and is the most general expression of a conservation law.

This form has some remarkable properties:

- Equation (28) is valid for any fixed surface S and volume $\mathscr{V}$.
- The internal variation of u with time depends solely on the flux across the boundary S and not on any flux inside the volume $\mathscr{V}$.
- The fluxes do not appear under a derivative (or gradient) operator and may therefore be discontinuous, as in the case of shock waves and contact discontinuities.

4.2 Differential form of a conservation law

Let us now try to obtain a local differential form of the conservation law. It can be derived by applying Gauss' theorem to the surface integral term of the flux, assuming that the flux is continuous. We also assume sufficient smoothness in u that allows one to bring the derivative operator inside the integral operator.

Gauss' theorem states that the surface integral of the flux is equal to the volume integral of the divergence of this flux:

$$\oint_{S} \overline{F} \cdot \hat{n}\, dS = \int_{\mathscr{V}} \nabla \cdot \overline{F}\, d\mathscr{V}$$

for any volume $\mathscr{V}$, enclosed by the surface S. Since the control volume would not change with respect to time, the Leibnitz rule of differentiation under integration sign[1] can be applied to the first term on the left-hand side for interchanging the differential and integral operators. Further, using divergence theorem the surface integral can be transformed into volume integrals. We then have

$$\int_{\mathscr{V}} \frac{\partial u}{\partial t}\, d\mathscr{V} + \int_{\mathscr{V}} \nabla \cdot \overline{F}\, d\mathscr{V} = 0 \tag{29a}$$

We now combine the volume integrals into one,

$$\int_{\mathscr{V}} \left(\frac{\partial u}{\partial t} + \nabla \cdot \overline{F} \right) d\mathscr{V} = 0 \tag{29b}$$

Finally, we argue that the equation (29b) must hold for any arbitrary control volume regardless of its size and shape. This is possible only if the integrand is identically zero (Dubois–Reymond lemma). This leads to the differential form of the conservation law,

$$\frac{\partial u}{\partial t} + \nabla \cdot \overline{F} = 0 \tag{30}$$

This differential form is more restrictive than the integral form, as it requires the fluxes to be differentiable, i.e., having at least C^1 continuity, which is not the case in presence of shock waves, for instance.

Now let us take a closer look at the fluxes. The fluxes are generated from two types of phenomena: the one due to the bulk motion of the fluid called the convective flux and that due to molecular motion called diffusive flux. The diffusive flux can be present even when the fluid is at rest. The convection flux $\overline{F}_C$, associated to the quantity u in a flow of velocity $\overline{V}$, represents the amount of u that is carried away or transported by the flow. The amount of u convected through a surface element dS per unit time is

$$\overline{F}_C \cdot \hat{n}\, dS \tag{31}$$

[1] Leibnitz's rule of differentiation under integral operator is given by:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} u(x,t)\, dx = \int_{a}^{b} \frac{\partial u(x,t)}{\partial t} dx + u(b,t)\frac{db}{dt} - u(a,t)\frac{da}{dt}$$

If $d\dot{\mathscr{V}}$ is the elemental volume flow rate through a surface element dS, the amount of u convected through dS per unit time can also be written as

$$u\,d\dot{\mathscr{V}} = u\overline{V}\cdot\hat{n}\,dS \tag{32}$$

Comparing the equations (31) and (32), we have

$$\overline{F}_C = u\overline{V} = \rho\phi\overline{V} \tag{33}$$

If mass is the conserved quantity, the corresponding $u = \rho$, the fluid density, and $\phi = 1$. The associated convective flux

$$(\overline{F}_C)_{\text{mass}} = \rho\overline{V}$$

is the mass flux.

In the absence of diffusive flux, the differential form of conservation law (30) can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0 \tag{34}$$

where f, g, h are the components of the convective flux vector $\overline{F}_C$. For a one-dimensional case, equation (34) takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{35}$$

where $u(x,t)$ is the conserved quantity per unit volume and $f(u)$ is the corresponding flux. Equation (35) is the one-dimensional conservation law. Equation (35) can also be written in an equivalent nonconservation form as follows

$$\frac{\partial u}{\partial t} + \frac{df}{du}\frac{\partial u}{\partial x} = 0 \tag{36}$$

Comparing with the nonlinear transport equation (10) (Burgers' equation), we can infer that for Burgers' equation

$$\frac{df}{du} = u \tag{37}$$

The derivative df/du is called *characteristic speed* or *flux Jacobian function*. The flux corresponding to the conserved variable u for the Burgers' equation is given by

$$f = \frac{1}{2}u^2 \tag{38}$$

Thus, the Burgers' equation (10) can also be written in the following conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial(\frac{1}{2}u^2)}{\partial x} = 0 \tag{39}$$

The integral form of the one-dimensional conservation law can be either directly obtained from the general conservation law (28) or by integrating the one-dimensional differential conservation law (35) as follows. Integrate (35) over a region $a < x < b$ to obtain

$$0 = \int_a^b \left[\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x}\right] dx = \frac{d}{dt}\int_a^b u(x,t)\,dx + \left[f(u)\right]_a^b$$

or

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + f\left[u(b,t)\right] - f\left[u(a,t)\right] = 0 \tag{40}$$

Equation (40) represents the integral form of the one-dimensional conservation law. The reason we have rewritten equation (35) in the integral form of equation (40) is that the latter does not assume that the solution is differentiable with respect to x. Thus, the solution class of equations has been markedly broadened. The integral form of the conservation law says that the rate of change of the conserved quantity u in the fixed region (a, b) is equal to the net flux into the interval. Integral conservation law should be viewed as a fundamental physical principle.

4.3 Rankine–Hugoniot jump condition

As we have seen, the solution of nonlinear wave equation can develop discontinuities, even when provided with continuous initial data. We would like to determine how a discontinuity (shock) will propagate once it is formed. The Rankine–Hugoniot jump condition is a weak solution of the conservation law (40) and determines the position of a discontinuity at a given time. For the interval $[a, b]$, (40) takes the form:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = f\left[u(a,t)\right] - f\left[u(b,t)\right] \tag{40}$$

Suppose that there is a discontinuity at $x_s(t)$ within the interval $a < x_s < b$. Further, suppose u has finite limits as $x \to x_s$ from left and right. Then we can split the left-hand side integral of equation (40) in the following way:

$$\frac{d}{dt}\int_a^{x_s(t)} u(x,t)\,dx + \frac{d}{dt}\int_{x_s(t)}^b u(x,t)\,dx = f\left[u(a,t)\right] - f\left[u(b,t)\right] \tag{41}$$

Using Leibnitz's rule for differentiation under the integral operator, we have

$$\frac{d}{dt}\int_a^{x_s(t)} u(x,t)\,dx = \int_a^{x_s(t)} \frac{\partial u}{\partial t}dx + u(x_s^-,t)\frac{dx_s}{dt}$$

Similarly, we have

$$\frac{d}{dt}\int_{x_s(t)}^b u(x,t)\,dx = \int_{x_s(t)}^b \frac{\partial u}{\partial t}dx - u(x_s^+,t)\frac{dx_s}{dt}$$

where $u(x_s^-,t)$ and $u(x_s^+,t)$ are the left and right limit of $u(x,t)$ for $x \to x_s(t)$. Plugging the above two equations into (41) to obtain

$$f\left[u(a,t)\right] - f\left[u(b,t)\right] = \int_a^{x_s(t)} \frac{\partial u}{\partial t}dx + \int_{x_s(t)}^b \frac{\partial u}{\partial t}dx + \frac{dx_s}{dt}\left[u(x_s^-,t) - u(x_s^+,t)\right]$$

Since $\partial u/\partial t$ is bounded in each of the interval separately, the integrals tend to zero in the limit as $a \to x_s(t)$, $b \to x_s(t)$ and, therefore,

$$f\left[u(x_s^-,t)\right] - f\left[u(x_s^+,t)\right] = S \times \left[u(x_s^-,t) - u(x_s^+,t)\right]$$

$$f(u_1) - f(u_2) = S\,(u_1 - u_2) \tag{42a}$$

$$[f] = S[u] \tag{42b}$$

where $u_1 = u(x_s^-)$, $u_2 = u(x_s^+)$, $S = dx_s/dt$ is the shock speed, and $[\cdot]$ indicates the jump across the discontinuity. Equation (42) is called the Rankine–Hugoniot jump condition and gives the relation between the shock speed S and the states u_1 and u_2 across the discontinuity. Rearranging the above expression we get the expression for shock speed

$$S = \frac{f(u_1) - f(u_2)}{u_1 - u_2} = \frac{[f]}{[u]} \tag{43}$$

4.4 Weak solution from Rankine–Hugoniot jump condition

We have seen that for an initial condition in the form of a ramp function (18) (see Fig. 14),

$$F(x) = u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_l - \alpha x & \text{if } 0 \le x \le x_r \\ u_r & \text{if } x > x_r \end{cases} \tag{18}$$

the Burgers' equation (10) has classical solution for $t < 1/\alpha$ as given by (21) where $\alpha = (u_l - u_r)/x_r$. We repeat it below:

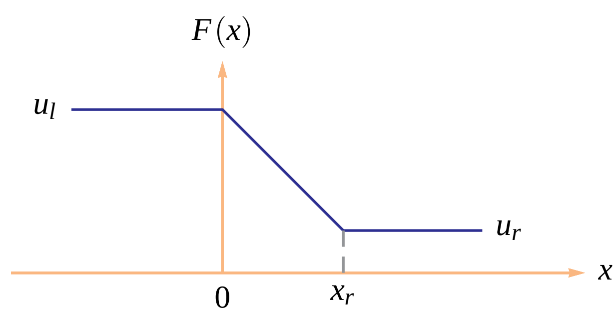

Figure 14: Initial waveform with a ramp function.

$$u(x,t) = \begin{cases} u_l & \text{if } x < u_l t \\ \dfrac{u_l - \alpha x}{1 - \alpha t} & \text{if } u_l t \le x \le x_r + u_r t \\ u_r & \text{if } x > x_r + u_r t \end{cases} \tag{21}$$

for $t \ge 0$. The wave breaks at time $t^* = 1/\alpha = x_r/(u_l - u_r)$ and location $x^* = u_l t^*$. Thus, beyond the breaking let's look for a weak solution of (10) that satisfies the initial condition (18). The weak solution must satisfy the Rankine–Hugoniot jump condition (43). That is, we need

$$f(u_1) - f(u_2) = S(u_1 - u_2)$$

or more specifically

$$\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 = S(u_1 - u_2)$$

or the shock speed is

$$S = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{u_1 + u_2}{2} \tag{44}$$

Thus, a shock which satisfies the Rankine–Hugoniot jump condition has a speed equal to the average of upstream and downstream values of u. Since for the present problem, $u_1 = u_l$ and $u_2 = u_r$, we have

$$S = \frac{u_l + u_r}{2} \tag{45}$$

Notice that the initial data for $x < 0$ wants $u = u_l$, while the initial data for $x > x_r$ wants $u = u_r$ when $t \geq 1/\alpha$. Now the equation of the characteristic curve $x = x_s(t)$, along which the discontinuity (shock) propagates, can be obtained as follows. We have from (45)

$$\frac{dx_s}{dt} = S = \frac{u_l + u_r}{2}$$

Integrating this equation to obtain

$$x_s = \left(\frac{u_l + u_r}{2}\right) t + c$$

The integration constant c can be obtained by noting that the curve $x = x_s(t)$ should contain the point (x^*, t^*). Plugging the values of t^* and x^* into the above equation, we get $c = x_r/2$. Therefore, the curve of discontinuity must be given by

$$x_s = \left(\frac{u_l + u_r}{2}\right) t + \frac{x_r}{2} = St + \frac{x_r}{2} \tag{46}$$

Therefore, for time $t \geq t* = x_r/(u_l - u_r)$, we have

$$u(x,t) = \begin{cases} u_l & \text{if } x < St + x_r/2 \\ S & \text{if } x = x_s = St + x_r/2 \\ u_r & \text{if } x > St + x_r/2 \end{cases} \tag{47}$$

Along the curve of discontinuity, $x_s = St + \frac{x_r}{2}$, the solution is given by

$$u(x,t) = S = \frac{u_l + u_r}{2}$$

Equation (47) is a classical solution of Burgers' equation (10) on either side of the curve of discontinuity, $x_s = St + x_r/2$, and the solution u satisfies the Rankine–Hugoniot jump condition along the curve of discontinuity (see figure 15). Without the discontinuity curve, the solution would have been multivalued and this curve prevents that from happening. Therefore, $u(x,t)$ defined by (47) is a weak solution of (10) for $t \geq t* = 1/\alpha$. Figure 16 shows the weak solution of Burgers' equation and the bounding characteristics for the special case of $u_r = 0$.

Finally, we conclude that the solution of (10) with initial condition (18) is given by equation (21) for $t < t^*$ and equation (47) for $t \geq t^*$. Note that, once a shock forms, it cannot suddenly disappear; the discontinuity remains as the solution propagates. The solution at various time for the special case of $u_r = 0$ is illustrated in the figure 17.

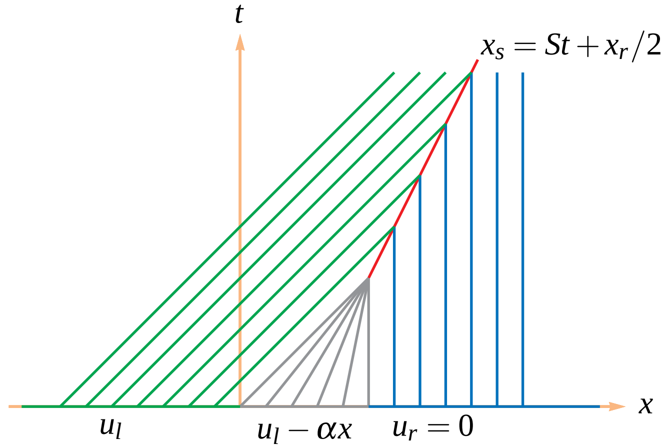

Figure 15: Characteristic curves for Burgers' equation for initial condition (18) for the case of $u_r = 0$.

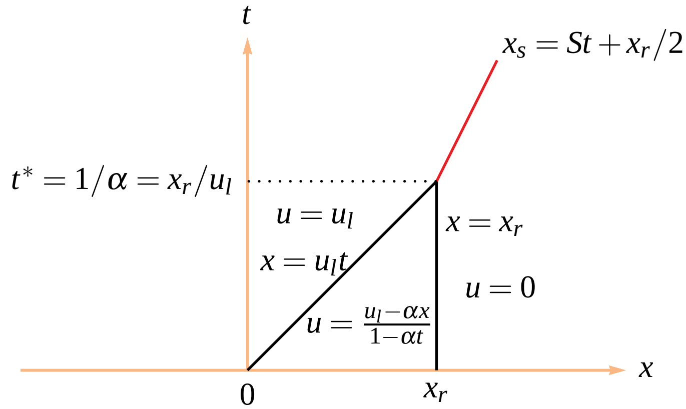

Figure 16: Weak solution of Burgers' equation showing the breaking time for the case of $u_r = 0$.

4.4.1 Non-uniqueness of the weak solutions

A mathematical difficulty associated with the weak formulation of conservation laws is that the possibility of more than *one* weak solution for the same initial data. We demonstrate this with the following example. A subtle fact about the hyperbolic conservation law is that transforming the differential form into what appears to be an equivalent differential equation may not give an equivalent equation in the context of weak solutions. Consider the inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{48a}$$

We multiply the above equation by u to obtain

$$u\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0 \tag{48b}$$

The differential equations (48a) and (48b) are equivalent in the sense that they have precisely the same smooth solutions. However, they have different weak solutions as we can see by computing the shock speed. The conservation forms of the equations (48a) and (48b) are given

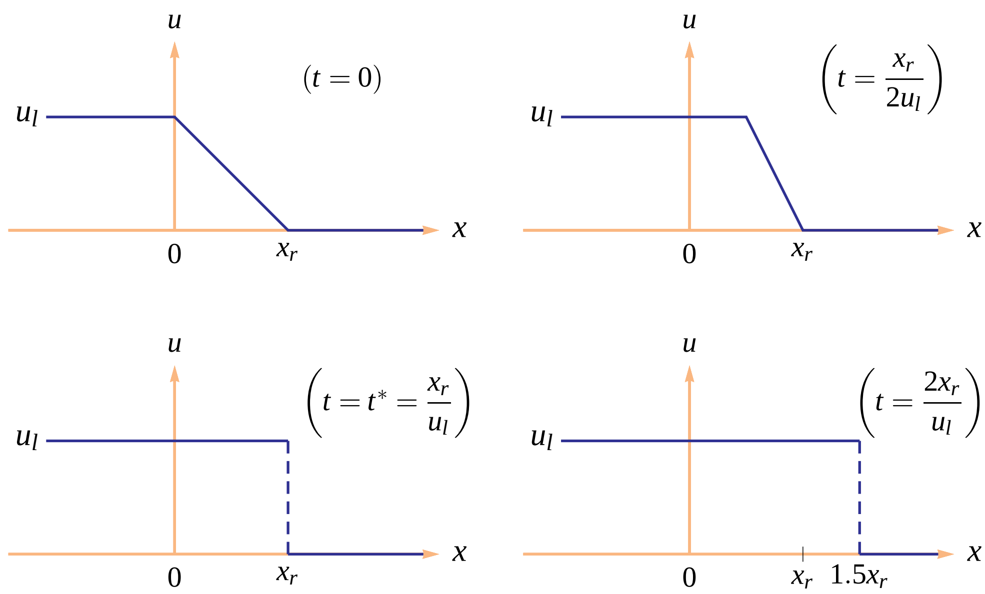

Figure 17: Weak solution of Burgers' equation for the case of $u_r = 0$.

by

$$\frac{\partial u}{\partial t} + \frac{\partial(\frac{1}{2}u^2)}{\partial x} = 0 \tag{49a}$$

$$\frac{\partial(u^2)}{\partial t} + \frac{\partial(\frac{2}{3}u^3)}{\partial x} = 0 \tag{49b}$$

Equation (49b) can be considered as a hyperbolic conservation law for u^2 rather than u itself, with flux function $f(u^2) = \frac{2}{3}u^3$. Now, for the initial condition

$$F(x) = u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

the unique shock solution of equation (49a) is shock travelling at speed

$$S_1 = \frac{[\frac{1}{2}u^2]}{[u]} = \frac{1}{2}\frac{u_l^2 - u_r^2}{u_l - u_r} = \frac{u_l + u_r}{2}$$

whereas the unique weak solution to equation (49b) is a shock travelling at speed

$$S_2 = \frac{[\frac{2}{3}u^3]}{[u^2]} = \frac{2}{3}\frac{u_l^3 - u_r^3}{u_l^2 - u_r^2} = \frac{2}{3}\frac{u_l^2 + u_l u_r + u_r^2}{u_l + u_r}$$

Clearly, $S_2 \neq S_1$, and thus the two equations have different weak solutions. Note that the derivation of equation (49b) from (49a) requires manipulating derivatives in a manner that is valid only when u is smooth. Although this example concerns Burgers' equation, in general, changing u and flux function f in scalar conservation laws always alters the weak solution.

4.5 Initial discontinuous data – shock

Consider the Burgers' equation again,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{50}$$

with an initial condition in the form of a step function with a single jump discontinuity at $x = x_r$:

$$F(x) = u(x,0) = \begin{cases} u_l & \text{if } x < x_r \\ u_r & \text{if } x > x_r \end{cases} \tag{51}$$

The form of the solution depend on the relation between u_l and u_r. Let us first consider the case when $u_l > u_r$ so that the initial profile is already in the form of a shock. From (13), the equation of the characteristics is given by

$$x = F(\xi)t + \xi$$

Using the initial condition (51), the equation of the characteristics become

$$x = \begin{cases} u_l t + \xi & \text{if } \xi < x_r \\ u_r t + \xi & \text{if } \xi > x_r \end{cases} \tag{52}$$

The equation for characteristic lines can also be written as

$$t = \begin{cases} \dfrac{1}{u_l}(x-\xi) & \text{if } \xi < x_r \\ \dfrac{1}{u_r}(x-\xi) & \text{if } \xi > x_r \end{cases} \tag{53}$$

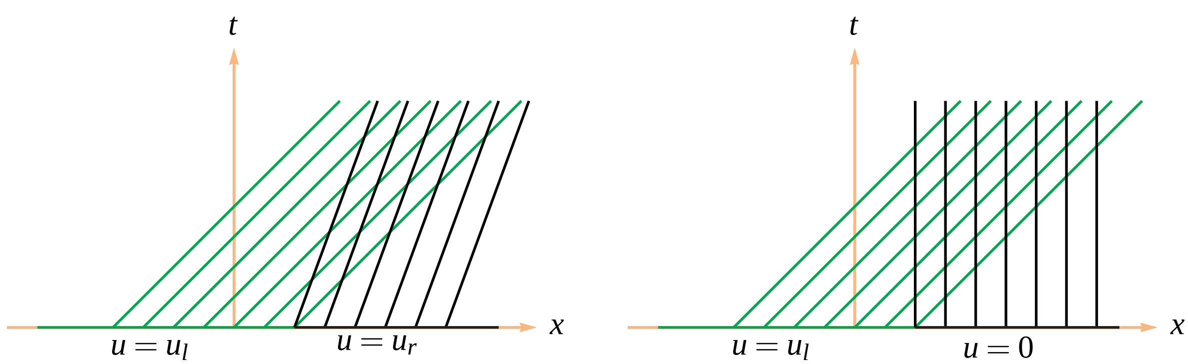

Figure 18: Characteristic lines for initial condition (51).

The plot of the equation for characteristics line for the initial condition (51) is shown in figure 18 for the cases $u_r \neq 0$ and $u_r = 0$. The characteristics originating from region where $x < x_r$ have positive slopes equal to $1/u_l$ and the solution should equal to u_l along these lines. But, characteristics originating from region where $x > x_r$ have positive slopes equal to $1/u_r$ and the solution should equal to u_r along these lines. Since these two families of characteristics intersect, we cannot hope to find any classical solution which solves this problem. Hence, we look for a weak solution, by looking for a piecewise continuously differentiable function which satisfy the Rankine–Hugoniot jump condition and initial condition (51). That is, we need

$$f(u_1) - f(u_2) = S(u_1 - u_2)$$

or more specifically

$$\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 = S(u_1 - u_2)$$

Since for the present problem, $u_1 = u_l$ and $u_2 = u_r$, the shock speed is given by

$$S = \frac{u_l + u_r}{2}$$

The shock propagates with a speed equal to the average of u at the jump discontinuity. Since $S = dx_s/dt$, the equation of characteristic curve along which the shock propagate can be obtained by solving the following ODE:

$$\frac{dx_s}{dt} = \frac{u_l + u_r}{2}$$

Integrating this equation to obtain

$$x_s = \left(\frac{u_l + u_r}{2}\right)t + c$$

The integration constant c can be obtained by noting that the curve $x = x_s(t)$ should contain the point $(x,t) = (x_r, 0)$. Plugging these values into the above equation, we get $c = x_r$. Therefore, the curve of discontinuity must be given by

$$x_s = \left(\frac{u_l + u_r}{2}\right)t + x_r = St + x_r \qquad \Rightarrow \qquad t = \frac{x_s - x_r}{S} \tag{54}$$

where S is the shock speed (see figure 19). Therefore, the weak solution that satisfies the Rankine–Hugoniot jump condition is given by

$$u(x,t) = \begin{cases} u_l & \text{if } x < x_s = St + x_r \\ S & \text{if } x = x_s = St + x_r \\ u_r & \text{if } x > x_s = St + x_r \end{cases} \tag{55a}$$

for $t \geq 0$. Or, we may write the solution as

$$u(x,t) = \begin{cases} u_l & \text{if } t > (x - x_r)/S \\ S & \text{if } t = (x - x_r)/S \\ u_r & \text{if } t < (x - x_r)/S \end{cases} \tag{55b}$$

Figure 20 shows the solution at time $t = 0$ and $t = 1$ for the case of $u_r = 0$.

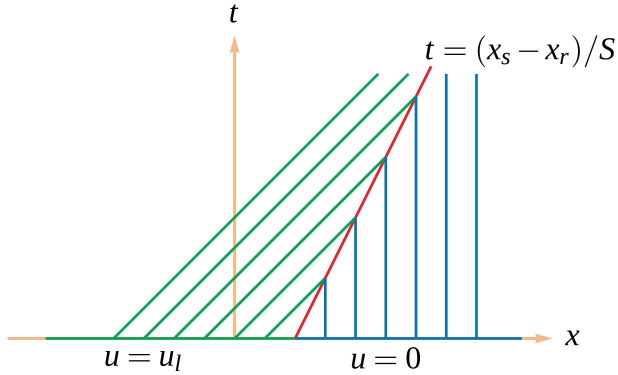

Figure 19: The characteristic curves for initial discontinuous data (51) when $u_r = 0$.

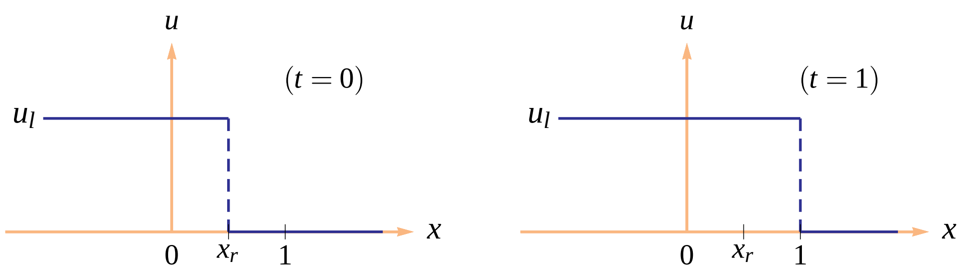

Figure 20: The solution of Burgers' equation for initial discontinuous data (51) when $u_r = 0$.

4.6 Initial discontinuous data – rarefaction

It is to be noted that the method of characteristics may not give a solution for all values of x and t, but it may be possible to extend such a solution to encompass all values of the independent variables. This will be an important consideration in gas dynamics applications. To illustrate this we again consider the Burgers' equation,

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{56}$$

where the flux function $f(u) = u^2/2$ and an initial condition in the form of a step function with a single jump discontinuity at $x = x_r$:

$$F(x) = u(x,0) = \begin{cases} u_l & \text{if } x < x_r \\ u_r & \text{if } x > x_r \end{cases} \tag{57}$$

Here we consider the case when $u_l < u_r$. From (13), the equation of the characteristics is given by

$$x = F(\xi)t + \xi$$

Using the initial condition (57), the equation of the characteristics become

$$x = \begin{cases} u_l t + \xi & \text{if } \xi < x_r \\ u_r t + \xi & \text{if } \xi > x_r \end{cases} \tag{58}$$

The equation for characteristic lines can also be written as

$$t = \begin{cases} \dfrac{1}{u_l}(x-\xi) & \text{if } \xi < x_r \\ \dfrac{1}{u_r}(x-\xi) & \text{if } \xi > x_r \end{cases} \tag{59}$$

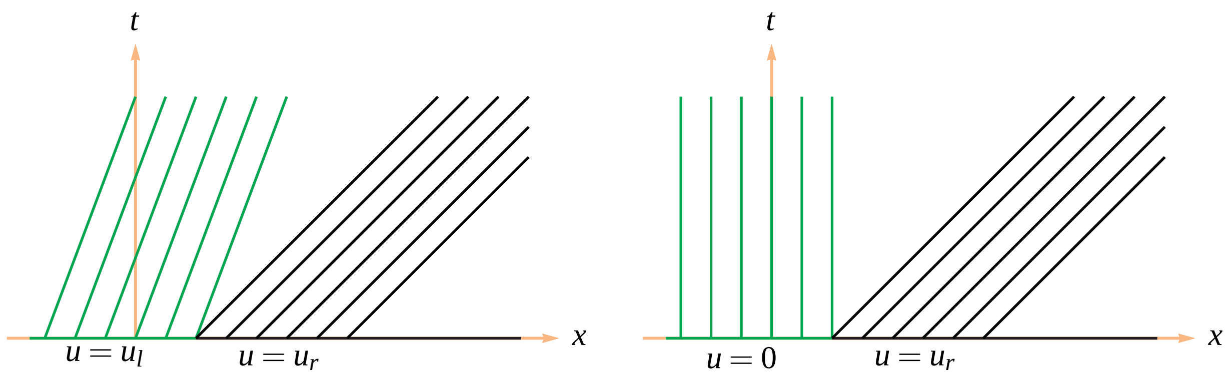

Figure 21: Characteristic lines for initial condition (57) when $u_l = 0$.

The plot of the equation for characteristics line for the initial condition (57) is shown in figure 21 for the cases $u_l \neq 0$ and $u_l = 0$. The characteristics originating from region where $x < x_r$ have slopes equal to $1/u_l$ and the solution should equal to u_l along these lines. But, the characteristics originating from region where $x > x_r$ have slopes equal to $1/u_r$ and the solution should equal to u_r along these lines. Since $u_l < u_r$, the left family of characteristics is steeper than the right family of characteristics. Consequently, in this case, we have no intersecting of characteristics.

However, we still have a problem. Notice that in this case the waves originating from the region $\xi > x_r$ move to the right faster than the waves origination from the region $\xi < x_r$. This leads to the formation of an ever increasing gap between the faster moving wavefront and the slower moving wavefront. Thus the characteristics diverges from the initial discontinuity and a wedge-shaped region is formed on which we do not have enough information! How should we define our solution in this region?

Let us first look for a weak solution in the form of a piecewise continuously differentiable function which satisfy the Rankine–Hugoniot jump condition and initial condition (57). That is, we need

$$\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 = S(u_1 - u_2)$$

Since for the present problem, $u_1 = u_l$ and $u_2 = u_r$, the shock speed is given by

$$S = \frac{u_l + u_r}{2}$$

From the work of previous case, it is easy to see that the curve of discontinuity must be given by

$$x_s = \left(\frac{u_l + u_r}{2}\right)t + x_r = St + x_r \qquad \Rightarrow \qquad t = \frac{x_s - x_r}{S} \tag{60}$$

Therefore, the weak solution that satisfies the Rankine–Hugoniot jump condition is given by

$$u^a(x,t) = \begin{cases} u_l & \text{if } x < St + x_r \\ u_r & \text{if } x > St + x_r \end{cases} \tag{61}$$

for $t \geq 0$. Clearly, u^a is a classical solution on either side of the curve of discontinuity $x_s = St + x_r$ (see figure 22 where u_l is assumed to be zero). Note that characteristics now go out of the shock. Since u^a satisfies the Rankine-Hugoniot jump condition along the curve of discontinuity, it is also a weak solution of (56) satisfying (57). However, this is not the only possible solution.

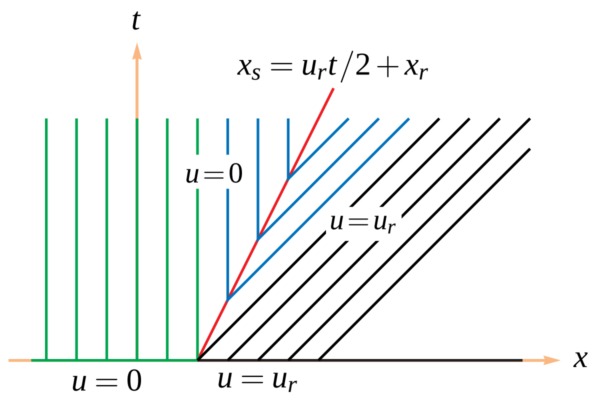

Figure 22: The incorrect rarefaction shock solution for $u_l = 0$.

As we have seen earlier, the solution is well defined outside the wedge-shaped region

$$x = \begin{cases} u_l t + \xi & \text{if } \xi < x_r \\ u_r t + \xi & \text{if } \xi > x_r \end{cases}$$

The equation of these characteristic lines can also be written as

$$t = \begin{cases} \dfrac{1}{u_l}(x - \xi) & \text{if } \xi < x_r \\ \dfrac{1}{u_r}(x - \xi) & \text{if } \xi > x_r \end{cases}$$

Now our task is to connect the two regions where the solution is well defined. The simplest way to connect is using straight lines passing through the point $(\xi = x_r)$. Thus, we have

$$x = ut + x_r \qquad \text{or} \qquad t = \frac{x - x_r}{u}$$

which produces a solution

$$u(x,t) = \frac{x - x_r}{t}$$

Note that this solution within the wedge-shape region satisfies both differential equation (54) and the required values $u[(u_l t + x_r), t] = u_l$ and $u[(u_r t + x_r), t] = u_r$ at the two edges of the wedge. Accordingly, the equation of characteristics for the complete region can be written as

$$t = \begin{cases} \dfrac{1}{u_l}(x - \xi) & \text{if } \xi < x_r \\ \dfrac{1}{u}(x - x_r) & \text{if } \xi = x_r \\ \dfrac{1}{u_r}(x - \xi) & \text{if } \xi > x_r \end{cases} \tag{62}$$

for $t \geq 0$ and the complete solution is given by

$$u^b(x,t) = \begin{cases} u_l & \text{if } x < (u_l t + x_r) \\ (x - x_r)/t & \text{if } (u_l t + x_r) \leq x \leq (u_r t + x_r) \\ u_r & \text{if } x > (u_r t + x_r) \end{cases} \tag{63}$$

for $t \geq 0$. Notice that $u^b(x,t)$ is a continuous solution of (56) and satisfies the initial condition (57). This type of solution which "fans" the wedge $(u_l t + x_r) < x < (u_r t + x_r)$ is called a *rarefaction wave*.

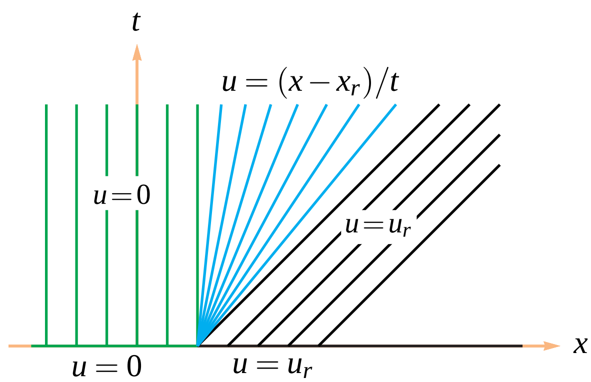

Figure 23: Rarefaction fan solution when $u_l = 0$.

We have found two different solutions. Clearly only one of these solution is physically meaningful. The question is which? There are several considerations towards picking one weak solution or another. In general, a more regular weak solution may be preferred. Below we introduce the notion of an entropy condition.

5 Entropy condition

Let us consider the Burgers' equation written in the conservation form (35). We allow for a curve of discontinuity in the solution $u(x,t)$ if the wave on the left of discontinuity is moving faster than the wave on the right. That is, we allow for a curve of discontinuity between u_l and u_r only if

$$f'(u_l) > S > f'(u_r) \tag{64a}$$

This is known as the *entropy condition*. Since $f'(u_l) = u_l$, $f'(u_r) = u_r$ and $u_l < u_r$ in the problem defined by equations (56) and (57), equation (64a) reduces to

$$u_l > S > u_r \tag{64b}$$

If the solution satisfies the entropy condition (64) then the characteristics starting from either side of the discontinuity when continued in the direction of increasing time t intersect the line of discontinuity. Clearly this condition is violated in the solution given by $u^a(x,t)$ and thus we reject this solution.

If all discontinuities in the solution satisfies the entropy condition (64), no characteristics drawn in the direction of decreasing t intersects a line of discontinuity. This shows that for such solution every point (x,t) can be connected to a point in the initial data-line by moving along the characteristics in backward time. In other words, the entropy condition ensures that no characteristics can "spring" out of a curve of discontinuity.

A discontinuity satisfying the Rankine–Hugoniot jump condition (42) and the entropy condition (64) is called a *shock*. Therefore, we only accept such solutions u for which curves of discontinuity in the solution are shock curves. Coming back to the problem posed by equations (56) and (57). we say u is an admissible solution only if u is a weak solution such that any curve of discontinuity for u is a shock curve.

For this case the shock speed, $S(t) = (u_l + u_2)/2$. According to the entropy condition (64b), we have

$$u_l > S > u_r$$

It is clear that u^a does not satisfy the entropy condition along the curve of discontinuity $x_s = St + x_r$. Consequently, $x = St + x_r$ is not a shock curve, and, therefore, u^a is not an admissible solution. Solution u^b, however, is a continuous solution. Therefore, we accept this solution as the physically relevant one.

6 Boundary conditions

Until now all of the wave equations we have examined were considered on an infinite domain $-\infty < x < \infty$. To find the solution to (1) in a practical problem, we need to specify the boundary conditions. To illustrate the basic points it is sufficient to study the simple linear wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

with initial conditions

$$u(x,0) = F(x)$$

We have already seen that on the interval $(-\infty, \infty)$ the solution to the problem (for $c > 0$) was a unidirectional right travelling wave

$$u(x,t) = F(x - ct)$$

On a semi-infinite interval, $0 < x < \infty$ with $c > 0$, we must specify a boundary condition at $x = 0$, say

$$u(0,t) = v(t) \qquad \text{for} \quad t > 0$$

in addition to the specified initial condition. The solution is then

$$u(x,t) = \begin{cases} v(t - x/c) & \text{if} \quad x \le ct \\ F(x - ct) & \text{if} \quad x > ct \end{cases} \tag{65a}$$

which is illustrated in Fig. 24. The solution in part of the domain, namely points that can be reached by characteristics from the positive x-axis, is given by the initial data; the solution in the remainder of the domain is given by tracing back along characteristics to the boundary data on the positive t-axis.

If the boundary value v is independent of time, equation (65a) becomes

$$u(x,t) = \begin{cases} v & \text{if} \quad x \le ct \\ F(x - ct) & \text{if} \quad x > ct \end{cases} \tag{65b}$$

Note that, since the wave travels from left to right (information travels from left to right), the boundary condition is needed at the left boundary and no boundary condition required on the right boundary, which is at infinity.

Now we consider a bounded domain $0 < x < L$ with $c > 0$. We make reference to Fig. 25. Since u is given by the initial condition $F(x)$ along the initial line $t = 0$, $0 < x < L$, data cannot be prescribed arbitrarily on the segment 'A' along the boundary $x = L$. This is because the characteristics $x - ct =$ constant carry the initial data to the segment 'A'. Boundary data can be imposed along the line $x = 0$, since those data would be carried along the forward-going characteristics to the segment 'B' along $x = L$. Then boundary conditions along 'B' cannot be prescribed arbitrarily.

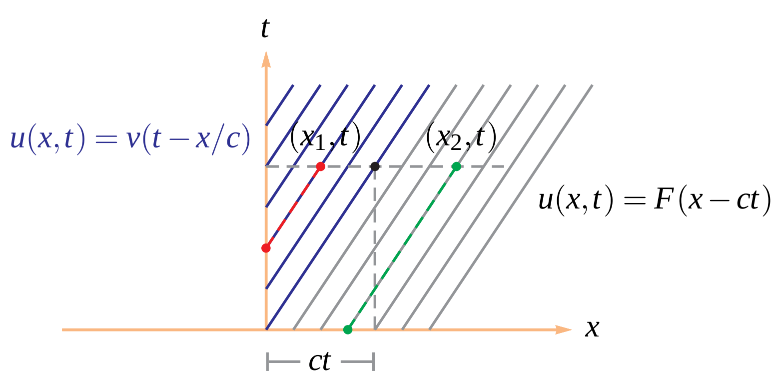

Figure 24: Characteristics of linear wave equation on a half-line.

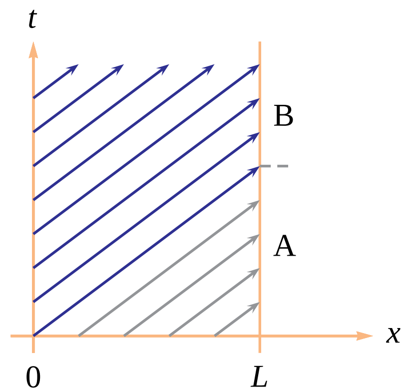

Figure 25: Characteristic diagram of linear wave equation on a finite domain.

Thus it is clear that the problem

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0, \qquad && 0 < x < L, \quad c > 0 \\ u(x,0) &= F(x), && 0 \le x \le L \\ u(0,t) &= v(t), && t > 0 \end{aligned}$$

is properly posed in the sense that there is a unique solution. There are no backward-going characteristics in this problem so there are no left traveling waves. Thus waves are not reflected from the boundary $x = L$. In summary, care must be taken to properly formulate boundary value problems for unidirectional wave equation. In fact, the solution would be impossible to determine had the boundary conditions been given on the right boundary $x = L$ as the problem would then be ill-posed for lack of proper boundary conditions. We shall see that the situation is much different for second order hyperbolic partial differential equations like $u_{tt} - u_{xx} = 0$. In this case both forward- and backward-going characteristics exist, and so left traveling waves are also possible as well as a mechanism for reflections from boundaries.

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