Inverse Source Problems for the Damped Euler-Bernoulli Beam and Kirchhoff Plate Equations

A thesis submitted

in partial fulfillment for the award of the degree of

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by

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Certificate

This is to certify that the thesis titled ***Inverse Source Problems for the Damped Euler-Bernoulli Beam and Kirchhoff Plate Equations*** submitted by **Anjuna Dileep**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a bona fide record of the original work carried out by her under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Declaration

I declare that this thesis titled ***Inverse Source Problems for the Damped Euler-Bernoulli Beam and Kirchhoff Plate Equations*** submitted in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a record of the original work carried out by me under the supervision of **Dr. K. Sakthivel**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

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This thesis is dedicated to my son Jaan Fabio (Achu)

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Abstract

The direct and inverse problems for the Euler-Bernoulli beam and Kirchhoff-Love plate models have been extensively studied over the years, and they continue to be an active area of research due to their applications in science and engineering. In this aspect, we mainly focus on the inverse source problems of the Euler-Bernoulli beam and Kirchhoff-Love plate equations with various damping mechanisms. More precisely, the research reported in the thesis mainly deals with the inverse problems of identifying unknown source terms in the Euler-Bernoulli beam with viscous and Kelvin-Voigt dampings, rectangular Kirchhoff-Love plate with viscous damping, thermoelastic plate with structural damping.

As explained in the introduction of the thesis, the unique determination of a spatial load in the undamped beam equation from final time measurement is not a feasible problem. We study the effect of viscous damping in the unique determination of unknown spatial load in a simply supported Euler-Bernoulli beam from measured final time displacement. By considering two specific temporal loads, we obtain sufficient conditions on the damping parameter and admissible final time interval to uniquely express the spatial load in terms of an infinite series using the Singular Value Decomposition (SVD) method. Next, we discuss the inverse problem of determining the unknown transverse shear force (boundary data) in the Euler-Bernoulli beam in the presence of the Kelvin-Voigt damping from measured deflection and bending moment. The inverse boundary value problem of determining the shear force acting on the inaccessible tip of the microcantilever, one of the key components of Transverse Dynamic Force Microscopy (TDFM), is important for understanding biological specimen images at submolecular precision. The considered inverse problems are transformed into minimization problems for Tikhonov functionals and show that the regularized functionals admit a unique solution for the inverse problems. In this work, we also prove remarkable Lipschitz stability estimates for the transverse shear force in terms of the given measurement by a feasible condition only on the Kelvin-Voigt damping coefficient using the variational methods. The required solvability of direct and adjoint problems is obtained under the minimal regularity of the admissible shear force, which turns out to be the regularizing effect of the Kelvin–Voigt damping.

The analysis of the inverse source problem of the beam is further explored for the unique reconstruction of spatial load and the stability of reconstructing the spatial load

in the Kirchhoff-Love plate in the presence of viscous damping using the regularization technique and spectral method. In this study, the inverse problem is first posed as a minimization problem of a regularized Tikhonov functional and obtained a unique solution to the minimization problem. We established a stability estimate under feasible conditions on final time and damping parameter. Then, the same inverse source problem is studied by the SVD method, and we concluded that the solution obtained by these two methods is equivalent. Besides, with the help of singular values of the input-output operator and regularity assumption on temporal load, we derived stability estimates for the regularized and SVD solutions of the inverse problem.

In the final work, we further extend the study for the inverse problem of simultaneously identifying the mechanical load and heat source in a structurally damped thermoelastic system describing a homogeneous and elastically and thermally isotropic plate from the vertical displacement measured at the final time. The inverse problem is reformulated as a minimization problem for the Tikhonov functional using the Tikhonov regularization method. We prove that the regularized Tikhonov functional admits a unique solution in the naturally defined set of admissible sources. An upper bound for the final time is established to derive a stability estimate for the inverse problem by invoking a first-order necessary optimality condition for the minimization problem. This stability result also gives rise to the uniqueness of the solution to the inverse problem. The results established in this work help to analyze the influence of thermal and mechanical loading that results in materials deflection, which, in turn, is vital for material science and engineering applications.

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Nomenclature

E	Young's modulus
I	Moment of inertia
ℓ	Length of the beam
h	Thickness of the plate
ν	Poisson ratio
ρ	Mass density per unit area/volume
D	Modulus of flextural rigidity
C	Specific heat of the body
η	Coefficient of thermal expansion
γ_0	Reference temperature
χ_0	Thermal conductivity
χ_1	Ratio of external thermal conductivity to the thermal conductivity of the plate
μ	Viscous damping coefficient
κ	Kelvin-Voigt damping coefficient
w	Structural damping coefficient
T	Final time
T	Transpose of a matrix
n	Outward normal vector to the boundary of a domain
$R(A)$	Range space of an operator A
$N(A)$	Null space of an operator A
A^*	Adjoint of an operator A

Chapter 1

Introduction

The *inverse problems* are the counterpart of the *direct problems* that consist of determining unmeasurable system parameters from measurable parameters by using a mathematical model associated with a physical system. For instance, one of the classical inverse problems is: *Can one hear the shape of a drum?* ([59]). In other words, can we figure out the shape of a drum based on the sound it emits? It is clear that the corresponding direct problem here is to determine the sound emitted by a drum of known shape. The mathematical theory of inverse problems for differential and integral equations is being developed rapidly within the framework of mathematical physics. It is known that in direct problems, we determine the solution of a differential equation utilizing given inputs, namely, coefficients, source functions, and initial data to describe the physical phenomena, while in the case of inverse problems, one may need to find the output functions as well as the input data. A mathematical problem is said to be well-posed in the sense of Hadamard ([40]), if a solution to this problem exists, the solution is unique, and that solution continuously depends on the data, that is, small deviations in the data lead to small deviations in the solution. A major issue with inverse problems is that even if the direct problem is well-posed, the corresponding inverse problems are mostly ill-posed ([57]). Moreover, without entering into further intricacies of mathematical terminology, it is necessary to emphasize that, in most situations, inverse and ill-posed problems have a crucial characteristic, namely, instability. The ill-posedness due to instability makes the inverse problems challenging and mathematically interesting, and the instability may be due to measurement noise and errors.

In spite of the challenges, inverse problems occur in almost all areas of science and engineering, for example, geophysical problems, medical imaging, astronomy, remote sensing, signal processing, inverse scattering problems, aerodynamics, electrodynamics, structural engineering, machine learning, and so on (see, [4, 23, 26, 61, 77, 28]). Given its broad range of applications, it is hardly surprising that the theory of inverse and ill-posed problems has developed into one of the fastest-growing fields of modern science and engineering. It is almost impossible to estimate the total number of scientific publications that directly or indirectly deal with inverse and ill-posed problems.

1.1 Vibrations of beams and plates

This section mainly discusses the physical models considered for our research work, namely, the Euler-Bernoulli beam, Kirchhoff-Love plate, and thermoelastic plate equations.

A beam is a structural element resisting forces acting laterally to its axis. Daniel Bernoulli derived the equation of motion for the transverse vibration of thin beams in 1735, while Euler obtained the solutions for various support conditions in 1744. The Euler-Bernoulli beam theory is a simplified version of linear elasticity theory that describes the relationship between deflection and applied load. Many mechanical systems from industry and engineering use the Euler–Bernoulli beam equation to represent bending vibration ([73]). Analysis and simulation of such systems have become key research areas because of the necessity to manage the dynamics of these systems. The static equation $(EIu''(x))'' = q(x)$, where u is the beam deflection, EI is the flexural rigidity and q is the distributed load (see,[79, 91]), is a basic model for the Euler-Bernoulli beam equation subject to external load.

A general dynamic model of the Euler-Bernoulli beam subject to external load is given by the equation $\rho(x)u_{tt} + (r(x)u_{xx})_{xx} = g(x,t)$, where ρ is mass density, $r(x) = EI(x)$ is the flexural rigidity, and $g(x,t)$ is the distributed load, has huge fundamental applications in civil, mechanical, and aeronautical engineering (see, [19, 31, 32, 98]). As an extension to beam theory, in 1888, Love developed a model to determine the stress and deformations of a thin plate subject to external forces and moments using the assumption proposed by Kirchhoff. This is the so-called Kirchhoff-Love plate equation (see, [69]) whose governing equation is given by $u_{tt} + D\Delta^2 u = g(x,t)$, where D is the modulus of flextural rigidity.

The study of thermoelasticity is a vital field in material science and engineering that deals with the coupling between thermal and mechanical responses of materials. The thermoelastic plate model is an interconnected system of the Kirchhoff-Love plate and heat equations. The Kirchhoff model describes the vibration of the plate, and the temperature distribution of the plate is described by the heat equation, which is modeled using Fourier's law of heat conduction. It is well known that a plate's temperature gradient will contribute to plate deformation and can lead to changes in stiffness, vibration frequencies, and even buckling.

1.2 Damping mechanism in beams

The vibration of a beam occurs due to internal and exterior forces, including external forces like wind, earthquakes, and machinery, depending on the beam's natural frequency. The inhabitants may be in danger if the structure is worn down or fails due to these vibrations. Controlling and reducing this vibrational energy in structural systems like beam necessitates using damping mechanisms. Besides, the damping phenomena also arise in various physical systems, such as viscous drag in mechanical systems, resistance in electrical oscillators, light absorption and scattering in optical oscillators, etc. In dynamic systems, damping mechanisms dissipate mechanical energy, usually into heat or sound. The nature of the damping mechanisms drastically changes the nature of the solution to the vibration problem and hence controls the response of the beam. The classical Euler-Bernoulli beam model with appropriate damping mechanisms such as viscous (air) damping, strain rate damping, spatial hysteresis, and time hysteresis play a significant role in applications (see, [6]). A general model of the damped beam equation with generic damping is given by

$$u_{tt}(x,t) + L_1 u_t(x,t) + L_2 u(x,t) + ((EI(x)/\rho)u_{xx})_{xx} = q(x,t),$$

where the term $L_1 u_t(x,t) + L_2 u(x,t)$ accounts for the damping mechanisms of this model. External damping mechanisms usually determine the nature of coefficient L_1 while the internal damping mechanisms often determine the coefficient L_2. In this thesis, we mainly focused on three types of dampings, namely, viscous damping, internal damping as Kelvin-Voigt damping, and structural damping in thermoelastic plate model (see, [6, 53, 83]).

Next, we briefly discuss the three damping mechanisms used in our study. Vibration analysis most frequently uses *viscous damping* as a damping method. When mechanical systems vibrate in a fluid medium like air, gas, water, or oil, the fluid's resistance to the moving body results in energy loss. The vibrating body's size and shape, the fluid's viscosity, the vibration frequency, and the velocity of the vibrating body are just a few of the variables that affect how much energy is lost in this situation. In viscous air damping, it is assumed that the damping force is proportional to the velocity of the vibrating body (see, [6, 78]), and hence $L_1 = \mu I_0$, where $\mu > 0$ is the viscous damping constant of proportionality. The term *Kelvin-Voigt damping* refers to the damping of the form $L_2 = c_d \, I \frac{\partial^5}{\partial x^4 \partial t}$, where I is the moment of inertia, and c_d is the strain-rate damping coefficient. This type of damping represents the energy dissipation due to internal friction in the beam. It is also commonly employed in finite element modeling and compatible with theoretical modal analysis. Due to its strain-rate dependence, unlike viscous external damping, this type of damping impacts the free-end boundary conditions (see, [6]). *Structural damping* is a consequence of mechanical-energy dissipation due to rubbing friction resulting from a relative motion between components and intermittent contact at the joints in a mechanical structure. A large portion of mechanical-energy dissipation in tall buildings, bridges, and many other civil engineering structures occurs through the structural-damping mechanism (see, [83]). From a mathematical point of view, structural damping is given by $-\mathrm{w}\frac{\partial^3}{\partial x^2 \partial t}$ has half of the order of the Kelvin-Voigt damping term, which has the same order as the leading elastic term. Further, structural damping describes a scenario where higher-order frequencies are more strongly damped than lower frequencies (see, [25]).

1.3 Methods for solving inverse problems

As in the case of the solvability of direct problems for Partial Differential Equations (PDEs), several methods (see, [7, 11, 24, 30, 54, 62, 90, 94]) are also used in the literature for inverse problems for PDEs. Since each method has its strengths and weaknesses, the choice of a method depends on the nature of the problem at hand, the quality of the available data, and the available computational resources. In this thesis, we mainly focused on two methods, namely, SVD and Tikhonov regularization.

1.3.1 Singular value decomposition method

Singular Value Decomposition (SVD) is a robust method for solving linear inverse problems. It is possible to define the SVD for a large class of linear operators, which offers a deeper knowledge of the operator's underlying structure and can result in more precise and reliable solutions to inverse problems. We briefly explain the method as follows (see, [50, 58]): Let $A : H \mapsto \tilde{H}$ be a linear compact operator from infinite-dimensional Hilbert space H into $\tilde{H}$. If $\tilde{H} = H$ and A is a self-adjoint compact operator, that is for all $u \in H$ and $v \in H$, $(Au, v) = (u, Av)$, then we may use the spectral representation

$$Au = \sum_{n=1}^{\infty} \lambda_n (u, u_n) u_n, \;\; \forall u \in H,$$

where $\lambda_n, \; n = 1, 2, 3, ...$, are nonzero real eigenvalues and $\{u_n\} \subset H$ is the complete set of corresponding orthonormal eigenvector. Then the set consisting of all pairs of nonzero eigenvalues and corresponding eigenvectors is defined as an *eigensystem* of the self-adjoint operator A.

If A is not self-adjoint, we introduce a singular system of the non-self-adjoint operator A with adjoint $A^* : \tilde{H} \mapsto H$. To describe this system we use the operators A^*A and AA^*, where $A^*A : H \mapsto H$ and $AA^* : \tilde{H} \mapsto \tilde{H}$ are compact self adjoint nonnegative operators. Let $\{\mu_n, u_n\}$ denote the eigensystem of the operator A^*A. Then $A^*Au_n = \mu_n u_n$, for all $u_n \in H$, which implies $(A^*Au_n, u_n) = \mu_n(u_n, u_n)$. Hence $\|Au_n\|^2_{\tilde{H}} = \mu_n \|u_n\|^2_H \geq 0$, that is, all the nonzero eigenvalues are positive: $\mu_n > 0,\ n \in N$, where $N := \{n \in \mathbb{N} : \mu_n \neq 0\}$.

Definition 1.1. Let $A : H \mapsto \tilde{H}$ be a linear compact operator with adjoint $A^* : \tilde{H} \mapsto H$, where H and $\tilde{H}$ are Hilbert spaces. The square root $\sigma_n := \sqrt{\mu_n}$ of the eigenvalue $\mu_n > 0$ of the self adjoint operator $A^*A : H \mapsto H$ is called the singular value of the operator A and $\{\sigma_n, u_n, v_n = \frac{Au_n}{\|Au_n\|}\}$ is the corresponding *singular system* of the operator.

If A is self-adjoint, then the singular system is $\{|\lambda_n|, u_n, \frac{\lambda_n u_n}{|\lambda_n|}\}$. Using this singular system, we can discuss the solution to the operator equation $Au = f$. In general, for a linear compact operator A, the equation $Au = f$ has a solution if and only if $f \in N(A^*)^\perp$ and $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} |(f, v_n)|^2 < \infty$, where $N(A^*)$ is the null space of A^*. In this case

$$u := A^\dagger f = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} (f, v_n) u_n$$

is the solution of the equation $Au = f$, where $A^\dagger$ is the generalized inverse of A (see, chapter 2, [50]). One can refer to [7, 16] for the spectral methods of solving inverse problems of beam equation.

1.3.2 Tikhonov regularization method

Tikhonov regularization is a powerful method for solving inverse problems with noisy and incomplete data. The regularization strategy works by adding a penalty term to the objective function, which controls the complexity of the solution, and the impact of noise and errors in the observed data is minimized by this penalty component.

Consider the linear injective bounded operator $A : H \mapsto \tilde{H}$, where H and $\tilde{H}$ are infinite-dimensional Hilbert spaces and

$$Au = f,\ u \in H,\ f \in R(A). \tag{1.1}$$

Since the data $f \in R(A)$ always contain random noise, the operator equation (1.1) is ill possed because the solution $u \in H$ does not depend continuously on the data. Let $f^{\delta_1} \in \tilde{H}$ be the noisy data and assume that $\|f^{\delta_1} - f\|_{\tilde{H}} \leq \delta_1,\ f \in\ R(A),\ f^{\delta_1} \in \tilde{H},\ \delta_1 > 0$. Then the exact equality in the equation $Au = f^{\delta_1}$ may not be satisfied due to the noisy data f^{δ_1}. We consider the minimization problem defined as follows:

$$\mathcal{J}(u) = \inf_{v \in H} \mathcal{J}(v), \tag{1.2}$$

where $\mathcal{J}(u)$ is the Tikhonov functional

$$\mathcal{J}(u) = \frac{1}{2}\|Au - f^{\delta_1}\|^2_{\tilde{H}},\ u \in H,\ \ f^{\delta_1} \in \tilde{H}. \tag{1.3}$$

A solution $u \in H$ of the minimization problem (1.2)-(1.3) is called as quasisolution or least square solution of (1.1). Since A is compact, $\mathcal{J}(u)$ doesn't depend continuously on the data $f^{\delta_1} \in R(A)$.

In order to stabilize the functional (1.3), we add a penalty term $\alpha\|u\|^2_H$, and consider the minimization problem for the regularized Tikhonov functional

$$\mathcal{J}_\alpha(u) := \frac{1}{2}\|Au - f^{\delta_1}\|^2_{\tilde{H}} + \frac{\alpha}{2}\|u\|^2_H,\ \ u \in H,\ f^{\delta_1} \in \tilde{H},$$

where $\alpha > 0$ is the parameter of regularization. This procedure is the so-called *Tikhonov regularization method* (see, [90]), and under the conditions of the operator A described above, the regularized Tikhonov functional $\mathcal{J}_\alpha$ has a unique minimum $u_\alpha^{\delta_1} \in H$, for all $\alpha > 0$. The minimum is the solution to the linear equation

$$(A^*A + \alpha I)u_\alpha^{\delta_1} = A^* f^{\delta_1},\ \ u_\alpha^{\delta_1} \in H,\ \ f^{\delta_1} \in \tilde{H},\ \alpha > 0$$

and has the form

$$u_\alpha^{\delta_1} = (A^*A + \alpha I)^{-1} A^* f^{\delta_1}. \tag{1.4}$$

Moreover, the operator $A^*A + \alpha I$ is boundedly invertible. Hence the solution $u_\alpha^{\delta_1}$ continuously depends on f^{δ_1} (see, [29, 50]).

It is worth noting that the presented formula (1.4) for the minimum of the regularized Tikhonov functional is valid for the case when the regularization term is taken in L^2 norm. Further, the uniqueness of the minimum generally depends on the underlying direct problem, the nature of the inverse problem, and the measured data.

1.4 Motivation and objectives

The classical beam and plate equations are fundamental for modeling of deformation of thin and flexible structures. Besides, recent advances in nanotechnology include nanobeams and nanoplates, which have found applications in new medical diagnostics and nanoscale measurement systems, such as Atomic Force Microscopy (AFM) and Transverse Dynamic Force Microscopy (TDFM) (see, [3, 17]). Since beams and plates are crucial models, designing various mechanical structures in engineering and reducing damages in these structures necessitate the study of the mechanical properties of beam and plate. Hence, the direct and inverse problems for the Euler-Bernoulli beam and plate equations have been extensively studied over the decades, and they continue to be an active area of research in science and engineering (see, [2, 6, 31, 33, 43, 52, 60, 68, 70, 88, 82]).

One of the reasons motivating our study is the non-uniqueness of the final time data inverse source problem for the undamped wave equation. It is shown in [50] that *for unique determination of the unknown source* $F(x)$ in the undamped wave equation

$$\begin{cases} u_{tt} = u_{xx} + F(x),\ (x,t) \in (0,\ell) \times (0,T), \\ u(x,0) = u_t(x,0) = 0,\ x \in (0,\ell), \\ u(0,t) = u(\ell,t) = 0,\ t \in [0,T], \end{cases} \tag{1.5}$$

from the final time data $u_T(x) := u(x,T),\ x \in (0,\ell)$, the final time must satisfy the following condition

$$T \neq \frac{2m}{n}, \text{ for all } m,n = 1,2,3, \dots . \tag{1.6}$$

Otherwise, that is, when $T = 2m/n$, an infinite number of singular values κ_n defined as (see formula (3.2.12) in [50])

$$\kappa_n = \frac{1}{\lambda_n}\left[1 - \cos\left(\sqrt{\lambda_n}\,T\right)\right],\ \text{for all } n = 1,2,3, \dots,\ \lambda_n = n^2\pi^2$$

in the singular value decomposition

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{\kappa_n} u_{T,n}\psi_n(x),\ x \in (0,\ell)$$

vanish, where $\{\lambda_n, \psi_n(x)\}_{n=1}^{\infty}$ is the eigensystem of the operator $-u''(x)$ subject to the boundary conditions given in (1.5). As a consequence, the Picard criterion

$$\sum_{n=1}^{\infty} \frac{u_{T,n}^2}{\kappa_n^2} < \infty$$

is not satisfied since this condition implicitly assumes the requirement $\kappa_n \neq 0,$ for all $n = 1, 2, 3, \ldots$. Therefore, if $\kappa_m = 0$ for some $m = 1, 2, 3, \ldots$, then the mth Fourier coefficient $F_m := (F, \psi_m)_{L^2(0,\ell)}$ of the unknown function $F(x)$ can not be determined uniquely.

From condition (1.6), it follows that the final time $T > 0$ can not be a rational number. In practice, fulfilling this necessary condition for the unique determination of $F(x)$ is impossible. For this reason, the above final data inverse problems for the undamped wave equation were defined as infeasible in [50]. The same conclusion for the inverse problem (1.5) with the final state over-determination $u_T(x)$ replaced by the final velocity over-determination $\nu_T(x) := u_t(x, T)$ holds true. Furthermore, a similar scenario occurs for the inverse problem of identifying the unknown source in an undamped beam equation from final time data.

The damping terms added to the basic governing equations of beam or plate models not only lead to the loss of energy to the system but also provides a certain smoothness effect for the direct problems of the model (see, [86]). This smoothing effect, in turn, further helps us to analyze the uniqueness and stability of solutions to the inverse source problems associated with the damped vibration models. In fact, the damping terms play a similar role to the regularization parameters used in the classical Tikhonov functional ([90]) to get a unique solution for some linear inverse problems by the quasi-solution method. The importance of damping phenomena in direct and inverse problems leads us to contemplate the effect of different types of dampings, like viscous damping, Kelvin-Voigt damping, and structural damping in vibration models. By considering this motivating factor, we investigate a unique reconstruction of spatial load in the Euler-Bernoulli beam and the stability of reconstructing the spatial load in the Kirchoff-Love plate equation in the presence of viscous damping. Moreover, we also investigate the role of internal damping $(\kappa(x)u_{xxt})_{xx}$ (Kelvin-Voigt damping) in the inverse problem of determining the unknown transverse shear force in the Euler-Bernoulli beam. This inverse boundary value problem of determining the shear force acting at the inaccessible tip of the microcantilever in TDFM is important for understanding the specimen images and mechanical properties at a submolecular precision ([75]).

It is known that the temperature gradients in a plate will contribute to plate deformation and can cause changes in stiffness and vibration frequencies, and even buckling. We study the thermoelastic plate, a coupled system consisting of heat, and the Kirchoff-Love plate equations ([65]). As this advanced physical and engineering science model is relevant in many real-life applications, the thermoelastic behavior of structures made of advanced composite material needs to be thoroughly investigated. The coupled effect between deformation and temperature has been a critical factor in thermal shock problems as well (see, [14, 76, 92]). Further, in this model, damping is an essential consideration in the design of plates as it helps to improve the structural integrity and performance and can help prevent damage or failure due to excessive vibrations. The influence of different damping mechanisms in thermoelastic plate equation in the analysis of qualitative properties of the solution has been extensively studied, see for example, the plate equation with frictional damping u_t, structural damping $-\Delta u_t$, Kelvin-Voigt type damping or viscoelastic damping $\Delta^2 u_t$ (see, [18] and references therein). All these factors inspire us to finally study the inverse problem of simultaneously reconstructing the spatial load and heat source in the thermoelastic system with structural damping.

1.5 Contributions of the thesis

This section mainly focuses on the previous studies on the inverse problems of Euler-Bernoulli beam and plate equations and our contributions to these models.

1.5.1 Inverse problems in Euler-Bernoulli beam

The inverse problems of the Euler-Bernoulli beam have been well-studied over the decades. Let us recall some of the literature, starting, for instance, from ([33]), where the author determines the cross-section and moment of inertia from spectral data. The uniqueness study of determining flexural rigidity of the classical steady-state Euler-Bernoulli beam equation was discussed in ([67]). The paper ([16]) studied the identification of spatial density $\rho(x)$ and inertia $r(x)$ from the boundary data. For the simplest Euler-Bernoulli beam equation: $m(x)u_{tt} + (EI(x)u_{xx})_{xx} = f(x,t)$ with $(x,t) \in (0,l) \times (0,T)$, the author determined the unknown spatial load from the measured output data $u(x,T)$ or $u_t(x,T)$ by using the least square and adjoint problem approach in [43]. The same theory was applied in the paper [45], to identify the unknown spatial and temporal load from the measured slope $u_x(0,t)$, and also developed the numerical algorithm to reconstruct the unknown sources. In the paper [48], two inverse source problems of identifying asynchronously distributed spatial loads governed by the Euler-Bernoulli beam equation $\rho(x)u_{tt}+\mu(x)u_t+$

$(k(x)u_{xx})_{xx} - T_r u_{xx} = \sum_{m=1}^{M} h_m(t) f_m(x)$ with hinged-clamped ends were investigated. The first inverse problem is to find $(f_1, f_2, ..., f_m)$ from the measured deflection and second is to find $(f_1, f_2, ..., f_m)$ from measured slope. For further results on the inverse source problems of the Euler-Bernoulli beam and plate equations, one may refer to [37], [60], [74]. Apart from the solvability of inverse problems on the Euler-Bernoulli beam, for the solvability of the direct problem with different boundary and initial conditions, one can refer to (see, [9], [64]).

Next, let us review some recent papers on the Euler-Bernoulli equation with internal/external damping mechanism. In the paper [47], the authors determine the unknown transverse shear force by using measured boundary deflection $u(\ell, t)$, and in the article [46], they consider the same inverse problem based on measured bending moment $-r(0)u_{xx}(0, t)$. In addition to this literature, there are some classic papers on the inverse problem of the Euler-Bernoulli beam equation with the Kelvin-Voigt damping or viscous damping. In [36] and [5], the parameter identification of the Euler-Bernoulli beam equation with structural or viscous damping was investigated, and the numerical approximations of those quantities were studied. The paper [55] determined the stiffness $EI(x)$, damping coefficient $DI(x)$, and initial data of the Euler-Bernoulli beam equation $u_{tt} + (EI(x)u_{xx} + DI(x)u_{xxt})_{xx} = f(x, t),\ t > 0$ using the spectral data of the model problem.

However, the above studies dealt with something other than the investigation of the role of various damping mechanisms in the inverse problems of the Euler-Bernoulli beam. Our first work focused on analyzing the viscous damping in the unique determination of unknown spatial load in a damped, simply supported, non-homogeneous Euler-Bernoulli beam from the measured final time displacement or velocity. We considered two cases, namely, pure spatial load and exponentially decaying temporal load, and derived the singular value expansion for the unknown spatial load explicitly for both temporal loads under some feasible conditions on damping coefficient and final time T. Furthermore, this study provides a method to determine the permissible and optimal final time interval for measuring the final time output.

Then, we study the inverse problems of determining the unknown transverse shear force in a system governed by the damped Euler-Bernoulli beam with more general physical coefficients and Kelvin-Voigt damping from the measured deflection at the right end of the beam and the measured bending moment at the left end of the beam. The main purpose of this work is to analyze the Kelvin-Voigt damping effect on determining the unknown transverse shear force (boundary input) through the given boundary measurements. The considered inverse problems are transformed into minimization problems for Tikhonov functionals, showing that the regularized functionals admit a unique solution for the inverse problems. By suitable regularity on the admissible class of shear force, we prove that these functionals are Fréchet differentiable. The derivatives are expressed through the solutions of corresponding adjoint problems posed with measured data as boundary data associated with the direct problem. The solvability of these adjoint problems is obtained under the minimum regularity of the boundary data, which turns out to be the regularizing effect of the Kelvin-Voigt damping in the direct problem. Furthermore, using the Fréchet derivative of the more regularized Tikhonov functionals, we obtain remarkable Lipschitz stability estimates for the transverse shear force in terms of the given measurement by a feasible condition only on the Kelvin-Voigt damping coefficient.

1.5.2 Inverse problems in Kirchhoff-Love plate

As explained in the previous section, coefficient or source identification problems for the Euler-Bernoulli beam equation have been fairly done to a great extent, whereas for the Kirchhoff-Love plate equation, a very limited number of studies have only been done so far. The determination of Young's modulus E and Poisson ratio ν of the classical plate equation from the Dirichlet to Neumann map was studied in ([52]). The dynamic plate equation in simple form $u_{tt} + a^2\Delta^2 u = g$ has been considered with Dirichlet and normal boundary conditions $u = 0$ and $\frac{\partial^2 u}{\partial n^2} = 0$ respectively in ([39]). The unique determination of source function $f(x)$, over the separable force $g(x,t) = f(x)R(t)$ from either interior measurement or boundary measurements was discussed using the fundamental solution method combined with the Tikhonov regularization technique. The paper ([71]) considered the determination of the bending stiffness $D(x, y)$ in the steady-state Kirchhoff-Love equation $\Delta(D\Delta u) = q$ in a unit square, where $q(x, y)$ is the distributed transverse load applied to the plate, using the technique called the method of variational embedding, where the original problem is transferred into a minimization problem. A numerical algorithm for solving this problem was also developed in this paper. In the paper [2], the problem of determining unknown source in plate equation with boundary conditions $u = 0$ and $\Delta u = 0$ was considered to study the general framework of allowing to use the exact observability of infinite dimensional systems to solve a class of inverse source problems.

Unlike the works mentioned above, we discuss the unique determination of an unknown spatial load in the non-homogeneous isotropic simply supported rectangular Kirchhoff plate equation with viscous damping from final time measured deflection. Under some acceptable conditions on the coefficients, we proved the well-posedness of the direct problem. The inverse problem is posed as a minimization problem of the regularized Tikhonov functional. Since this inverse problem is linear, by using the classical calculus of variations methods, we have obtained the existence of a unique minimizer to the regularized functional, which indeed gives the solution to the inverse problem. The Lipschitz continuity of the Fréchet gradient of the Tikhonov functional is obtained in terms of direct and adjoint problems. The Lipschitz constant is useful for deriving a gradient-type algorithm for the inverse problem. We have obtained an upper limit for the final time and a lower bound for the damping coefficient, which leads to the stability estimates for the source term in terms of measured data.

We also studied the inverse source problem by the SVD method, and the series representation of a unique minimum of regularized Tikhonov functional is established. Using the representation formula for the regularized solution, we observed that the solution obtained by these two widely used methods, Tikhonov regularization and SVD, are equivalent. Finally, with the help of singular values of the input-output operator and regularity assumption on the temporal load, we derived stability estimates for the regularized and SVD solutions of the inverse problem. These results clearly express that a very small value of the regularization parameter magnifies the error between the measured outputs.

1.5.3 Inverse source problem in thermoelastic plate

In the previous sections, we discussed the literature on inverse source and inverse boundary value problems on the Euler-Bernoulli beam and plate equations. Next, we briefly discuss the literature on inverse problems of classical thermoelastic systems. The identification of two coefficients in a coupled system of the hyperbolic equation for displacement, and the heat equation for temperature from measured displacement in a subdomain along a sufficiently large time interval was discussed in [100]. By the Carleman estimate method, the authors discussed the Lipschitz stability estimate for the solution to the inverse problem. The inverse source problem of a generalized thermoelastic system was analyzed by the method of Carleman estimates in [13]. In [99], the authors studied the determination of spatially varying unknown source term in a thermoelastic system with memory from measured displacement in a subdomain along a sufficiently large time interval. They provide the Hölder stability estimate by the method of the Carlemen estimate. The inverse problem of space-dependent vector source in a thermoelastic system from measured final time deflection was studied in [96]. They proved the uniqueness of the solution to the direct problem by variational approach and provided a numerical reconstruction of the solution to the inverse problem.

However, to our knowledge, the inverse problem of simultaneously identifying the mechanical load and heat source in structurally damped thermoelastic plate equations describing a homogeneous and elastically as well as thermally isotropic plate from the vertical displacement measured at the final time has not been studied. By Galerkin's approximation method, we establish the well-posedness of the thermoelastic plate equation and the corresponding adjoint problem. Unlike the single plate equation, the coupled effect between the plate and heat equations demands specific methods for the solvability of this system in both forward and backward in time. The inverse problem is transformed into a minimization problem for Tikhonov functional using the Tikhonov regularization method. We prove the regularized Tikhonov functional admits a unique solution for the inverse problem. We prove that this functional is Fréchet differentiable, and the gradient is written in terms of the adjoint problem associated with the thermoelastic plate equation. We establish an upper bound for the final time to derive the stability estimate for the source terms by invoking a first-order necessary optimality condition of the minimization problem. This stability result also gives the uniqueness of the solution to the inverse problem. These findings in this paper help to analyze the effect of thermal and mechanical load that results in material deflection, which is vital from the perspective of physical applications.

1.6 Preliminaries

This section briefly lists standard function spaces, inequalities, and embedding theorems used throughout the thesis. For more details, we refer to [27].

1.6.1 Function spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set and $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω.

Definition 1.2. Let $u, v \in L^1_{\text{loc}}(\Omega)$, and r is a multiindex. We say that v is the weak derivative of u denoted by $v = D^{\text{r}}u$, if

$$\int_\Omega u D^{\text{r}}\varphi dx = (-1)^{|\text{r}|}\int_\Omega v\varphi dx, \text{ for all test functions } \varphi \in C_c^\infty(\Omega).$$

Fix $1 \leq p \leq \infty$, and let m be a non-negative integer. We define certain function spaces on the domain $\Omega \subset \mathbb{R}^n$ as follows.

Definition 1.3. The function space which consists of all locally summable functions $u : \Omega \mapsto \mathbb{R}$ such that for each multiindex r with $|\mathrm{r}| \leq m$, $D^{\mathrm{r}}u$ exists in the weak sense and belongs to $L^p(\Omega)$ is called the Sobolev space and it is denoted by $W^{m,p}(\Omega)$. If $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega), \;\; m = 0, 1, 2, ...$

Definition 1.4. For any $u \in W^{m,p}(\Omega)$, norm of u is defined as follows:

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\mathrm{r}| \leq m} \int_\Omega |D^{\mathrm{r}}u|^p dx \right)^{1/p}, \;\; 1 \leq p < \infty, \tag{1.7}$$

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\mathrm{r}| \leq m} \text{ess } \sup_\Omega |D^{\mathrm{r}}u|, \qquad p = \infty. \tag{1.8}$$

Definition 1.5. The set of all real valued locally summable functions $u \in H^1(\Omega)$ with zero trace on the boundary Γ is denoted by $H_0^1(\Omega)$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ with norm

$$\|u\|_{H^{-1}(\Omega)} = \sup\{< u, v >: v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq 1\}.$$

The time dependent function space $L^2(0, T; \; H^m(\Omega))$ consists of all measurable functions $u : [0, T] \mapsto H^m(\Omega)$ such that

$$\|u\|_{L^2(0,T;H^m(\Omega))} = \left(\int_0^T \|u(t)\|^2_{H^m(\Omega)} dt \right)^{\frac{1}{2}} < \infty,$$

and $C([0, T]; \; H^m(\Omega))$ consists of all continuous functions $u : [0, T] \mapsto H^m(\Omega)$ such that

$$\|u\|_{C([0,T];H^m(\Omega))} = \max_{t \in [0,T]} \|u(t)\|_{H^m(\Omega)} < \infty.$$

Next, we introduce some notations involving function spaces used in the subsequent chapters. For any $l > 0$, we introduce the function spaces

$$\mathcal{V}(0, \ell) = \{v \in H^2(0, \ell) : \; v(0) = v(\ell) = 0\}, \tag{1.9}$$

$$\mathcal{V}_1^2(0, \ell) = \{v \in H^2(0, \ell) : v(0) = v_x(0) = 0\}, \tag{1.10}$$

with standard Sobolev space norm

$$\|v\|_{\mathcal{V}_1^2(0,\ell)} := \Big(\int_0^\ell \left(v^2 + v_x^2 + v_{xx}^2\right) dx \Big)^{\frac{1}{2}},$$

while $\mathcal{V}_1^2(0,\ell)'$ denotes the dual space of $\mathcal{V}_1^2(0,\ell)$. The space $\mathcal{V}(0,\ell)$ will also be endowed with the same norm.

Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ. The function space $\mathcal{V}_1^2(\Omega)$ is given by

$$\mathcal{V}_1^2(\Omega) = \{v \in H^2(\Omega) : v(x) = 0, \frac{\partial v(x)}{\partial \mathrm{n}} = 0,\ x \in \Gamma\}, \tag{1.11}$$

with the standard Sobolev norm (1.7) corresponding to $H^2(\Omega)$, and $\mathcal{V}_1^2(\Omega)'$ is the dual space. If we consider the domain Ω as a rectangular domain in $\mathbb{R}^2$, specifically,

$$\begin{aligned}
&\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \in (0,\ell_1),\ x_2 \in (0,\ell_2)\},\ \ell_1, \ell_2 > 0,\\
&\Gamma := \overline{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4},\\
&\Gamma_1 := \{(x_1, 0) \,:\, x_1 \in (0,\ell_1)\},\ \Gamma_2 := \{(\ell_1, x_2) \,:\, x_2 \in (0,\ell_2)\},\\
&\Gamma_3 := \{(x_1, \ell_2) \,:\, x_1 \in (0,\ell_1)\},\ \Gamma_4 := \{(0, x_2) \,:\, x_2 \in (0,\ell_2)\},
\end{aligned} \tag{1.12}$$

then the function space $\mathcal{V}^2(\Omega)$ is introduced by

$$\mathcal{V}^2(\Omega) = \{v \in H^2(\Omega) : v(x) = 0,\ x \in \Gamma\}, \tag{1.13}$$

with norm defined through (1.7) and the dual space is $\mathcal{V}^2(\Omega)'$.

Remark 1.1. We use the following convention for the space and time dependent fuction $u(x,t)$. Assume that $t \in [0,T]$ and for every t, the function $u(.,t)$ belongs to a Hilbert space $\mathcal{V}^2(\Omega)$, that is $u : [0,T] \mapsto \mathcal{V}^2(\Omega)$. By this convention, we will write $u(t)$ instead of $u(x,t)$.

1.6.2 Basic inequalities

Following is a list of fundamental inequalities (see, [27], Appendix B), continually employed throughout the work.

Cauchy's inequality: For $a, b \in \mathbb{R}$,

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}. \tag{1.14}$$

Cauchy's inequality with ε: For any $a, b > 0$ and $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}. \tag{1.15}$$

Hölder's inequality: Suppose $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we get

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Cauchy-Schwarz inequality: For $x, y \in \mathbb{R}^n$, we have

$$|x \cdot y| \leq |x||y|.$$

Gronwall's inequality: Let ξ, G be continuous functions on $[0, T]$, with G nondecreasing and $C_1 > 0$. If $\xi(t) \leq G(t) + C_1 \int_0^t \xi(s) ds, \text{ for all } t \in [0, T]$, then

$$\xi(t) \leq G(t) \exp(C_1 t), \text{ for all } t \in [0, T].$$

In particular, if $\xi(t) \leq C_1 \int_0^t \xi(s) ds, \text{ for all } t \in [0, T]$, then $\xi(t) = 0$. (see, [87], section 10.3.2).

1.6.3 Embedding theorems

In this subsection, we list the important embedding theorems used in the thesis.

Definition 1.6. Let X and Y be Banach spaces, $X \subset Y$. The space X is said to be compactly embedded in Y, that is, $X \subset\subset Y$, if

1. $\|f\|_Y \leq C\|f\|_X, \ (f \in X), \text{ for some constant C}$, and
2. each bounded sequence in X is precompact in Y, (see, [27], section 5.7).

Theorem 1.1. *(Rellich-Kondrachov Compactness Theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with C^1 boundary Γ. Suppose $1 \leq p < n$, then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega),$$

for each $1 \leq q < p^*$, $p^* := \frac{np}{n-p}$, (see, [27], section 5.7, Theorem 1).

Theorem 1.2. *(Poincaré's inequality) Let Ω be a bounded Lipschitz domain. Then for any $u \in H_0^1(\Omega)$, there exists a Poincaré constant C_p such that*

$$\|u\|_{L^2(\Omega)} \leq C_p \|\nabla u\|_{L^2(\Omega)}.$$

(see, [87], section 7.10.2).

For any $v \in \mathcal{V}_1^2(0,\ell)$, we obtain the Poincaré inequalities, $\|v\|_{L^2(0,\ell)} \leq 2\ell\|v_x\|_{L^2(0,\ell)} \leq 4\ell^2\|v_{xx}\|_{L^2(0,\ell)}$. Thus, we have

$$\|v\|_{\mathcal{V}_1^2(0,\ell)} \leq \sqrt{C^*}\|v_{xx}\|_{L^2(0,\ell)},\ C^* = 4\ell^2(1+4\ell^2)+1. \tag{1.16}$$

It is clear that the norm $\|v\|_{\mathcal{V}_1^2(0,\ell)}$ is equivalent to $\|v_{xx}\|_{L^2(0,\ell)}$.

Applying classical regularity results for elliptic PDE given by $\Delta v = \Delta u$ in $\Omega,\ v = 0$ on $\partial\Omega$ (see, [35], Corollary 8.7 and Theorem 8.12), we get that the norms $\|u\|_{\mathcal{V}^2(\Omega)}$ and $\|\Delta u\|_{L^2(\Omega)}$ are equivalent in $\mathcal{V}^2(\Omega)$. This implies that

$$\|u\|_{\mathcal{V}^2(\Omega)} \leq \sqrt{C'}\|\Delta u\|_{L^2(\Omega)}, \tag{1.17}$$

where C' is a positive constant (see, also [38], Theorem 1). This result also holds for the norm defined in the space $\mathcal{V}_1^2(\Omega)$.

Theorem 1.3. *Let $I = (a,b) \subset \mathbb{R}$, then the elements of $H^1(a,b)$ are continuous in $[a,b]$. Furthermore,*

$$\|v\|_{L^\infty(a,b)} \leq C_1^*\|v\|_{H^1(a,b)},$$

where $C_1^* = \sqrt{2}\max\{(b-a)^{-\frac{1}{2}},(b-a)^{\frac{1}{2}}\}$, (see, [87], Section 7.10.4).

Theorem 1.4. *Let $I = (a,b) \subset \mathbb{R}$ be an open interval. For every integer j, $1 \leq j \leq m-1$, and for every $\epsilon > 0$, there exists a constant $C(\epsilon,|I|)$ such that (see, [15], page 217)*

$$\|D^j u\|_{L^p(I)} \leq \epsilon\|D^m u\|_{L^p(I)} + C\|u\|_{L^p(I)},\ \textit{for all}\ u \in W^{m,p}(I).$$

Theorem 1.5. *(Ehrling's lemma) Suppose X, Y and Z be Banach spaces, and X is compactly embedded in Y, Y is continuously embedded in Z, that is, $X \subset\subset Y \subset Z$. Then for every $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that (see, [80], Theorem 7.30)*

$$\|f\|_Y \leq \epsilon\|f\|_X + C(\epsilon)\|f\|_Z,\ \textit{for every}\ f \in X.$$

The following two theorems play a crucial role in the verification of the initial conditions.

Theorem 1.6. *Let $u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;H^{-1}(\Omega))$, then $u \in C([0,T];L^2(\Omega))$, (see, [27], section 5.9, Theorem 3).*

Theorem 1.7. *Assume that Ω is an open bounded domain with smooth boundary Γ. Let m be a nonnegative integer. Suppose $u \in L^2(0,T;H^{m+2}(\Omega))$ with $u' \in L^2(0,T;H^m(\Omega))$. Then, we have (see, [27], section 5.9, Theorem 4)*

$$u \in C([0,T];H^{m+1}(\Omega)).$$

Finally, we state theorems on the existence and uniqueness of solutions to abstract minimization problems.

Theorem 1.8. *(Generalized Weierstrass existence theorem) Suppose that the functional $\mathcal{J} : M \mapsto \mathbb{R}$ satisfies the following properties:*

1. *M is a nonempty closed convex subset of the real Hilbert space X.*
2. *$\mathcal{J}$ is weakly sequentially lower semicontinuous.*
3. *If the set M is unbounded, then $\mathcal{J}$ is weakly coercive.*

Then there exists a minimum u in M such that $\mathcal{J}(u) = \min_{\tilde{u}\in M} \mathcal{J}(\tilde{u})$. Further, if $\mathcal{J}$ is strictly convex, then the minimization problem has a unique solution, (see, [101], Theorem 2.D).

Theorem 1.9. *Let V be a nonempty, closed, convex subset of the real reflexive Banach space $\mathcal{B}$. Assume that the functional $\mathcal{J} : V \subset \mathcal{B} \mapsto \mathbb{R}$ is continuous and convex. Then, the minimization problem*

$$V_* := \{v \in V \ : \ \mathcal{J}(v) = \mathcal{J}_* := \inf_{u\in V} \mathcal{J}(u)\} \tag{1.18}$$

has a solution. Furthermore, if the Găteaux derivative $\mathcal{J}'(v)$ exists for all $v \in V$, then the minimization problem (1.18) is equivalent to the following variational inequality:

$$\langle \mathcal{J}'(v_*), v - v_* \rangle \geq 0, \ \textit{for all} \ \ v \in V. \tag{1.19}$$

(see, [101], Theorem 2.E or [97]).

Chapter 2

Determination of a spatial load in the Euler-Bernoulli beam with viscous damping

2.1 Introduction

This chapter studies the Inverse Source Problem (ISP) for the Euler-Bernoulli beam with viscous external damping. More precisely, consider the problem of determining the spatial load $F(x)$ in the simply supported damped Euler-Bernoulli beam

$$\begin{cases} u_{tt} + \mu u_t + (r(x)u_{xx})_{xx} = F(x)G(t),\ (x,t) \in \Omega_T, \\ u(x,0) = u_t(x,0) = 0,\ x \in (0,\ell), \\ u(0,t) = u_{xx}(0,t) = 0,\ u(\ell,t) = u_{xx}(\ell,t) = 0,\ t \in [0,T], \end{cases} \tag{2.1}$$

from the final time measured displacement

$$u_T(x) := u(x,T),\ x \in (0,\ell), \tag{2.2}$$

or the final time measured velocity $\nu_T(x) := u_t(x,T),\ x \in (0,\ell)$,

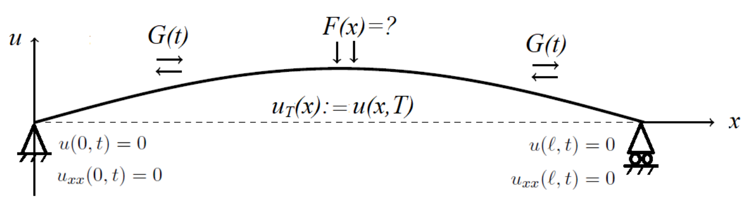

Figure 2.1: Geometry of ISP

where $\Omega_T = \{(x,t) \in \mathbb{R}^2 : 0 < x < \ell,\ 0 < t < T\}$. The coefficients $\mu > 0$ and $r(x) > 0$ denote the viscous damping coefficient and spatially varying flexural stiffness of the beam, respectively. The source functions $F(x) \not\equiv 0$ corresponds to spatial load, and $G(t) > 0$ denotes the temporal load. The main focus of this chapter is the term μu_t and its role in the uniqueness of solutions to the inverse problem.

In the subsections, 1.4 and 1.5, we discussed the motivation behind this study and previous studies on the inverse problems of the Euler-Bernoulli beam equation, respectively, but none of these studies explore the importance of damping mechanism in the unique determination of unknown source term. However, recently, in [95], a set of sufficient conditions on the temporal load $G(t)$ and coefficients are established for the unique determination of spatial load $F(x)$ in the source term of the form $F(x)G(t)$ in a damped Kirchhoff-Love plate equation from final time measurements using a variational approach. For the study of the uniqueness of the inverse source problem for the damped Euler-Bernoulli beam equation, we are invoking the spectral method rather than the variational approach. Using the eigensystem of the Euler-Bernoulli operator, we computed the singular system of the input-output operator Φ. When the temporal load $G(t)$, the damping coefficient μ and the final time T guarantee the strict positivity of the singular values, the unknown spatial load is uniquely expressed as an infinite series in terms of the eigensystem of Φ and the Fourier coefficients of the measured data $u_T(x)$ (see, Theorem 2.4). This result is a unique feature of the spectral method, which is unavailable in [95].

Since the main motive of this work is to explore the importance of the damping parameter in the unique recovery of spatial load, we considered two specific temporal loads $G(t) = 1$ and $G(t) = \exp(-\eta t),\ \eta > 0$ and determined the sufficient conditions relating $\mu, T,$ and η by which we have shown that the singular values of Φ are strictly positive. These conditions are obtained for the physically relevant values of the damping parameter μ, namely, underdamping case $\mu < 2\sqrt{\lambda_1}$ and critically damped case $\mu = 2\sqrt{\lambda_{n^*}}$, where $\{\lambda_n\}$ is the eigenvalues of the Euler-Bernoulli operator. Consequently, we found the singular value expansion for the unknown spatial load $F(x)$ explicitly for both temporal loads (see, Propositions 2.1, 2.2). Another added advantage of the spectral method is that the spectral series representation of $F(x)$ can be utilized to develop the Truncated Singular Value Decomposition (TSVD) algorithm for the numerical reconstruction of the spatial load. The study of this method has been done for the heat and wave equations in [50], where it is explored that the numerical study of these problems using the Conjugate Gradient Algorithm (CGA) with TSVD initialization is more robust than implementing with CGA alone.

This chapter is organized as follows. In Section 2.2, the input-output operator corresponding to the inverse source problem (2.1)-(2.2) is introduced, and existence of a quasi-solution to the inverse problem is proved. The singular value decomposition (SVD) of the input-output operator is derived in Section 2.3. Sufficient conditions for the uniqueness of specific applied problems are discussed in Section 2.4.

2.2 Existence of a quasi-solution to the inverse problem

Assume that the inputs in (2.1) satisfy the following conditions:

$$\begin{cases} 0 < \mu < \mu^*, \; 0 < r_0 \leq r(x) \leq r_1, \; x \in (0, \ell), \\ F \in L^2(0, \ell), \; F(x) \not\equiv 0, \; G \in L^2(0, T), \; G(t) > 0. \end{cases} \tag{2.3}$$

Theorem 2.1. *Let conditions (2.3) hold. Then there exists a unique weak solution* $u \in L^2(0, T; \mathcal{V}(0, \ell))$ *with* $u_t \in L^2(0, T; L^2(0, \ell))$ *and* $u_{tt} \in L^2(0, T; \mathcal{V}'(0, \ell))$ *of the direct problem (2.1). Moreover, the following estimate holds:*

$$\begin{aligned} \|u\|^2_{L^2(0,T;\mathcal{V}(0,\ell))} + \|u_t\|^2_{L^2(0,T;L^2(0,\ell))} + \|u_{tt}\|^2_{L^2(0,T;\mathcal{V}'(0,\ell))} \\ \leq C^2 \|F\|^2_{L^2(0,\ell)} \|G\|^2_{L^2(0,T)}, \end{aligned} \tag{2.4}$$

where $\mathcal{V}(0, \ell)$ *is defined by* (1.9) *and the constant* $C > 0$ *depends on the constants introduced in (2.3), also on* $\ell, T > 0$.

Proof. The proof follows from the similar arguments to those given in [9]. Although the results in [9] are proved for Dirichlet type boundary conditions (clamped beam), the same analysis applies to the current set of boundary conditions. □

Introduce the set of admissible sources

$$\mathcal{F} = \{F \in L^2(0, \ell) : \; \|F\|_{L^2(0,\ell)} \leq \gamma, \; \gamma > 0\}$$

and denote by $u(x, t; F)$ the weak solution of the direct problem (2.1) corresponding to given $F \in \mathcal{F}$. Let us define input-output operator:

$$\Phi : \mathcal{F} \subset L^2(0, \ell) \mapsto L^2(0, \ell), \quad (\Phi F)(x) := u(x, T; F). \tag{2.5}$$

We reformulate the inverse problem (2.1)-(2.2) in terms of the operator equation:

$$\Phi F = u_T, \; F \in \mathcal{F}, \; u_T \in L^2(0, \ell). \tag{2.6}$$

The equality in (2.6) holds only for a noiseless measured output u_T. However, in practice the measured output usually contains noise and as a consequence the exact equality in (2.6) is not possible in practice. Hence one needs to introduce the Tikhonov functional

$$\mathcal{J}(F) := \frac{1}{2}\left\|\Phi F - u_T\right\|^2_{L^2(0,\ell)},\ F \in \mathcal{F},$$

and reformulate the inverse source problem (2.1)-(2.2) as a minimization problem for this functional:

$$\mathcal{J}(F) = \inf_{\tilde{F}\in\mathcal{F}} \mathcal{J}(\tilde{F}). \tag{2.7}$$

A solution of the minimization problem (2.7) is a quasi-solution of the inverse source problem (2.1) -(2.2).

Using the results given in [44] we can show that under conditions (2.3) the input-output operator defined in (2.5) is a linear compact operator, which implies that the inverse problem (2.1) -(2.2) is ill-posed.

Theorem 2.2. *Assume that conditions (2.3) hold. Then the minimization problem (2.7) has a solution in the set of admissible inputs $\mathcal{F}$.*

Proof. Let $F_1, F_2 \in \mathcal{F}$ be the two admissible sources from $\mathcal{F}$ and $u(x,t;F_1)$, $u(x,t;F_2)$ be the corresponding solutions of the direct problem (2.1). Then the function

$$\delta u(x,t) := \delta u(x,t;\delta F),\ \ \delta u(x,t;\delta F) = u(x,t;F_1) - u(x,t;F_2)$$

solves the direct problem (2.1) with $F(x)$ replaced by $\delta F(x) = F_1(x) - F_2(x)$.

Now, we employ the identity

$$|\mathcal{J}(F_1) - \mathcal{J}(F_2)|^2 = \left|\sqrt{\mathcal{J}(F_1)} + \sqrt{\mathcal{J}(F_2)}\right|^2 \left|\sqrt{\mathcal{J}(F_1)} - \sqrt{\mathcal{J}(F_2)}\right|^2,$$

to deduce that

$$\begin{aligned}\left|\sqrt{\mathcal{J}(F_1)} - \sqrt{\mathcal{J}(F_2)}\right|^2 &= \frac{1}{2}\left|\|\Phi(F_1) - u_T\|_{L^2(0,\ell)} - \|\Phi(F_2) - u_T\|_{L^2(0,\ell)}\right|^2 \\ &\leq \frac{1}{2}\|\Phi(F_1) - \Phi(F_2)\|^2_{L^2(0,\ell)}.\end{aligned}$$

Further, in view of the definition of the operator Φ and the trace inequality

$$\|u(\cdot,T)\|^2_{L^2(0,\ell)} \leq T\|u_t\|^2_{L^2(0,T;L^2(0,\ell))} \tag{2.8}$$

with $u(x,t)$ replaced by $\delta u(x,t)$ we conclude that

$$\begin{aligned}
&\|\Phi(F_1)-\Phi(F_2)\|^2_{L^2(0,\ell)}\\
&\quad=\|u(\cdot,T;F_1)-u(\cdot,T;F_2)\|^2_{L^2(0,\ell)}\\
&\quad=\|\delta u(\cdot,T;\delta F)\|^2_{L^2(0,\ell)}\le C^2T\,\|\delta F\|^2_{L^2(0,\ell)}\|G\|^2_{L^2(0,T)}.
\end{aligned}$$

This implies that

$$|\mathcal{J}(F_1)-\mathcal{J}(F_2)|^2\le\frac{TC^2}{2}\,\|\delta F\|^2_{L^2(0,\ell)}\|G\|^2_{L^2(0,T)}\left|\sqrt{\mathcal{J}(F_1)}+\sqrt{\mathcal{J}(F_2)}\right|^2. \tag{2.9}$$

Applying the triangle inequality and using the estimate (2.8), we obtain

$$\begin{aligned}
&\left|\sqrt{\mathcal{J}(F_1)}+\sqrt{\mathcal{J}(F_2)}\right|^2\\
&\quad\le 2\left(\|\Phi F_1\|^2_{L^2(0,\ell)}+\|\Phi F_2\|^2_{L^2(0,\ell)}+2\|u_T\|^2_{L^2(0,\ell)}\right)\\
&\quad\le 4(1+TC^2)\left(\gamma_F^2\|G\|^2_{L^2(0,T)}+\|u_T\|^2_{L^2(0,\ell)}\right),
\end{aligned}$$

since $\|F_n\|_{L^2(0,\ell)}\le\gamma$ for all $F_n\in\mathcal{F}$. Substituting this in (2.9) we find that

$$|\mathcal{J}(F_1)-\mathcal{J}(F_2)|\le L\|F_1-F_2\|_{L^2(0,\ell)}\,,$$

i.e. the functional $\mathcal{J}$ is Lipschitz continuous with the Lipschitz constant

$$L=\left(2TC^2(1+TC^2)\left(\gamma_F^2\|G\|^2_{L^2(0,T)}+\|u_T\|^2_{L^2(0,\ell)}\right)\right)^{1/2}.$$

This implies that $\mathcal{J}$ is a lower semi-continuous functional and hence it is weakly lower-semi continuous on nonempty closed convex set $\mathcal{F}$.

Then, by the generalized Weierstrass theorem, (see, Theorem 1.8), we conclude that the functional $\mathcal{J}(F)$ has a minimizer $F\in\mathcal{F}$. □

Remark 2.1. By the linearity of the direct problem (2.1), we have

$$u(x,t;\nu F_1+(1-\nu)F_2)=\nu u(x,t;F_1)+(1-\nu)u(x,t;F_2),\ \nu\in(0,1),$$

and hence, one can get that

$$\mathcal{J}(\nu F_1+(1-\nu)F_2)\le\nu\mathcal{J}(F_1)+(1-\nu)\mathcal{J}(F_2),\forall\,F_1,F_2\in\mathcal{F},\nu\in(0,1).$$

It shows that the functional $\mathcal{J}(F)$ is convex. Since the functional is not strictly convex, we cannot guarantee the uniqueness of the inverse problem.

Corollary 2.1. *Assume that conditions (2.3) hold. Then the regularized Tikhonov functional*

$$\mathcal{J}_\alpha(F) = \frac{1}{2}\,\|\Phi F - u_T\|^2_{L^2(0,\ell)} + \frac{1}{2}\,\alpha\|F\|^2_{L^2(0,\ell)},\ F \in \mathcal{F} \tag{2.10}$$

has a unique minimizer $F_\alpha \in \mathcal{F}$.

2.3 Singular value decomposition of the input-output operator

Consider the eigenvalue problem

$$\begin{cases} (\mathcal{L}w)(x) = \lambda w(x),\ x \in (0,\ell), \\ w(0) = w_{xx}(0) = w(\ell) = w_{xx}(\ell) = 0, \end{cases} \tag{2.11}$$

associated with the Euler-Bernoulli operator $(\mathcal{L}w)(x) := (r(x)w''(x))''$ defined on

$$D(\mathcal{L}) = \{v \in \mathcal{V}(0,\ell) \cap H^4(0,\ell) : v_{xx}(0) = v_{xx}(\ell) = 0\},$$

that is, $\mathcal{L} : D(\mathcal{L}) \subset L^2(0,l) \mapsto L^2(0,l)$. Denote the symmetric bilinear form associated by the Euler-Bernoulli operator by

$$B[w,v] := \int_0^\ell r(x)w''(x)v''(x)dx,\ w,v \in \mathcal{V}(0,\ell). \tag{2.12}$$

Then the energy norm $B[w,w]^{1/2}$ is equivalent to the norm $\|w\|_{\mathcal{V}(0,\ell)}$, by condition (2.3).

In view of the results given in [34], Chapter 13, the Euler-Bernoulli operator is self-adjoint and positive definite. Furthermore, there exist eigenfunctions $\{\psi_n\}_{n=1}^\infty$,

$$\begin{cases} (\mathcal{L}\psi_n)(x) = \lambda_n\psi_n(x),\ x \in (0,\ell), \\ \psi_n(0) = \psi_n''(0) = \psi_n(\ell) = \psi_n''(\ell) = 0, \end{cases} \tag{2.13}$$

corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^\infty$, $0 < \lambda_1 < \lambda_2$ ..., with the asymptotic property $\mathcal{O}(n^4)$ ([8]). In addition, the system $\{\psi_n\}_{n=1}^\infty$, forms an orthonormal basis for $L^2(0,l)$.

Using this basis, we can write the Fourier series expansion

$$w(x) = \sum_{n=1}^{\infty} w_n \psi_n(x)\,,\; w_n := (w, \psi_n)_{L^2(0,\ell)} \tag{2.14}$$

for the weak solution $w \in \mathcal{V}(0,\ell)$ of the problem (2.11).

Remark 2.2. In the case when $r(x) = 1$ and $\ell = \pi$ in (2.13), one can prove that (see, for instance [79])

$$\lambda_n = n^4,\; n = 1, 2, ...,\; \psi_n(x) = \sqrt{2/\pi}\,\sin nx,\; x \in (0, \pi).$$

Lemma 2.1. *Assume that conditions (2.3) hold and the eigensystem* $\{\lambda_n, \psi_n\}_{n=1}^{\infty}$ *is defined as above. Then for any* $w \in \mathcal{V}(0,\ell)$*, there exists an orthonormal basis* $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=1}^{\infty}$ *for* $\mathcal{V}(0,\ell)$*, and the series (2.14) converges in* $\mathcal{V}(0,\ell)$*. Furthermore, the principal eigenvalue* $\lambda_1 > 0$ *of the Euler-Bernoulli operator can be defined through the norm of the bilinear form* $B[w,v]$ *as follows:*

$$\lambda_1 = \min\{B[v,v] :\, v \in \mathcal{V}(0,\ell),\; \|v\|_{L^2(0,\ell)} = 1\}. \tag{2.15}$$

Proof. Let $w \in \mathcal{V}(0,\ell)$. Then we can write the Fourier series expansion (2.14) as $\{\psi_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(0,l)$. Furthermore, from (2.12) and (2.13) it follows that

$$B[\psi_n, \psi_m] = \lambda_n (\psi_n, \psi_m)_{L^2(0,\ell)} = \lambda_n \delta_{n,m},\; n, m = 1, 2, ..., \tag{2.16}$$

where $\delta_{n,m}$ is the Kronecker symbol. This implies that $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=1}^{\infty}$ is an orthonormal subset of $\mathcal{V}(0,\ell)$ endowed with the new inner product (2.12), since

$$B\left[\frac{\psi_n}{\sqrt{\lambda_n}}, \frac{\psi_m}{\sqrt{\lambda_n}}\right] = \delta_{n,m},\; n, m = 1, 2, ...,$$

by (2.16). We prove that $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=1}^{\infty}$ is in fact an orthonormal basis of $\mathcal{V}(0,\ell)$. To this end, we need to show that $B[\psi_n/\sqrt{\lambda_n}, w] = 0$, for all $n = 1, 2, 3,$ ... , implies $w \equiv 0$. But this assertion is evidently holds since $B[\psi_n/\sqrt{\lambda_n}, w] = \sqrt{\lambda_n}(\psi_n, w)_{L^2(0,\ell)}$ by (2.16), and the conditions

$$(\psi_n, w)_{L^2(0,\ell)} = 0, \text{for all } n = 1, 2, 3, \; ...$$

imply $w(x) \equiv 0$, as $\{\psi_n\}_{n=1}^{\infty}$ is a basis for $L^2(0,\ell)$. Thus, $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{V}(0,\ell)$ and, as a consequence, the series

$$\sum_{n=1}^{\infty} \widehat{w}_n \frac{\psi_n}{\sqrt{\lambda_n}}\,,\ \widehat{w}_n := B\left[w,\, \frac{\psi_n}{\sqrt{\lambda_n}}\right],$$

converges in $\mathcal{V}(0,\ell)$. Comparing this series with the series (2.14) we find that $\widehat{w}_n = \sqrt{\lambda_n}\, w_n$. This means that the series (2.14) in fact converges also in $\mathcal{V}(0,\ell)$.

To prove the second part of the theorem, we employ the Fourier series

$$\mathring{w} = \sum_{n=1}^{\infty} \mathring{w}_n\, \psi_n\,,\ \mathring{w}_n = (\mathring{w}, \psi_n)_{L^2(0,\ell)}$$

of the element $\mathring{w} \in \mathcal{V}(0,\ell)$ with $\|\mathring{w}\|_{L^2(0,\ell)} = 1$. Then, by Parseval's equality,

$$\sum_{n=1}^{\infty} \mathring{w}_n^2 = \|\mathring{w}\|^2_{L^2(0,\ell)} = 1.$$

In view of $B[\mathring{w}\,, \mathring{w}] = \lambda_n \|\mathring{w}\|^2_{L^2(0,\ell)} = \lambda_n$ we obtain that

$$B[\mathring{w}\,, \mathring{w}] = \sum_{n=1}^{\infty} \mathring{w}_n^2\, \lambda_n \geq \lambda_1 \sum_{n=1}^{\infty} \mathring{w}_n^2 = \lambda_1\,.$$

The above result leads to (2.15). □

Theorem 2.3. *Assume that conditions (2.3) hold. Then the input-output operator* Φ *introduced in (2.5), and corresponding to the inverse problem (2.1)-(2.2) is self-adjoint. Furthermore,*

$$(\Phi\psi_n)(x) = \sigma_n \psi_n(x), \tag{2.17}$$

that is, $\{\sigma_n, \psi_n\}_{n=1}^{\infty}$ *is the eigensystem of the input-output operator, where*

$$\sigma_n = \frac{1}{\omega_n} \int_0^T e^{-\mu(T-t)/2} \sin(\omega_n(T-t))\, G(t)dt,$$
$$\omega_n = \frac{1}{2}\sqrt{4\lambda_n - \mu^2}, \qquad \textit{if}\ \ \mu < 2\sqrt{\lambda_n}, \tag{2.18}$$

$$\sigma_{n_*} = \int_0^T (T-t)e^{-\mu(T-t)/2} G(t)dt, \qquad \textit{if}\ \ \mu = 2\sqrt{\lambda_{n_*}}, \tag{2.19}$$

$$\sigma_n = \frac{1}{2\widehat{\omega}_n} \int_0^T e^{-\mu(T-t)/2} \left[e^{\widehat{\omega}_n(T-t)} - e^{-\widehat{\omega}_n(T-t)} \right] G(t)dt,$$
$$\widehat{\omega}_n = \frac{1}{2}\sqrt{\mu^2 - 4\lambda_n}, \quad \textit{if } \mu > 2\sqrt{\lambda_n} \tag{2.20}$$

while $\{\lambda_n, \psi_n\}_{n=1}^{\infty}$ *is the eigensystem of the Euler-Bernoulli operator.*

Proof. We use the Fourier series expansion

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\psi_n(x), \; u_n(t) := (u(\cdot,t), \psi_n)_{L^2(0,\ell)}, \tag{2.21}$$

for the weak solution of the initial boundary value problem (2.1) and then take the L^2-inner product between (2.1) and $\psi_n(x)$. In view of (2.13) we arrive at the problem:

$$\begin{cases} (u_{tt}(t), \psi_n) + \mu(u_t(t), \psi_n) + \lambda_n(u(t), \psi_n) = (F, \psi_n)\, G(t), \\ (u(0), \psi_n) = (u_t(0), \psi_n) = 0. \end{cases}$$

This implies that

$$\begin{cases} u_n''(t) + \mu u_n'(t) + \lambda_n u_n(t) = F_n G(t), \\ u_n(0) = u_n'(0) = 0, \end{cases} \tag{2.22}$$

for each $n = 1, 2, 3, ...$. The characteristic equation associated with (2.22) is given by $\beta_{1,2}^2 + \mu\, \beta_{1,2} + \lambda_n = 0$ and it's root is $\beta_{1,2} = (-\mu \pm \sqrt{\mu^2 - 4\lambda_n})/2$. There are three possible cases depending on the sign of the discriminant $\mu^2 - 4\lambda_n$ and determined by the conditions $\mu < 2\sqrt{\lambda_n}$, $\mu = 2\sqrt{\lambda_n}$ and $\mu > 2\sqrt{\lambda_n}$. The solutions of the Cauchy problem (2.22) corresponding to these cases are

$$u_n(t) = \begin{cases} \dfrac{F_n}{\omega_n} \displaystyle\int_0^t e^{-\mu(t-\tau)/2} \sin(\omega_n(t-\tau))\, G(\tau)d\tau, \; t \in [0,T], \\ \qquad\qquad \omega_n = \frac{1}{2}\sqrt{4\lambda_n - \mu^2} \;\; \text{if } \mu < 2\sqrt{\lambda_n}, \\ F_n \displaystyle\int_0^t (t-\tau)\, e^{-\mu(t-\tau)/2} G(\tau)d\tau, \; t \in [0,T] \qquad \text{if } \mu = 2\sqrt{\lambda_n}, \\ \dfrac{F_n}{2\widehat{\omega}_n} \displaystyle\int_0^t e^{-\mu(t-\tau)/2} \left[e^{\widehat{\omega}_n(t-\tau)} - e^{-\widehat{\omega}_n(t-\tau)} \right] G(\tau)d\tau, \\ \qquad\qquad \widehat{\omega}_n = \frac{1}{2}\sqrt{\mu^2 - 4\lambda_n} \;\; \text{if } \mu > 2\sqrt{\lambda_n}. \end{cases} \tag{2.23}$$

Substituting $t = T$ in (2.21) and (2.23) we arrive at the (formal) Fourier series expansion of the input-output operator:

$$(\Phi F)(x) = \sum_{n=1}^{\infty} \sigma_n F_n \psi_n(x), \tag{2.24}$$

where σ_n, $n = 1, 2, 3, ...$ are defined by expressions (2.18) to (2.20). Evidently ψ_n are eigenfunctions of the operator Φ corresponding to the eigenvalues σ_n, $n = 1, 2, ...$ Indeed,

$$(\Phi \psi_m)(x) = \sum_{n=1}^{\infty} \sigma_n (\psi_m(x), \psi_n(x))\, \psi_n(x) = \sigma_m \psi_m(x).$$

Furthermore,

$$\begin{aligned}(\Phi F, \tilde{F})_{L^2(0,l)} &:= \left(\sum_{n=1}^{\infty} \sigma_n F_n \psi_n(x), \sum_{m=1}^{\infty} \tilde{F}_m \psi_m(x) \right) \\ &= \sum_{n=1}^{\infty} \sigma_n F_n \tilde{F}_n = (F, \Phi \tilde{F}) \quad \forall\, F, \tilde{F} \in L^2(0,l),\end{aligned}$$

where $\tilde{F}_n := (\tilde{F}, \psi_n)_{L^2(0,l)}$. Hence, the input-output operator is self-adjoint, that is $\Phi = \Phi^*$, where $\Phi^* : L^2(0,l) \mapsto L^2(0,l)$ is the adjoint operator. This completes the proof. □

Remark 2.3. The cases $\mu < 2\sqrt{\lambda_n}$, $\mu = 2\sqrt{\lambda_n}$ and $\mu > 2\sqrt{\lambda_n}$ defined in (2.18) to (2.20), correspond to underdamped, critically damped and overdamped vibrating systems, according to the commonly accepted classification [79].

It should be emphasized that only one term σ_{n_*} associated with the case $\mu = 2\sqrt{\lambda_{n_*}}$ can appear in the expansion (2.24). If $\mu = 2\sqrt{\lambda_{n_*}}$ and $n_* > 1$, then the terms σ_n, $n = 1, 2, , .. n_* - 1$ associated with the case $\mu > 2\sqrt{\lambda_n}$, appear in the expansion (2.24), due to the fact that the sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ increases monotonically as $n \to \infty$. Furthermore, the case $\mu \in (2\sqrt{\lambda_m}, 2\sqrt{\lambda_{m+1}})$ means that the terms $\sigma_1, ..., \sigma_m$ in the expansion (2.24) are defined by formula (2.20).

Next, we derive some important consequences of Theorem 2.3. Formula (2.24) implies that

$$(\Phi^* \Phi F)(x) = \sum_{n=1}^{\infty} \sigma_n^2 F_n \psi_n(x).$$

By definition, the square root of eigenvalues σ_n^2 of the self-adjoint operator $\Phi^*\Phi$ are defined as the singular values of the self-adjoint input-output operator Φ. Hence σ_n, $n = 1, 2, 3, ...$ defined by formula (2.17) are the singular values of the input-output operator Φ. Accordingly, the triple $\langle \sigma_n, \psi_n, \psi_n \rangle$ defines the singular system for the self-adjoint input-output operator Φ.

Theorem 2.4. *Let conditions (2.3) hold. Assume that the noise free final output introduced in (2.2) satisfies the condition $u_T \in L^2(0, \ell)$. Suppose that the temporal load $G(t)$ is such that $\sigma_n > 0$ for all $n = 1, 2, 3, \ldots$.*

Then for the unique minimum $F_\alpha \in \mathcal{F}$ of the regularized Tikhonov functional $\mathcal{J}_\alpha(F)$ defined in (2.10) the following singular value decomposition holds:

$$F_\alpha(x) = \sum_{n=1}^{\infty} \frac{q(\alpha; \sigma_n)}{\sigma_n} u_{T,n} \, \psi_n(x), \; x \in (0, \ell), \tag{2.25}$$

where

$$q(\alpha; \sigma) = \frac{\sigma^2}{\sigma^2 + \alpha}, \; \alpha > 0$$

is the filter function, $\alpha > 0$ is the parameter of the regularization, $\{\sigma_n, \, \psi_n(x)\}_{n=1}^{\infty}$ is the eigensystem of the input-output operator Φ and $u_{T,n}$ is the nth Fourier coefficient of the output $u_T(x)$.

Proof. Proof of this theorem follows from the similar arguments to those given in Theorem 3.1.3 of [50]. □

Remark 2.4. The inverse problem with the final time velocity is analyzed in a similar way. In this case one needs to replace the additional condition (2.2) by the condition $\nu_T(x) = u_t(x, T), \; x \in (0, \ell)$.

2.4 Sufficient conditions for pure spatial and exponentially decaying loads

In this section, we derive sufficient conditions for the positivity of the singular values σ_n, $n = 1, 2, \ldots$ corresponding to the input-output operator (2.5) for the common dynamic loading cases: the pure spatial loading, $G(t) \equiv 1$, and the exponentially decay loading $G(t) = e^{-\eta t}$, where $\eta > 0$ is the decay rate of the applied temporal load. In applications, the most widely used range of values of the damping parameter is $\mu \in (0, 1)$ ([81]). Hence, we will examine the underdamped, $\mu < 2\sqrt{\lambda_n}$, and, as a limit case, critically damped, $\mu = 2\sqrt{\lambda_n}$ cases. Notice also that the condition $\mu < 2\sqrt{\lambda_1}$ implies $\mu < 2\sqrt{\lambda_n}$ for all $n = 2, 3, \ldots,$, since $\lambda_1 > 0$ is the minimal eigenvalue of the Euler-Bernoulli operator.

2.4.1 Forced vibration under pure spatial load

Assuming $G(t) \equiv 1$ in (2.18)-(2.19) and calculating the integrals, we find the singular values corresponding to underdamped and critically damped cases as follows:

$$\sigma_n = \frac{1}{\lambda_n}\left\{1 - \left[\cos(T\omega_n) + \frac{\mu}{2\omega_n}\sin(T\omega_n)\right]e^{-\mu T/2}\right\}, \qquad \omega_n = \frac{1}{2}\sqrt{4\lambda_n - \mu^2},\ 0 < \mu < 2\sqrt{\lambda_1}, \tag{2.26}$$

$$\sigma_{n_*} = \frac{1}{\lambda_{n_*}}\left\{1 - \left[1 + T\sqrt{\lambda_{n_*}}\right]e^{-\mu T/2}\right\},\ \mu = 2\sqrt{\lambda_{n_*}}. \tag{2.27}$$

Under the assumption $\mu < 2\sqrt{\lambda_1}$, we need to find a relationship between the final time $T > 0$ and the damping parameter that provide the positivity conditions $\sigma_n > 0$, for all $n = 1, 2, 3, \ldots$, that is, a sufficient condition for unique determination of the spatial load $F(x)$ from final time output $u_T(x)$ through (2.25) and (2.26). To this end, we introduce the function

$$g_\sigma(n;\mu,T) = \left[\cos(T\omega_n) + \frac{\mu}{2\omega_n}\sin(T\omega_n)\right]\exp\left(-\mu T/2\right)$$

and rewrite formula (2.26) for σ_n in the form: $\sigma_n = \frac{1}{\lambda_n}\left\{1 - g_\sigma(n;\mu,T)\right\}$.

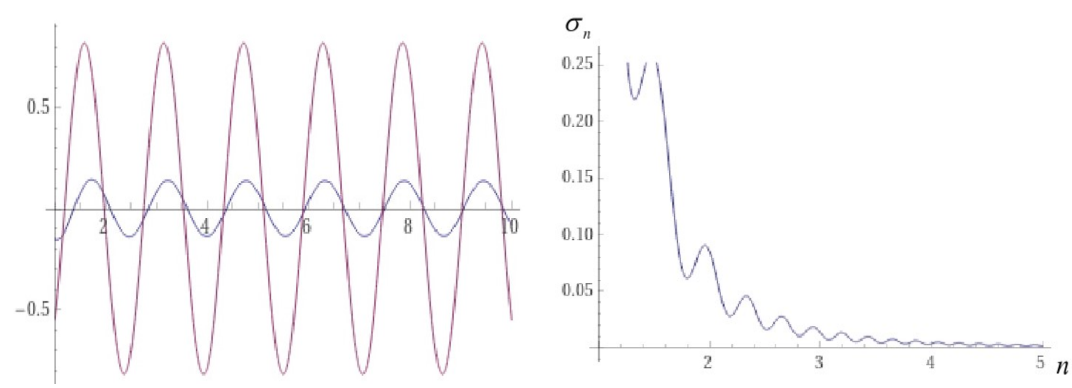

Figure 2.2: Behaviour of the function $g_\sigma(n;\mu,T)$ depending on the values of the damping parameter and the final time (the figure on the left: $\mu = 0.1$ (purple line) and $\mu = 1$ (blue line) for $T = 4$), and behaviour of the singular values (the figure on the right: σ_n: $\mu = 0.5$, for $T = 4$).

Behaviours of the function $g_\sigma(n;\mu,T)$ and the singular values σ_n are plotted on the left and right in Figure 2.2, depending on the values of the damping parameter and the final time. The left figure shows how the function $g_\sigma(n;\mu,T)$ generates an oscillating behavior of the singular values, even when there is no temporal component of the load ($G(t) \equiv 1$). Moreover, it is also evident that the oscillation is drastically reduced when the effect of the damping parameter increases. However, this oscillating behavior is weakly reflected in the behavior of the singular values due to the absence of the temporal component of the load ($G(t) \equiv 1$). The oscillating behavior together with the decay rate of the order $\mathcal{O}(n^{-4})$ can be clearly seen in the right figure.

In view of the inequality $|a \cos \alpha + b \sin \alpha| \leq \sqrt{a^2 + b^2}$ and the relation $4\omega_n^2 = 4\lambda_n - \mu^2$, the positiveness $\sigma_n > 0$ is guaranteed by the conditions

$$\exp(\mu T) > 1 + \frac{\mu^2}{4\lambda_n - \mu^2} := \Lambda_\sigma(\lambda_n),\ n = 1, 2, 3, \dots .$$

The function $\Lambda_\sigma(\lambda_n)$ decreases monotonically as $\lambda_n > 0$ increases, hence all the above conditions hold, if

$$T > \frac{1}{\mu} \ln\left(1 + \frac{1}{(2\sqrt{\lambda_1}/\mu)^2 - 1}\right) := g_T(\mu; \lambda_1),\ \ 2\sqrt{\lambda_1}/\mu > 1.$$

With the inequality $\ln(1 + x) < x$, for all $x > 0$, this leads to the following sufficient condition for the positivity of the singular values defined by formula (2.26):

$$T > \frac{\mu}{4\lambda_1 - \mu^2} := T_*,\ 0 < \mu < 2\sqrt{\lambda_1}. \tag{2.28}$$

Let us analyze what does the condition (2.28) means in the sense of admissible lower limit $T_* > 0$ of the final time $T > 0$. For near-zero values of the damping parameter, the corresponding value of the lower limit is the same order. Namely, for the case $\lambda_1 = 1$ considered in the Remark 2.2, $T_* \approx 10^{-2}$ for $\mu = 10^{-2}$. Hence, in this case, no condition other than the positivity is imposed in (2.28) on the final time $T > 0$.

Furthermore, for the most important and critical case, when the difference between the values μ and $2\sqrt{\lambda_1}$ is of the order 10^{-1}, the corresponding value of the admissible lower limit T_* is reasonable. Specifically, for $\lambda_1 = 1$ and $\mu = 1.9$, we have $2\sqrt{\lambda_1} - \mu = 10^{-1}$, and $T_* \approx 4.9$, by formula for T_* in (2.28). This means that one needs to take the value of the final time $T \geq 5$.

Proposition 2.1. *Assume that conditions (2.3) hold and $G(t) \equiv 1$ in (2.1). Suppose that the final time $T > 0$ and the damping coefficient $\mu > 0$ satisfiy the conditions (2.28). Then $\sigma_n > 0$ for all $n = 1, 2, 3, \dots$, and the inverse problem (2.1) - (2.2) has a unique solution $F \in \mathcal{F}$.*

Moreover, for this solution the following singular value expansion holds:

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} u_{T,n} \, \psi_n(x), \tag{2.29}$$

where σ_n, $n = 1, 2, 3, ...$ are defined by formula (2.26).

Consider now the critically damped case, when $\mu = 2\sqrt{\lambda_{n_*}}$. From formula (2.27) it follows that the sufficient condition for the positivity of the singular value σ_{n_*} is

$$e^{T\sqrt{\lambda_{n*}}} > 1 + T\sqrt{\lambda_{n_*}}.$$

Since the inequality $\exp(y) > 1 + y$ holds for all $y > 0$, the above inequality holds for all $T > 0$. Therefore, the singular value σ_{n_*} corresponding to the critically damped case $\mu = 2\sqrt{\lambda_{n_*}}$ is positive for all $T > 0$.

2.4.2 A beam subjected to exponentially decaying temporal load

Assume in (2.18) that $G(t) = e^{-\eta t}$, where $\eta > 0$ is the decay rate of the applied temporal load. Consider again the underdamped case: $\mu < 2\sqrt{\lambda_1}$.

Substituting $G(t) = \exp(-\eta t)$ in formula (2.18) we get:

$$\begin{aligned} \sigma_n = \frac{1}{\omega_n} \int_0^T e^{-\mu(T-t)/2} \sin(\omega_n(T-t)) \, e^{-\eta t} dt, \\ \omega_n = \tfrac{1}{2}\sqrt{4\lambda_n - \mu^2}, \; \mu < 2\sqrt{\lambda_n}, \end{aligned} \tag{2.30}$$

Assuming $0 < 2\eta < \mu$, we calculate integral (2.30) to find:

$$\begin{aligned} \sigma_n = \frac{4}{(4\lambda_n - \mu^2) + (\mu - 2\eta)^2} \Big\{ e^{-\eta T} \\ - \left[\cos(\omega_n T) + \frac{\mu - 2\eta}{2\omega_n} \sin(\omega_n T) \right] e^{-\mu T/2} \Big\}. \end{aligned} \tag{2.31}$$

Then for the condition $\sigma_n > 0$ holds, if $0 < 2\eta < \mu$ and

$$e^{(\mu - 2\eta)T} > 1 + \frac{(\mu - 2\eta)^2}{4\lambda_n - \mu^2}, \; n = 1, 2, 3, ... \, .$$

Since the right-hand-side expression is a monotone decreasing function of $\lambda_n > 0$ and

$\lambda_1 < \lambda_n$, for all $n > 1$, the above inequalities hold, if

$$T > \frac{1}{\mu - 2\eta} \ln\left(1 + \frac{(\mu - 2\eta)^2}{4\lambda_1 - \mu^2}\right).$$

By the inequality $\ln(1 + x) < x$, for all $x > 0$, a sufficient condition to achieve this inequality is the following condition:

$$T > \frac{\mu - 2\eta}{4\lambda_1 - \mu^2} := T_*(\mu; \eta, \lambda_1). \tag{2.32}$$

Proposition 2.2. Assume that conditions (2.3) hold and $G(t) = e^{-\eta t}, \eta > 0$ *in (2.1). Suppose that the final time* $T > 0$*, the damping coefficient* $\mu > 0$ *and the decay rate* $\eta > 0$ *satisfy the following conditions:*

$$\begin{cases} T > \dfrac{\mu - 2\eta}{4\lambda_1 - \mu^2}, \\ 0 < 2\eta < \mu < 2\sqrt{\lambda_1}. \end{cases}$$

Then the singular values $\sigma_n > 0$, $n = 1, 2, 3, \ldots$ *, defined by formula (2.31) are positive, and the inverse problem (2.1) - (2.2) has a unique solution* $F \in \mathcal{F}$*. Moreover, for this solution the singular value expansion (2.29) holds.*

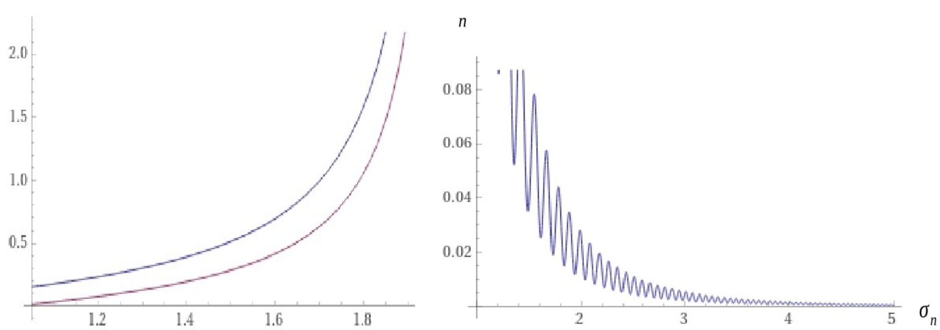

Figure 2.3: The figure on the left: the function $T_*(\mu; \eta, \lambda_1)$ with $\lambda_1 = 1$, depending on the damping parameter: $\eta = 0.3$ (top curve) and $\eta = 0.5$ (bottom curve). The figure on the right: behaviour of the singular values: for $\lambda_n = n^4$, $T = 4$.

The right-hand-side in (2.32) determines the admissible lower limit $T_*(\mu; \eta, \lambda_1) > 0$ of the final time $T > 0$. This is an easily testable and feasible condition, and does not impose a very strong constraint on the admissible values of the final time. So, $T_*(\mu; \eta, \lambda_1) < 4$, even for very close to $2\sqrt{\lambda_1}$ (limit) values of the damping parameter. For λ_1, and for the two values $\eta = 0.3$ and $\eta = 0.5$ of the decay rate parameter, the function $T_*(\mu; \eta, \lambda_1)$ is plotted on the left in Figure 2.3.

As soon as the temporal component $G(t)$ in the Euler-Bernoulli equation (2.1) is activated, even as an *exponentially decaying* temporal load $G(t) = e^{-\eta t}$, the oscillations of the singular values σ_n, as a function of $n = 1, 2, 3, \ldots$, occur, and indicates a real vibration of the beam. This is clearly seen from the graph of this function in Figure 2.3 on the right.

Finally, consider the critically damped case, when $\mu = 2\sqrt{\lambda_{n_*}}$. From the integral in (2.19) with $G(t) = e^{-\eta t}$ it follows that in this case the formula for the singular value σ_{n_*} is

$$\sigma_{n_*} = \frac{1}{(\mu/2 - \eta)^2} \left\{ e^{-\eta T} - [1 + (\mu/2 - \eta)T] \, e^{-\mu T/2} \right\}, \mu/2 = \sqrt{\lambda_{n_*}}. \qquad (2.33)$$

By the same argument in the previous critically damped case, σ_{n_*} given in (2.33) is positive for all $T > 0$.

Remark 2.5. This work provides the uniqueness of the infeasible inverse source problem by considering the damping term in the Euler-Bernoulli beam model. With the help of the damping term, conditions on the final time T, and specifically selecting a temporal load $G(t) > 0$, we ensured the positivity of singular values σ_n, and thereby we obtained the uniqueness of the solution. It is emphasized that the analysis for pure spatial load and exponentially decaying temporal load is comparatively straightforward and verifiable. Furthermore, as an extension of these findings, the paper [51] addresses the unique identification of spatial load in the Euler-Bernoulli beam with a harmonic temporal load $cos(wt)$. Nevertheless, deriving a more comprehensive set of sufficient conditions for the unique recovery of the spatial load using SVD from $u_T(x)$ for any given temporal load $G \in L^2(0, T)$ is still an open problem.

Chapter 3

Identification of a transverse shear force in the Euler-Bernoulli beam with Kelvin-Voigt damping

In Chapter 2, we proved that in the presence of viscous external damping in the Euler-Bernoulli beam, we can uniquely determine the source function from the final time measured data under suitable conditions on the final time T and damping coefficient μ by a rigorous analysis involving the SVD. In this chapter, we consider a more general form of the Euler-Bernoulli beam by including all the possible physical coefficients together with both damping effects given by external damping as well as Kelvin-Voigt damping. It is understood from the previous chapter that the nature of the damping term drastically changes the nature of the solutions to the direct beam model, which in turn helps to obtain a feasible solution to the inverse problems as well. By taking this motivating factor into account, we analyze the Inverse Boundary Value Problem (IBVP) of determining the unknown transverse shear force (boundary data) in the presence of both external viscous damping and internal Kelvin-Voigt damping factors.

Consider the Euler-Bernoulli beam with more general physical coefficients and damping effects as follows (see, [20], Chapter 17, Section 4 and also refer to [6]):

$$\begin{cases} \rho(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} + (\kappa(x)u_{xxt})_{xx} = 0, \ (x,t) \in \Omega_T, \\ u(x,0) = 0, \ u_t(x,0) = 0, \quad x \in (0,\ell), \\ u(0,t) \ = 0, \ u_x(0,t) = 0, \quad t \in [0,T], \\ [r(x)u_{xx} + \kappa(x)u_{xxt}]_{x=\ell} \ = 0, \\ \qquad - \left[\left(r(x)u_{xx} + \kappa(x)u_{xxt}\right)_x\right]_{x=\ell} = g(t), \ t \in [0,T], \end{cases} \tag{3.1}$$

where $\Omega_T := (0,\ell) \times (0,T]$, $r(x) := E(x)I(x) > 0$ is the flexural rigidity (or bending stiffness) of a non homogeneous beam while $E(x) > 0$ is the elasticity modulus and $I(x) > 0$ is the moment of inertia. The coefficient $\kappa(x) := c_d I(x)$ represents energy dissipated by friction internal to the beam, where c_d is the strain-rate damping coefficient. The nature of the terms $\mu(x)u_t$ and $(\kappa(x)u_{xxt})_{xx}$ are determined by external and internal damping mechanisms, respectively. The non-negative coefficient $\mu(x)$ and the positive coefficient $\kappa(x)$ are called the viscous external damping and the strain-rate or Kelvin-Voigt damping coefficients, respectively. For this considered model, there can be only two types of force effects acting on the right end $x = \ell$, one is bending moment, and the other is transverse shear force. In this study, we look at the case in which the vibration is caused by an unknown transverse shear force $g(t) := -\left[(r(x)u_{xx} + \kappa(x)u_{xxt})_x\right]$ at $x = \ell$, which needs to be identified from either given measured deflection at $x = \ell$ or bending moment at $x = 0$.

The following two IBVPs are formulated to the model (3.1).

IBVP-1. *Find the unknown transverse shear force $g(t)$ from measured deflection $\nu(t)$ given at the right end of the beam $x = \ell$:*

$$\nu(t) := u(\ell, t),\ t \in [0,T]. \tag{3.2}$$

For a given $g(t)$ from the set of admissible transverse shear forces, the problem (3.1) with the solution $u(x,t) := u(x,t;g)$ is known as the direct problem. Here, the following tricky question arises. Can we swap the boundary conditions $-\left[\left(\kappa(x)u_{xxt} + r(x)u_{xx}\right)_x\right]_{x=\ell} = g(t)$, $\nu(t) := u(\ell,t)$, and solve the initial boundary value problem

$$\begin{cases} \rho(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} + (\kappa(x)u_{xx})_{xxt} = 0,\ (x,t) \in \Omega_T, \\ u(x,0) = 0,\ u_t(x,0) = 0,\ \ x \in (0,\ell), \\ u(0,t)\ = 0,\ u_x(0,t) = 0,\ \ t \in [0,T], \\ u(\ell,t)\ = \nu(t),\ [r(x)u_{xx} + \kappa(x)u_{xxt}]_{x=\ell} = 0,\ t \in [0,T] \end{cases} \tag{3.3}$$

and then find the unknown transverse shear force $g(t)$ using the formula

$$g(t) = -\left[\left(r(x)u_{xx} + \kappa(x)u_{xxt}\right)_x\right]_{x=\ell}?$$

Therefore, a situation arises as if there is no need for any inverse problem. Besides, swapping of the above mentioned boundary conditions is also a mathematically correct approach. However, in this chapter, we consider the inverse problem (3.1)-(3.2) as the

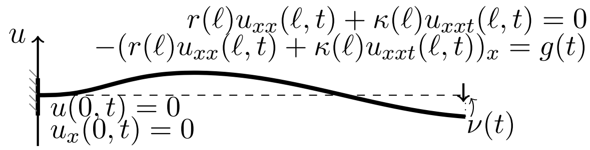

Figure 3.1: Geometry of the IBVP-1

physical justification to that model comes from the fact that problem (3.1) is a developed mathematical model of vibration of the cantilever tip due to the shear force interaction in TDFM (see, [3], [75] and references therein). As mentioned in the subsection1.4, the problem of determining the shear force is of great importance when specimen images and mechanical properties need to be computed at some submolecular precision (see, [75]).
The problem IBVP-1 defined by (3.1) and (3.2) can be reformulated as the invertibility of the Neumann-to-Dirichlet operator

$$\begin{aligned}&\Phi:\mathcal{G}_1\subset H^1(0,T)\mapsto L^2(0,T),\ \ (\Phi g)(t):=u(\ell,t;g),\ t\in[0,T],\\&\mathcal{G}_1=\{g\in H^1(0,T):g(0)=0,\ \|g\|_{H^1(0,T)}\le\mathfrak{K},\ \mathfrak{K}>0\},\end{aligned}\tag{3.4}$$

where $\mathcal{G}_1$ is called the set of admissible inputs (shear forces). With the help of noise free measured output $\nu(t)$, we can reformulate IBVP-1 in terms of functional equation as

$$\Phi g(t)=\nu(t),\ \nu\in L^2(0,T).\tag{3.5}$$

We note that the exact equality in (3.5) can hold only in the case of noiseless measured output $\nu(t)$. However, as noted in the previous chapter, in practice the measured output $\nu(t)$ always contains measurement errors, and hence exact equality in the functional equation (3.5) is not possible. Therefore, we introduce the Tikhonov functional to solve the minimization problem

$$\min_{g\in\mathcal{G}_1}\mathcal{J}_1(g),\ \ \mathcal{J}_1(g):=\frac{1}{2}\|\Phi g-\nu\|^2_{L^2(0,T)},\tag{3.6}$$

whose solution, according to [56], is defined as a quasi-solution of the inverse problem. Then we consider the same problem for the regularized Tikhonov functional

$$\mathcal{J}_{1\alpha}(g):=\frac{1}{2}\|\Phi g-\nu\|^2_{L^2(0,T)}+\frac{\alpha}{2}\|g'\|^2_{L^2(0,T)},\tag{3.7}$$

where $\alpha>0$ is the parameter of regularization.

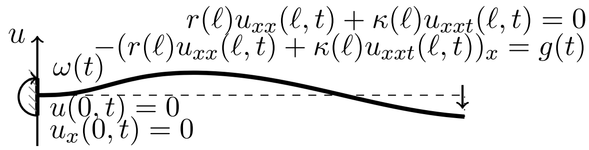

Figure 3.2: Geometry of the IBVP-2

The second inverse problem, we study in this chapter is formulated as follows.

IBVP-2. *Find the unknown transverse shear force $g(t)$ from measured bending moment $\omega(t)$ given at the beginning of the beam $x=0$:*

$$\omega(t) := -\left(r(0)u_{xx}(0,t)+\kappa(0)u_{xxt}(0,t)\right),\ t\in[0,T]. \tag{3.8}$$

The problem IBVP-2 defined by (3.1) and (3.8) is related to the invertibility of Neumann-to-Neumann operator

$$\begin{aligned}
&\Psi : \mathcal{G}_3 \subset H^3(0,T) \mapsto L^2(0,T),\\
&(\Psi g)(t) := -\left(r(0)u_{xx}(0,t;g)+\kappa(0)u_{xxt}(0,t;g)\right), t\in[0,T],\\
&\mathcal{G}_3 = \{g\in H^3(0,T) : g(0)=g'(0)=g''(0)=0, \|g\|_{H^3(0,T)} \le \mathfrak{K}_1, \mathfrak{K}_1>0\}.
\end{aligned} \tag{3.9}$$

In terms of the functional equation, we will again express the inverse problem (3.1) and (3.8) as follows

$$\Psi g(t) = \omega(t),\ \ \omega\in L^2(0,T). \tag{3.10}$$

As in the case of IBVP-1, when the measured data $\omega(t)$ contain random noise the exact equality in (3.10) is not feasible. In this case we solve the minimization problem for the Tikhonov functional

$$\min_{g\in\mathcal{G}_3} \mathcal{J}_2(g), \quad \mathcal{J}_2(g) := \frac{1}{2}\left\|\Psi g-\omega\right\|^2_{L^2(0,T)} \tag{3.11}$$

and the regularized Tikhonov functional is considered as follows

$$\mathcal{J}_{2\alpha}(g) := \frac{1}{2}\left\|\Psi g-\omega\right\|^2_{L^2(0,T)} + \frac{\alpha}{2}\left\|g'''\right\|^2_{L^2(0,T)}. \tag{3.12}$$

It is worth noting that IBVP-2 can be formulated using the admissible source $\mathcal{G}_1$ and the regularized Tikhonov functionals $\mathcal{J}_{1\alpha}$, $\mathcal{J}_{2\alpha}$ can be defined with usual L^2 norm regularizer $\|g\|^2_{L^2(0,T)}$. Further, the solvability of these inverse problems (see, Theorem 3.4 or

Remark 3.2) do not require the more regularized functionals as in (3.7) and (3.12), while these regularized functionals are crucial to derive the stability estimates (see, Theorems 3.7, 3.8). We study the inverse problems (3.1)-(3.2) and (3.1), (3.8) as a minimization problems for the Tikhonov functionals $\mathcal{J}_{1\alpha}(g)$ and $\mathcal{J}_{2\alpha}(g)$ on the set $\mathcal{G}_1$, $\mathcal{G}_3$ respectively. In the absence of the internal damping term $(\kappa(x)u_{xxt})_{xx}$ in (3.1), the inverse problems of (3.1) with measurements (3.2) and (3.8) were studied respectively in [47] and [46].

The presence of Kelvin-Voigt damping and the mixed boundary conditions, in turn, makes the problem more complicated, and the boundary data (shear force) determination under these conditions becomes difficult. Indeed, for the existence and uniqueness of the solution to the direct problem (3.1), we need to develop appropriate identities to handle these types of mixed boundary conditions for deriving priori estimates for the direct problem and proving the regularity of solutions. This makes the current work different from [47], [46] where only external damping $\mu(x)u_t$ is considered, and so the balancing effect of the Kelvin-Voigt damping term on the boundary conditions is also not needed on those papers. For a related model on the wave equation with Kelvin-Voigt damping in the bounded domain, one may refer to [1]. On the other hand, Kelvin-Voigt damping also has some sort of regularizing effect in proving the solutions of the direct problem. For instance, in [47], [46], the authors require higher regularity like $g \in H^2(0,T)$ to prove $u_t \in L^2(0,T;\mathcal{V}_1^2(0,\ell)), u_{tt} \in L^2(0,T;L^2(0,\ell))$, whereas in this chapter we prove those estimates with $g \in H^1(0,T)$ by coupling with appropriate identities and Sobolev embedding theorems. We also prove the existence of solutions to the inverse problems (3.1)-(3.2) and (3.1), (3.8) when the transverse shear force $g(t)$ belongs to the admissible inputs $\mathcal{G}_1 \subset H^1(0,T)$, as the detailed analysis shown in the following sections.

The main contributions of the chapter are summarized as follows:

1. The existence and uniqueness of the weak and regular weak solutions to the direct problem are proved. Furthermore, the necessary a priori estimates are derived.

2. Solvability of the inverse problems (3.1)-(3.2) and (3.1), (3.8), defined as IBVP-1 and IBVP-2, and governed by Neuman-to-Dirichlet and Neumann-to-Neumann operators, respectively, are studied in appropriate admissible set of transverse shear sources $\mathcal{G}_m \subset H^m(0,T)$, $m = 1, 3$. It is demonstrated that for compactness and Lipschitz continuity, the operator Ψ does not require $\mathcal{G}_3$ regularity. We just need $\mathcal{G}_2$ regularity for Ψ's compactness, and $\mathcal{G}_1$ regularity for Lipschitz continuity, while these results can be verified for the operator Φ on the admissible source $\mathcal{G}_1$ itself.

3. The Tikhonov functionals $\mathcal{J}_{m\alpha}(g)$, $m = 1, 2$ are introduced, and the Fréchet deriva-

tives of these functionals are derived in through the solutions of corresponding adjoint problems. It is shown that for IBVP-1, the admissible source $g(t)$ needs to be in $\mathcal{G}_1$, while for IBVP-2 the more regular set of admissible sources $\mathcal{G}_3 \subset H^3(0,T)$ is needed (see, Section 3.3). For IBVP-2, the regularity of the admissible sources is also needed for the solvability of the adjoint problem.

4. Other remarkable results are the Lipschitz type stability estimates for IBVP-1 and IBVP-2. We provide a local stability estimate for the unknown shear force $g \in \mathcal{G}_1$ when Kelvin-Voigt damping coefficient $\kappa(x) > 0$ satisfies a condition on its lower bound, while the external damping coefficient is nonnegative: $\mu(x) \geq 0$. It should be noted that this stability result is valid even when the external damping effect is not present, that is, when $\mu(x) = 0$. In the case of IBVP-2, we establish a stability estimate for $g \in \mathcal{G}_3$ under a feasible condition on the parameter of regularization α. Both the stability results are obtained when more smooth regularization terms are added to the Tikhonov functionals $\mathcal{J}_m(g), m = 1, 2$. These results give a new perspective for the stability analysis of the shear force determination in the presence of both damping terms.

This chapter is organized as follows. The existence and uniqueness, and regularity of solutions are given in Section 3.1. The solvability of inverse problems are given in Sections 3.2. The Section 3.3 is devoted to the Fréchet derivative of the Tikhonov functionals and Lipschitz continuity of the Fréchet derivatives. The monotonocity of the gradient based algorithm is analyzed in Section 3.4. The stability analysis of the inverse source problems are discussed in Section 3.5.

3.1 Existence and uniqueness of weak solutions to the direct problem

In this section, we consider problem (3.1) with non homogeneous initial conditions:

$$\begin{cases} \rho(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} + (\kappa(x)u_{xxt})_{xx} = 0,\ (x,t) \in \Omega_T, \\ u(x,0) = u_0,\ u_t(x,0) = v_0, \qquad x \in (0,\ell), \\ u(0,t)\ = 0,\ u_x(0,t) = 0, \qquad t \in [0,T], \\ [r(x)u_{xx} + \kappa(x)u_{xxt}]_{x=\ell}\ = 0, \\ \qquad -\left[\left(r(x)u_{xx} + \kappa(x)u_{xxt}\right)_x\right]_{x=\ell} = g(t),\ t \in [0,T]. \end{cases} \tag{3.13}$$

In the study of direct and inverse problems related to the model (3.1), the following basic assumptions are used:

$$\begin{cases} r, \kappa \in H^2(0,\ell),\ g \in H^1(0,T),\ g(0) = 0, \\ 0 < \rho_0 \le \rho(x) \le \rho_1, \quad 0 < r_0 \le r(x) \le r_1, \\ 0 \le \mu_0 \le \mu(x) \le \mu_1, \quad 0 < \kappa_0 \le \kappa(x) \le \kappa_1. \end{cases} \tag{3.14}$$

Definition 3.1. Let $0 < T < +\infty$, $u_0 \in \mathcal{V}_1^2(0,\ell)$ and $v_0 \in \mathcal{V}_1^2(0,\ell)$ be given. We say a function $u \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$ with $u_t \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$ and $u_{tt} \in L^2(0,T;L^2(0,\ell))$ is a weak solution of (3.13) provided

$$\begin{aligned} i) \qquad & (\rho\, u_{tt}(t), v) + (\mu\, u_t(t), v) + (r u_{xx}(t), v_{xx}) \\ & +(\kappa u_{xxt}(t), v_{xx}) = g(t)v(\ell),\ \forall\, v \in \mathcal{V}_1^2(0,\ell),\ \text{ a.e. } t \in [0,T]. \\ ii) \qquad & u(0) = u_0,\ u_t(0) = v_0, \end{aligned}$$

where $\mathcal{V}_1^2(0,\ell)$ is defined by (1.10).

From Definition 3.1, it is evident that $u \in H^1(0,T;\mathcal{V}_1^2(0,\ell))$, so that $u \in C([0,T];\mathcal{V}_1^2(0,\ell))$ and $u_t \in C([0,T];L^2(0,\ell))$. Consequently, the equalities $u(0) = u_0$ and $u_t(0) = v_0$ can be justified.

We apply the Faédo-Galerkin approximation method to illustrate that there exists a unique weak solution to direct problem (3.13). First, we choose a sequence of smooth functions $\{\xi_i\}_{i=1}^n$, which form an orthonormal and orthogonal basis for $L^2(0,\ell)$ and $\mathcal{V}_1^2(0,\ell)$ respectively. Then, we construct the n dimensional subspace $W_n := span\{\xi_1, \xi_2, ..., \xi_n\}$ of $\mathcal{V}_1^2(0,\ell)$ and seek the Faédo-Galerkin approximation $u_n(t) := u_n(x,t)$ of the form

$$u_n(t) = \sum_{i=1}^n d_{i,n}(t)\xi_i, u_{0,n} = \sum_{i=1}^n p_{i,n}\xi_i, \text{ and } v_{0,n} = \sum_{i=1}^n q_{i,n}\xi_i,$$

where we hope to find the coefficients $d_{i,n},\ p_{i,n}$ and $q_{i,n}$ so that

$$\begin{cases} (\rho\, u_n''(t), v) + (\mu\, u_n'(t), v) + (r\, u_{n,xx}(t), v_{xx}) \\ +(\kappa\, u_{n,xx}'(t), v_{xx}) = g(t)v(\ell),\ \ \forall\, v \in W_n,\ \ t \in [0,T], \\ u_n(0) = u_{0,n},\ \ u_n'(0) = v_{0,n}. \end{cases} \tag{3.15}$$

By inserting $v = \xi_j,\ j = 1, 2, 3, ...n$, and using the fact that $\xi_i,\ i = 1, 2, ...n$ are orthonormal, it is clear that the problem (3.15) corresponds to the following linear system of ordinary differential equations (ODEs):

$$\begin{cases} M^{\mathrm{T}}D_n''(t) + [N^{\mathrm{T}} + P^{\mathrm{T}}]D_n'(t) + Q^{\mathrm{T}}D_n(t) = \mathbb{G}_n(t), & \text{for } t \in [0,T], \\ D_n(0) = U_n, \quad D_n'(0) = V_n, \end{cases}$$

where $D_n(t) = (d_{1,n}(t), d_{2,n}(t), ..., d_{n,n}(t))^{\mathrm{T}}$, the entries of the matrix M, N, P, Q are

$$\begin{aligned} M &= [(\rho\,\xi_i, \xi_j)]_{n\times n}\,, \quad N = [(\mu\,\xi_i, \xi_j)]_{n\times n}\,, \quad P = [(\kappa\,\xi_{i,xx}, \xi_{j,xx})]_{n\times n}\,, \\ Q &= [(r\,\xi_{i,xx}, \xi_{j,xx})]_{n\times n}\,, \end{aligned}$$

and $\mathbb{G}_j(t) = g(t)\xi_j(\ell)$, $\mathbb{G}_n(t) = (\mathbb{G}_1(t), \mathbb{G}_2(t), ..., \mathbb{G}_n(t))^{\mathrm{T}}$, $U_j = (u_0, \xi_j)$, $V_j = (v_0, \xi_j)$, $U_n = (U_{1,}, U_2, ..., U_n)^{\mathrm{T}}$, $V_n = (V_1, V_2, ..., V_n)^{\mathrm{T}}$.

By the Carathéodory theorem for ODEs (see, [21], Chapter 2, Theorem 1.1) for every $n \geq 1$ there exists a unique solution $u_n \in C^1([0,T]; W_n)$ with $u_n'' \in L^2(0,T;W_n)$ of problem (3.15).

Theorem 3.1. *Let assumptions (3.14) hold true. Then, in the perspective of Definition 3.1, there exists a unique weak solution u to the direct problem (3.13). Moreover,*

$$\begin{aligned} \|u_{xx}\|^2_{L^\infty(0,T;L^2(0,\ell))} &\leq \frac{2(C_0^2+1)}{r_0}\left[\frac{2(1+T)\ell^3}{3r_0}\|g'\|^2_{L^2(0,T)} + R_0(u_0,v_0)\right], && (3.16) \\ \|u\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} &\leq \frac{2C^*C_0^2}{r_0}\left[\frac{2(1+T)\ell^3}{3r_0}\|g'\|^2_{L^2(0,T)} + R_0(u_0,v_0)\right], && (3.17) \\ \|u_t\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} &\leq \frac{C^*(C_0^2+1)}{2\kappa_0}\left[\frac{2(1+T)\ell^3}{3r_0}\|g'\|^2_{L^2(0,T)} + R_0(u_0,v_0)\right], && (3.18) \end{aligned}$$

and

$$\|u_{tt}\|^2_{L^2(0,T;L^2(0,\ell))} \leq \frac{C_1^2}{2\rho_0}\left[(1+T)\ell^3\|g'\|^2_{L^2(0,T)} + R_1(u_0,v_0)\right], \qquad (3.19)$$

where $R_0(u_0,v_0) := \rho_1\|v_0\|^2_{L^2(0,\ell)} + r_1\|u_{0,xx}\|^2_{L^2(0,\ell)}$,
$R_1(u_0,v_0) := \|v_0\|^2_{L^2(0,\ell)} + \|u_{0,xx}\|^2_{L^2(0,\ell)} + \|v_{0,xx}\|^2_{L^2(0,\ell)}$, $C_0^2 = (\exp(T)-1)$,
r_0, ρ_0, κ_0 *are the constants given in (3.14),* C^* *is from (1.16), and the constant* $C_1 > 0$ *is introduced in the proof.*

Proof. Consider the Galerkin approximation of (3.1), multiply it by $2d_{i,n}'(t)$ and sum over $i = 1, 2, 3..., n$. Further, instead of doing integration by parts as in (3.15), we use the formal identities

$$2\big(r(x)u_{n,xx}\big)_{xx}u_n' \equiv 2\big[(r(x)u_{n,xx})_x u_n' - r(x)u_{n,xx}u_{n,x}'\big]_x + \big(r(x)u_{n,xx}^2\big)',$$
$$2\big(\kappa(x)u_{n,xx}'\big)_{xx}u_n' \equiv 2\big[(\kappa(x)u_{n,xx}')_x u_n' - \kappa(x)u_{n,xx}'u_{n,x}'\big]_x + 2\kappa(x)(u_{n,xx}')^2.$$

Integrating by parts, using the initial and boundary conditions of (3.1), we obtain the following *energy identity*:

$$\int_0^\ell \Big(\rho(x)u_n'(t)^2 + r(x)u_{n,xx}^2(t)\Big)dx + 2\int_0^t\int_0^\ell \mu(x)(u_n')^2 dx d\tau$$
$$+2\int_0^t\int_0^\ell \kappa(x)(u_{n,xx}')^2 dx d\tau = 2g(t)u_n(\ell,t) - 2\int_0^t g'(\tau)u_n(\ell,\tau)$$
$$+\int_0^\ell \Big(\rho(x)v_{0,n}^2 + r(x)(u_{0,n})_{xx}^2\Big)dx. \tag{3.20}$$

We employ the ε-inequality (1.15) frequently in the proof. Apply this inequality in the first two terms of the right-hand side of (3.20), and then use the trace inequalities

$$u_n^2(\ell,t) \leq \frac{\ell^3}{3}\int_0^\ell u_{n,xx}^2(x,t)dx, \tag{3.21}$$
$$g^2(t) \leq T\|g'\|_{L^2(0,T)}^2, \quad \text{for all } t \in [0,T], \tag{3.22}$$

where the inequality (3.21) is a consequence of the identity

$$u_n(\ell,t) \equiv \int_0^\ell (\ell - x)u_{n,xx}(x,t)dx, \text{ for all } t \in [0,T].$$

Then we choose the arbitrary constant $\varepsilon > 0$ from the condition $r_0 - \ell^3\,\varepsilon/3 > 0$ as follows: $\varepsilon = 3r_0/(2\ell^3)$. After elementary transformations, we obtain the following main integral inequality

$$\rho_0\int_0^\ell u_n'(t)^2 dx + \frac{r_0}{2}\int_0^\ell u_{n,xx}^2(t)dx + 2\int_0^t\int_0^\ell \mu(x)(u_n')^2 dx d\tau$$
$$+2\int_0^t\int_0^\ell \kappa(x)(u_{n,xx}')^2 dx d\tau \leq \frac{r_0}{2}\int_0^t\int_0^\ell u_{n,xx}^2 dx d\tau + \frac{2\ell^3(1+T)}{3r_0}\|g'\|_{L^2(0,T)}^2$$
$$+\rho_1\|v_{0,n}\|_{L^2(0,\ell)}^2 + r_1\|u_{0,nxx}\|_{L^2(0,\ell)}^2. \tag{3.23}$$

The first consequence of the integral inequality (3.23) is that

$$\int_0^\ell u_{n,xx}^2(t)\,dx \;\;\leq\;\; \int_0^t\int_0^\ell u_{n,xx}^2\,dxd\tau + \frac{2}{r_0}R(g,u_0,v_0),$$

where

$$R(g,u_0,v_0) := \frac{2(1+T)\ell^3}{3r_0}\,\|g'\|^2_{L^2(0,T)} + \rho_1\|v_0\|^2_{L^2(0,\ell)} + r_1\|u_{0,xx}\|^2_{L^2(0,\ell)}.$$

By invoking the Grönwall-Bellmann inequality, we obtain

$$\int_0^\ell u_{n,xx}^2(t)\,dx \;\;\leq\;\; \frac{2}{r_0}\exp(t)R(g,u_0,v_0). \tag{3.24}$$

Integrating inequality (3.24) over $[0,T]$, we arrive at the first required estimate as follows:

$$\|u_{n,xx}\|^2_{L^2(0,T;L^2(0,\ell))} \;\;\leq\;\; \frac{2C_0^2}{r_0}R(g,u_0,v_0), \tag{3.25}$$

where $C_0^2 = (\exp(T)-1)$. Taking maximum over $t\in[0,T]$ in (3.24), we get

$$\max_{t\in[0,T]}\|u_{n,xx}(t)\|^2_{L^2(0,\ell)} \;\;\leq\;\; \frac{2(C_0^2+1)}{r_0}R(g,u_0,v_0). \tag{3.26}$$

Since $\|u_n(t)\|^2_{\mathcal{V}_1^2(0,\ell)} \leq C^*\|u_{n,xx}(t)\|^2_{L^2(0,\ell)}$ by (1.16), using the estimate (3.25), we obtain

$$\|u_n\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} \;\;\leq\;\; \frac{2C^*C_0^2}{r_0}R(g,u_0,v_0). \tag{3.27}$$

The second consequence of (3.23) and (3.25) is the inequality

$$\begin{aligned} 2\int_0^t\int_0^\ell \kappa(x)(u'_{n,xx})^2dxd\tau \;\;&\leq\;\; \frac{r_0}{2}\int_0^t\int_0^\ell u_{n,xx}^2dxd\tau + R(g,u_0,v_0) \\ &\leq\;\; (C_0^2+1)R(g,u_0,v_0). \end{aligned} \tag{3.28}$$

The estimate (3.28) and again the equality of norms lead to the inequality

$$\|u'_n\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} \;\;\leq\;\; \frac{C^*(C_0^2+1)}{2\kappa_0}R(g,u_0,v_0). \tag{3.29}$$

To determine the estimate $\|u''_n\|^2_{L^2(0,T;L^2(0,\ell))}$, we proceed as follows. Multiply the Galerkin approximation of (3.1) by $2d''_{i,n}(t)$ and use the following formal identities

$$\begin{aligned}2\big(r(x)u_{n,xx}\big)_{xx}u''_n &\equiv 2\big[(r(x)u_{n,xx})_x u''_n - r(x)u_{n,xx}u''_{n,x}\big]_x + 2\left(r(x)u_{n,xx}u''_{n,xx}\right),\\ 2\big(\kappa(x)u'_{n,xx}\big)_{xx}u''_n &\equiv 2\big[(\kappa(x)u'_{n,xx})_x u''_n - \kappa(x)u'_{n,xx}u''_{n,x}\big]_x + \left(\kappa(x)(u'_{n,xx})^2\right)'.\end{aligned}$$

Integrate by parts and invoking the initial and boundary conditions of (3.1), we obtain the following second energy identity

$$\begin{aligned}&\int_0^\ell \mu(x)u'_n(t)^2dx + \int_0^\ell \kappa(x)\left(u'_{n,xx}(t)\right)^2dx + 2\int_0^t\int_0^\ell \rho(x)\left(u''_n\right)^2dxd\tau\\ &= 2\int_0^t\int_0^\ell r(x)(u'_{n,xx})^2dxd\tau - 2\int_0^\ell r(x)u_{n,xx}(t)u'_{n,xx}(t)dx + 2g(t)u'_n(\ell,t)\\ &\quad -2\int_0^t g'(\tau)u'_n(\ell,\tau)d\tau + \int_0^\ell \kappa(x)v^2_{0,nxx}dx + \int_0^\ell \mu(x)v^2_{0,n}dx\\ &\quad +2\int_0^\ell r(x)u_{0,nxx}v_{0,nxx}dx. \qquad (3.30)\end{aligned}$$

By applying Cauchy's inequality (1.14), to the second, third, fourth and seventh terms on the right-hand side of (3.30), using the inequality (3.22) and

$$(u'_n(\ell,t))^2 \le \frac{\ell^3}{3}\int_0^\ell (u'_{n,xx})^2dx,$$

we get,

$$\begin{aligned}&\int_0^\ell \mu(x)u'_n(t)^2dx + \int_0^\ell \kappa(x)\left(u'_{n,xx}(t)\right)^2dx + 2\int_0^t\int_0^\ell \rho(x)(u''_n)^2dxd\tau\\ &\le \left(2r_1 + \frac{\ell^3\epsilon}{3}\right)\int_0^t\int_0^\ell (u'_{n,xx})^2dxd\tau + \frac{r_1^2}{\epsilon}\int_0^\ell u^2_{n,xx}(t)dx\\ &\quad + \left(\frac{\ell^3\epsilon}{3}+\epsilon\right)\int_0^\ell (u'_{n,xx}(t))^2dx\\ &\quad + \frac{1+T}{\epsilon}\int_0^t g'(\tau)^2d\tau + \mu_1\|v_0\|^2_{L^2(0,\ell)} + (\kappa_1+1)\|v_{0,xx}\|^2_{L^2(0,\ell)} + r_1^2\|u_{0,xx}\|^2_{L^2(0,\ell)}.\end{aligned}$$

Now choose $\epsilon = 3\kappa_0/2(\ell^3+3)$ from the condition $\kappa_0 - (\ell^3\epsilon/3+\epsilon) > 0$ and invoking the estimates (3.26), (3.28), we obtain

$$\begin{aligned}2\int_0^t\int_0^\ell \rho(x)\left(u''_n\right)^2dxd\tau &\le R_2\ell^3(1+T)\|g'\|^2_{L^2(0,T)} + R_3\|v_0\|^2_{L^2(0,\ell)}\\ &\quad + R_4\|u_{0,xx}\|^2_{L^2(0,\ell)} + (\kappa_1+1)\|v_{0,xx}\|^2_{L^2(0,\ell)},\end{aligned}$$

where

$$R_1 = \left(\frac{4r_1^2(\ell^3+3)}{3r_0} + \frac{\kappa_0\ell^3}{4(\ell^3+3)} + r_1\right)\frac{(C_0^2+1)}{\kappa_0}, \quad R_2 = R_1\frac{2}{3r_0} + \frac{2(l^3+3)}{3l^3\kappa_0},$$
$$R_3 = R_1\rho_1 + \mu_1, \quad R_4 = R_1 r_1 + r_1^2.$$

Choosing $C_1^2 = \max\{R_2, R_3, R_4, \kappa_1 + 1\}$, we get

$$\|u_n''\|_{L^2(0,T,L^2(0,\ell))} \leq \frac{C_1^2}{2\rho_0}\Big[\ell^3(1+T)\|g'\|^2_{L^2(0,T)} + \|v_0\|^2_{L^2(0,\ell)} + \|u_{0,xx}\|^2_{L^2(0,\ell)} + \|v_{0,xx}\|^2_{L^2(0,\ell)}\Big]. \tag{3.31}$$

Consequently, by using the estimates (3.27), (3.25), (3.29) and (3.31), we obtain that the sequences $\{u_n\}$, $\{u_{n,xx}\}$, $\{u'_{n,xx}\}$, $\{u'_n\}$, and $\{u''_n\}$ are bounded in $L^2(0,T;\mathcal{V}_1^2(0,\ell))$, $L^2(0,T;L^2(0,\ell))$, $L^2(0,T;L^2(0,\ell))$, $L^2(0,T;\mathcal{V}_1^2(0,\ell))$, $L^2(0,T;L^2(0,\ell))$ respectively. We now use the Banach-Alaoglu weak compactness theorem (see, Theorem 3.16, [15]) to deduce that there exists a subsequence $\{u_{n_k}\}$ of u_n and functions $u \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$, $u_{xx} \in L^2(0,T;L^2(0,\ell))$, $u'_{xx} \in L^2(0,T;L^2(0,\ell))$, $u' \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$, and $u'' \in L^2(0,T;L^2(0,\ell))$ such that

$$\begin{cases} u_{n_k} \rightharpoonup u & \text{weakly in } L^2(0,T;\mathcal{V}_1^2(0,\ell)) \\ u_{n_k,xx} \rightharpoonup u_{xx} & \text{weakly in } L^2(0,T;L^2(0,\ell)) \\ u'_{n_k,xx} \rightharpoonup u'_{xx} & \text{weakly in } L^2(0,T;L^2(0,\ell)) \\ u'_{n_k} \rightharpoonup u' & \text{weakly in } L^2(0,T;\mathcal{V}_1^2(0,\ell)) \\ u''_{n_k} \rightharpoonup u'' & \text{weakly in } L^2(0,T;L^2(0,\ell)). \end{cases} \tag{3.32}$$

We should be able to derive the weak solution u of the direct problem (3.1) by passing the limit on the weak form (3.15). The solution u also satisfies estimates (3.16)-(3.19).

The uniqueness of weak solution of direct problem (3.1) can be proved by using (3.17) and (3.18). Suppose that there are two weak solutions u_1 and u_2 in $\mathcal{V}_1^2(0,\ell)$ of the direct problem (3.1). Then a function $\mathcal{U}(x,t) = u_1(x,t) - u_2(x,t)$ that solves the direct problem (3.1) with homogeneous initial and boundary conditions. The estimates (3.17) and (3.18) applied to this problem imply that $\|\mathcal{U}\|_{L^\infty(0,T;\mathcal{V}_1^2(0,\ell))} = 0$. Hence the homogeneity of initial and boundary conditions imply that $\mathcal{U}(x,t) = 0$, $\forall\ (x,t) \in (0,\ell) \times (0,T)$.

It is still necessary to verify that $u(t)$ satisfy the initial condition $u(0) = u_0$ and $u_t(0) = v_0$. Taking $u \in C([0,T];\mathcal{V}_1^2(0,\ell))$ and $u_t \in C([0,T];L^2(0,\ell))$ into account, choosing a test function $v \in C^2([0,T];\mathcal{V}_1^2(0,\ell))$ with $v(T) = 0$ and $v'(T) = 0$ and arguing as in Theorem

3 in [9], one can verify the initial data. This completes the proof. □

3.1.1 Regularity of Weak Solutions

In this subsection, we study the regularity of the weak solution, which is required to show the compactness of the input output operator and the Fréchet derivative of the functionals. For simplicity, we take $u_0 = v_0 \equiv 0$.

Theorem 3.2. *Let conditions (3.14) hold. Assume that the following regularity and consistency conditions are also satisfied:*

$$\begin{cases} r,\ \kappa \in H^2(0,\ell),\ \|r\|_{H^2(0,\ell)} \le r_2,\ \|\kappa\|_{H^2(0,\ell)} \le \kappa_2 \\ g \in H^2(0,T),\ g(0)=0,\ g'(0)=0. \end{cases}$$

Then the following estimate holds for regular weak solution of (3.1) with enhanced regularity $u \in H^1(0,T;H^4(0,\ell))$, $u_{tt} \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$, $u_{ttt} \in L^2(0,T;L^2(0,\ell))$:

$$\|u_{ttt}\|^2_{L^2(0,T;L^2(0,\ell))} \le \frac{C_5^2}{2\rho_0}\exp(C_5^2 T)\|g\|^2_{H^2(0,T)}, \tag{3.33}$$

where

$$C_5^2 := \frac{2}{\kappa_0}\max\left(\frac{2(1+T)(\ell^3+3)}{3\kappa_0},\ \left[\frac{\kappa_0\ell^3}{2(\ell^3+3)} + 2r_1 + \frac{2Tr_1^2(\ell^3+3)}{3\kappa_0}\right]\right).$$

Proof. Multiply equation (3.1) by $2u_{xxxxt}$, integrate over $(0,\ell)\times(0,t)$ and apply Cauchy's ϵ-inequality with $\epsilon = \kappa_0/6$, where κ_0 is given in (3.14), we arrive at

$$\begin{aligned} &\int_0^\ell r(x)u_{xxxx}^2(t)dx + \kappa_0\int_0^t\int_0^\ell u_{xxxx\tau}^2 dxd\tau \\ \le\ & \frac{6}{\kappa_0}\Big[\int_0^t\int_0^\ell \rho^2 u_{\tau\tau}^2 dxd\tau + \int_0^t\int_0^\ell \mu^2 u_\tau^2 dxd\tau + 4\int_0^t\int_0^\ell (r')^2 u_{xxx}^2 dxd\tau \\ &+4\int_0^t\int_0^\ell (\kappa')^2 u_{xxx\tau}^2 dxd\tau + \int_0^t\int_0^\ell (r'')^2 u_{xx}^2 dxd\tau + \int_0^t\int_0^\ell (\kappa'')^2 u_{xx\tau}^2 dxd\tau\Big] \\ &=: \frac{6}{\kappa_0}\sum_{i=1}^6 I_i. \end{aligned} \tag{3.34}$$

Notice that,

$$I_3 \le 4\int_0^t \|r'\|^2_{L^\infty(0,\ell)}\|u_{xxx}(\tau)\|^2_{L^2(0,\ell)}d\tau \le 4\bar{C}\|r'\|^2_{H^1(0,\ell)}\|u_{xxx}\|^2_{L^2(0,T;L^2(0,\ell))}. \tag{3.35}$$

Similarly,

$$I_4 \leq 4\bar{C}\kappa_2^2\|u_{xxxt}\|^2_{L^2(0,T;L^2(0,\ell))},$$

where we used the fact that $H^1(0,\ell)$ is continuously embedded in $L^\infty(0,\ell)$, that is,

$$\|u\|_{L^\infty(0,\ell)} \leq \bar{C}(\ell)\|u\|_{H^1(0,\ell)}, \;\; \bar{C}(\ell) = \sqrt{2}\max\{1/\sqrt{\ell},\ell\},$$

(see, Theorem 1.3). Further, the integrals I_5 and I_6 can be estimated as follows

$$\begin{aligned} I_5 &\leq \int_0^t \|u_{xx}(\tau)\|^2_{L^\infty(0,\ell)}\|r''\|^2_{L^2(0,\ell)}d\tau \leq \bar{C}r_2^2\|u\|^2_{L^2(0,t;H^3(0,\ell))} \\ &= \bar{C}r_2^2\Big(\|u\|^2_{L^2(0,t;\mathcal{V}_1^2(0,\ell))}+\|u_{xxx}\|^2_{L^2(0,t;L^2(0,\ell))}\Big), \\ I_6 &\leq \bar{C}\kappa_2^2\|u_t\|^2_{L^2(0,t;H^3(0,\ell))} = \bar{C}\kappa_2^2\Big(\|u_t\|^2_{L^2(0,t;\mathcal{V}_1^2(0,\ell))}+\|u_{xxx\tau}\|^2_{L^2(0,t;L^2(0,\ell))}\Big). \end{aligned}$$

By invoking the result in Theorem 1.4, let us estimate the integrals $\|u_{xxx}\|^2_{L^2(0,T;L^2(0,\ell))}$ and $\|u_{xxxt}\|^2_{L^2(0,T;L^2(0,\ell))}$. For any $\epsilon_1 > 0$, we obtain that

$$\int_0^t\int_0^\ell u^2_{xxx\tau}dxd\tau \;\; \leq \;\; 2\epsilon_1\int_0^t\int_0^\ell u^2_{xxxx\tau}dxd\tau + \tilde{C}(\epsilon_1,\ell)\int_0^t\int_0^\ell u^2_\tau dxd\tau, \quad (3.36)$$

and similar estimate holds for $\|u_{xxx}\|^2_{L^2(0,T;L^2(0,\ell))}$ with $\epsilon_2 > 0$. Using the above estimates (3.35)-(3.36) in (3.34) and choosing $\epsilon_1 = \kappa_0^2/(120\kappa_2^2\bar{C})$, $\epsilon_2 = 1/2$, we obtain

$$\begin{aligned} \sum_{i=1}^6 I_i \;\; \leq \;\; & \frac{6}{\kappa_0}\Big(\rho_1^2\|u_{tt}\|^2_{L^2(0,T;L^2(0,\ell))} + \Big(\mu_1^2 + \bar{C}\kappa_2^2(1+5\tilde{C})\Big)\|u_t\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} \\ & +\bar{C}r_2^2(1+5\tilde{C})\;\|u\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} + 5\bar{C}r_2^2\int_0^t\int_0^\ell u^2_{xxxx}\,dxd\tau\Big) \\ & +\frac{\kappa_0}{2}\int_0^t\int_0^\ell u^2_{xxxx\tau}\,dxd\tau. \end{aligned}$$

Making use of the estimates (3.17)-(3.19), we further have

$$\begin{aligned} r_0\int_0^\ell u^2_{xxxx}(t)dx \;\; + \;\; & \frac{\kappa_0}{2}\int_0^t\int_0^\ell u^2_{xxxx\tau}dxd\tau \\ & \leq C_2^2\|g'\|^2_{L^2(0,T)} + \frac{30\bar{C}r_2^2}{\kappa_0}\int_0^t\int_0^\ell u^2_{xxxx}dxd\tau, \qquad (3.37) \end{aligned}$$

with the constant

$$C_2^2 = \frac{6\ell^3(1+T)}{\kappa_0} \times \max\left(\frac{\rho_1^2 C_1^2}{2\rho_0}, \frac{4\bar{C}r_2^2(1+5\tilde{C})C^*C_0^2}{3r_0^2}, \right.$$
$$\left. \left[\mu_1^2 + (1+5\tilde{C})\bar{C}\kappa_2^2\right]\frac{C^*(C_0^2+1)}{3r_0\kappa_0}\right).$$

Applying Grönwall's inequality and then integrating over $(0,T)$, we get

$$\|u_{xxxx}\|^2_{L^2(0,T;L^2(0,\ell))} \leq C_3^2\|g'\|^2_{L^2(0,T)}, \tag{3.38}$$

where $C_3^2 = \frac{C_2^2\kappa_0}{30\bar{C}r_2^2}\left[\exp\left(\frac{30\bar{C}r_2^2}{\kappa_0 r_0}T\right) - 1\right]$. Next, let us note that

$$\|u\|^2_{L^2(0,T;H^4(0,\ell))} = \|u\|^2_{L^2(0,T;H^3(0,\ell))} + \|u_{xxxx}\|^2_{L^2(0,T;L^2(0,\ell))}, \tag{3.39}$$
$$\|u_t\|^2_{L^2(0,T;H^4(0,\ell))} = \|u_t\|^2_{L^2(0,T;H^3(0,\ell))} + \|u_{xxxxt}\|^2_{L^2(0,T;L^2(0,\ell))}. \tag{3.40}$$

To estimate the first term on the right-hand side of (3.39), we use Ehrling's lemma (see, Theorem 1.5), that for any $\epsilon_3 > 0$, there exist $C(\epsilon_3)$ such that for any $u \in L^2(0,T;H^4(0,\ell))$, we get

$$\|u\|^2_{L^2(0,T;H^3(0,\ell))} \leq \epsilon_3\|u\|^2_{L^2(0,T;H^4(0,\ell))} + C(\lambda_3)\|u\|^2_{L^2(0,T;L^2(0,\ell))}.$$

Choosing $\epsilon_3 = \frac{1}{2}$, substituting this into (3.39) and using (3.38), 3.17, we obtain

$$\|u\|^2_{L^2(0,T;H^4(0,\ell))} \leq 2\left(C_3^2 + \frac{4CC^*C_0^2}{3r_0^2}(1+T)\ell^3\right)\|g'\|^2_{L^2(0,T)}. \tag{3.41}$$

The second consequence of (3.37) and (3.38) is the following

$$\|u_{xxxxt}\|^2_{L^2(0,T;L^2(0,\ell))} \leq \frac{2}{\kappa_0}\left(C_2^2 + \frac{30\bar{C}C_3^2r_2^2}{\kappa_0}\right)\|g'\|^2_{L^2(0,T)}.$$

Again invoking Ehrling's lemma for $u_t \in L^2(0,T;H^4(0,\ell))$, (3.40) and 3.18, we get

$$\|u_t\|^2_{L^2(0,T;H^4(0,\ell))} \leq C_4^2\|g'\|^2_{L^2(0,T)}, \tag{3.42}$$

where $C_4^2 = \frac{2}{\kappa_0}\left(2\left(C_2^2 + \frac{30\bar{C}C_3^2r_2^2}{\kappa_0}\right) + \frac{CC^*(C_0^2+1)\ell^3(1+T)}{3r_0}\right)$. Consequently, from (3.41) and (3.42), we infer that $u \in H^1(0,T;H^4(0,\ell))$.

To estimate $\|u_{ttt}\|^2_{L^2(0,T;L^2(0,\ell))}$ and $\|u_{xxtt}\|^2_{L^2(0,T;L^2(0,\ell))}$, we proceed as follows.

Formally differentiate (3.1) with respect to time and multiply by $2u_{ttt}$, use the crucial identities

$$2\big(r(x)u_{xx}\big)_{xxt}u_{ttt} \equiv 2\big[(r(x)u_{xx})_{xt}u_{ttt} - r(x)u_{xxt}u_{xttt}\big]_x + 2r(x)u_{xxt}u_{xxttt},$$
$$2\big(\kappa(x)u_{xxt}\big)_{xxt}u_{ttt} \equiv 2\big[(\kappa(x)u_{xxt})_{xt}u_{ttt} - \kappa(x)u_{xxtt}u_{xttt}\big]_x + \kappa(x)(u^2_{xxtt})_t,$$

and integrate over $(0,\ell)\times(0,t), t \in (0,T)$. Then integrating by parts using the initial and boundary conditions of (3.1), we obtain

$$\begin{aligned}
&\int_0^\ell \Big(\mu(x)u_{tt}(t)^2 + \kappa(x)u_{xxtt}(t)^2\Big)dx + 2\int_0^t\int_0^\ell \rho(x)u^2_{\tau\tau\tau}dxd\tau \\
&= \ -2\int_0^t\int_0^\ell r(x)(u_{xx\tau}u_{xx\tau\tau\tau})\ dxd\tau + 2\int_0^t g'(\tau)u_{\tau\tau\tau}(\ell,\tau)\ d\tau \\
&\quad + \int_0^\ell \mu(x)u^2_{tt}(x,0^+)dx + \int_0^\ell \kappa(x)u^2_{xxtt}(x,0^+)dx.
\end{aligned} \tag{3.43}$$

Let us evaluate the third and fourth right-hand side integrals by using the initial data given in (3.1). First, we deduce that

$$\begin{aligned}
\int_0^\ell \mu(x)u^2_{tt}(x,0^+)dx \ &= \ \int_0^\ell \frac{\mu(x)}{\rho^2(x)}\Big(\mu(x)u_t(x,0^+) + (r(x)u_{xx}(x,0^+))_{xx} \\
&\quad + (\kappa(x)u_{xxt}(x,0^+))_{xx}\Big)^2 dx = 0,
\end{aligned}$$

since $u_t(x,0^+) = u_{xx}(x,0^+) = u_{xxt}(x,0^+) = 0$. Similarly, we also obtain that

$$\int_0^\ell \kappa(x)u^2_{xxtt}(x,0^+)dx = 0.$$

Now the integration by parts with respect to time over the first two right-hand side integrals of (3.43) leads to the following

$$\begin{aligned}
&-2\int_0^t\int_0^\ell r(x)(u_{xx\tau}u_{xx\tau\tau\tau})\ dxd\tau + 2\int_0^t g'(\tau)u_{\tau\tau\tau}(\ell,\tau)\ d\tau \\
&= \ 2g'(t)u_{tt}(\ell,t) - 2\int_0^t g''(\tau)u_{\tau\tau}(\ell,\tau)d\tau + 2\int_0^t\int_0^\ell r(x)u^2_{xx\tau\tau}dxd\tau \\
&\quad -2\int_0^\ell r(x)u_{xxt}(t)u_{xxtt}(t)dx.
\end{aligned} \tag{3.44}$$

By substituting the identity (3.44) into (3.43), applying the trace inequalities (3.21), (3.22) and Cauchy's inequality, we obtain that

$$\int_0^\ell \kappa(x) u_{xxtt}(t)^2 dx + 2\int_0^t\int_0^\ell \rho(x) u_{\tau\tau\tau}^2 dx d\tau$$
$$\leq \frac{1+T}{\epsilon}\int_0^t g''(\tau)^2 d\tau + \Big(\frac{\ell^3\epsilon}{3}+\epsilon\Big)\int_0^\ell u_{xxtt}(t)^2 dx$$
$$+\Big(\frac{\ell^3\epsilon}{3}+2r_1\Big)\int_0^t\int_0^\ell u_{xx\tau\tau}^2\,dxd\tau + \frac{r_1^2}{\epsilon}\int_0^\ell u_{xxt}(t)^2 dx.$$

Taking $\epsilon = \frac{3\kappa_0}{2(\ell^3+3)}$ and employing the trace inequality $u_{xxt}^2(t) \leq T\int_0^t u_{xx\tau\tau}^2 d\tau$ give

$$\frac{\kappa_0}{2}\int_0^\ell u_{xxtt}(t)^2 dx + 2\rho_0\int_0^t\int_0^\ell u_{\tau\tau\tau}^2 dx d\tau \leq \frac{2(1+T)(\ell^3+3)}{3\kappa_0}\int_0^t g''(\tau)^2 d\tau$$
$$+\left(\frac{\kappa_0\ell^3}{2(\ell^3+3)} + 2r_1 + \frac{2Tr_1^2(\ell^3+3)}{3\kappa_0}\right)\int_0^t\int_0^\ell u_{xx\tau\tau}^2\,dxd\tau. \quad (3.45)$$

By setting $C_5^2 := \frac{2}{\kappa_0}\max\left(\frac{2(1+T)(\ell^3+3)}{3\kappa_0},\ \left[\frac{\kappa_0\ell^3}{2(\ell^3+3)} + 2r_1 + \frac{2Tr_1^2(\ell^3+3)}{3\kappa_0}\right]\right)$, and applying Grönwall's inequality, we get $\int_0^\ell u_{xxtt}(t)^2 dx \leq C_5^2\|g''\|_{L^2(0,T)}^2 \exp(C_5^2 t)$. This implies that

$$\|u_{xxtt}\|_{L^2(0,T;L^2(0,\ell))}^2 \leq \|g''\|_{L^2(0,T)}^2\left[\exp(C_5^2 T) - 1\right]. \quad (3.46)$$

Substituting (3.46) into (3.45), we conclude the proof of the estimate (3.33). □

Theorem 3.3. *Let the conditions of Theorem 3.2 hold. Additionally assume that, the input $g(t)$ and the coefficients meet the following regularity and consistency conditions:*

$$\begin{cases} r,\ \kappa \in H^4(0,\ell),\ \|r\|_{H^4(0,\ell)} \leq r_3,\ \|\kappa\|_{H^4(0,\ell)} \leq \kappa_3 \\ \rho,\ \mu \in H^2(0,\ell),\ \|\rho\|_{H^2(0,\ell)} \leq \rho_2,\ \|\mu\|_{H^2(0,\ell)} \leq \mu_2 \\ g \in H^3(0,T),\ g(0) = g'(0) = g''(0) = 0. \end{cases}$$

Then for the regular weak solution with improved regularity, we have the enhanced regularity $u \in H^1(0,T;H^6(0,\ell))$, $u_{tt} \in L^2(0,T;H^4(0,\ell))$, $u_{ttt} \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$, $u_{tttt} \in L^2(0,T;L^2(0,\ell))$, and the estimate

$$\|u_{tttt}\|_{L^2(0,T;L^2(0,\ell))}^2 \leq \frac{C_5^2}{2\rho_0}\exp(C_5^2 T)\|g\|_{H^3(0,T)}^2. \quad (3.47)$$

Proof. The proof of this theorem can be completed by following the lines of arguments of Theorem 3.2. □

Remark 3.1. From Theorem 3.1 to Theorem 3.3 it follows that the weak and regular weak solutions of the Euler-Bernoulli beam equation with Kelvin-Voigt damping term $(\kappa(x)u_{xxt})_{xx})$ has more enhanced regularity property than corresponding weak solutions of this equation without this term.

3.2 Solvability of regularized inverse problems

In this section, using the regularity of the solution to the direct problem (3.1), we prove the compactness as well as the Lipschitz continuity of the input-output operators Φ and Ψ. The lower semi-continuity of the Tikhonov functionals $\mathcal{J}_1$ and $\mathcal{J}_2$ will result from the Lipschitz continuity of operators which lead to the existence of a minimizer for these functionals. This will in turn solve IBVP-1 and IBVP-2.

3.2.1 Ill-posedness of the problems IBVP-1 and IBVP-2

In order to attain the compactness of the input-output operator Ψ associated with IBVP-2, we are employing more regularity on the solution to the direct problem (3.1) as stated in Theorem 3.2. The proof of Theorem 3.2 requires only $H^2(0,T)$ regularity of admissible source inputs, instead of $H^3(0,T)$ regularity introduced in (3.9), as follows

$$\mathcal{G}_2 = \{g \in H^2(0,T) : g(0) = g'(0) = 0,\ \|g\|_{H^2(0,T)} \le \mathfrak{K}_2\}.$$

But one may notice that the Lipschitz continuity of Ψ can be proved on the admissible source $\mathcal{G}_1$ itself.

Proposition 3.1. *Suppose the conditions of Theorem 3.2 hold. Then the Neumann-Dirchlet operator* $\Phi : \mathcal{G}_1 \subset H^1(0,T) \mapsto L^2(0,T)$ *and Neumann-Neumann operator* $\Psi : \mathcal{G}_2 \subset H^2(0,T) \mapsto L^2(0,T)$ *defined by (3.4), (3.9) respectively are compact operators. Furthermore* Φ *and* Ψ *are Lipschitz continuous:*

$$\begin{aligned} &\|\Phi(g_1) - \Phi(g_2)\|_{L^2(0,T)} \le L_0\|g_1 - g_2\|_{H^1(0,T)},\ \ \forall\, g_1, g_2 \in \mathcal{G}_1,\\ &\|\Psi(g_1) - \Psi(g_2)\|_{L^2(0,T)} \le L_1\|g_1 - g_2\|_{H^1(0,T)},\ \ \forall\, g_1, g_2 \in \mathcal{G}_1, \end{aligned} \tag{3.48}$$

with the Lipschitz constants
$L_0^2 = \frac{4\ell^6(1+T)C_0^2}{9r_0^2},\ L_1^2 = C_7^2\left[1 + \left(\frac{C^*\left(C_0^2+1\right)}{3\kappa_0 r_0} + \frac{C_1^2}{2\rho_0}\right)(1+T)\ell^3\right]$, *where*
$C_7^2 = 2\ell^2 \max\left(1, \frac{2\ell}{3}(\rho_1^2 + \mu_1^2)\right)$ *and* $C^*, C_0 > 0$ *are the constants defined in Theorem 3.1.*

Proof. Using the similar arguments given in Lemma 3 and Lemma 5 of [47], we can show that the input-output operator Φ is compact and Lipschitz continuous with Lipschitz constant L_0. We only show that the input-output operator Ψ is compact and Lipschitz continuous with Lipschitz constant L_1.

Denote by $\{u^{(m)}(x,t)\}$, where $u^{(m)}(x,t) := u(x,t;g_m)$, the sequence of regular weak solutions of (3.1) corresponding to the sequence of inputs $\{g_m\} \subset \mathcal{G}_2$, $m = 1,2,...$, bounded in the norm of $H^2(0,T)$. Then $-(r(0)u^{(m)}_{xx}(0,t) + \kappa(0)u^{(m)}_{xxt}(0,t))$ denotes the sequence of corresponding outputs. We need to prove that this sequence is relatively compact in $L^2(0,T)$, that is, by Rellich-Kondrachov compactness theorem (see, Theorem 1.1), it is enough to show that the above sequence of outputs is bounded in the norm of $H^1(0,T)$. To this end, we estimate the norms $\|r(0)u^{(m)}_{xx}(0,\cdot) + \kappa(0)u^{(m)}_{xxt}(0,\cdot)\|_{L^2(0,T)}$ and $\|r(0)u^{(m)}_{xxt}(0,\cdot) + \kappa(0)u^{(m)}_{xxtt}(0,\cdot)\|_{L^2(0,T)}$.

Using the identity

$$r(0)u^{(m)}_{xx}(0,t) + \kappa(0)u^{(m)}_{xxt}(0,t) = \ell g_m(t) + \int_0^\ell x\left[\rho(x)u^{(m)}_{tt} + \mu(x)u^{(m)}_t\right]dx, \quad (3.49)$$

which is obtained as a result of integrating (3.1) first over the interval (x,ℓ) and then over the interval $(0,\ell)$, and applying Hölder's inequality, we deduce that

$$\begin{aligned}&\|r(0)u^{(m)}_{xx}(0,.) + \kappa(0)u^{(m)}_{xxt}(0,.)\|^2_{L^2(0,T)}\\ &\quad\le C_7^2\left(\|g_m\|^2_{L^2(0,T)} + \|u^{(m)}_{tt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|u^{(m)}_t\|^2_{L^2(0,T;L^2(0,\ell))}\right), \end{aligned}\quad (3.50)$$

where $C_7^2 = 2\ell^2\max\{1, \frac{2\ell}{3}(\rho_1^2 + \mu_1^2)\}$. We infer from the estimates (3.18) and (3.19) that left-hand-side norm in (3.50) is bounded in the norm of $L^2(0,T)$.

Next, using the analogue

$$r(0)u^{(m)}_{xxt}(0,t) + \kappa(0)u^{(m)}_{xxtt}(0,t) = \ell g'_m(t) + \int_0^\ell x\left[\rho(x)u^{(m)}_{ttt} + \mu(x)u^{(m)}_{tt}\right]dx$$

of identity (3.49), we deduce the estimate

$$\begin{aligned}&\|r(0)u^{(m)}_{xxt}(0,.) + \kappa(0)u^{(m)}_{xxtt}(0,.)\|^2_{L^2(0,T)}\\ &\quad\le C_7^2\left(\|g'_m\|^2_{L^2(0,T)} + \|u^{(m)}_{ttt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|u^{(m)}_{tt}\|^2_{L^2(0,T;L^2(0,\ell))}\right), \end{aligned}\quad (3.51)$$

with the same constant $C_7 > 0$. By the estimates (3.19) and (3.33), the left-hand-side norm in (3.51) is also bounded in the norm of $L^2(0,T)$.

Thus, estimates (3.50) and (3.51) imply that $r(0)u_{xx}^{(m)}(0,t) + \kappa(0)u_{xxt}^{(m)}(0,t)$ is bounded in the norm of $H^1(0,T)$. This implies that Ψ is a compact operator.

In order to prove the Lipschitz continuity of Ψ, we proceed as follows. Let $g_1, g_2 \in \mathcal{G}_1$ be the given inputs and $u(x,t;g_1)$, $u(x,t;g_2)$ be the corresponding solutions of direct problem (3.1). Then $\delta u(x,t) = u(x,t;g_1) - u(x,t;g_2)$ solves the problem

$$\begin{cases} \rho(x)\delta u_{tt} + \mu(x)\delta u_t + (r(x)\delta u_{xx})_{xx} + (\kappa(x)\delta u_{xxt})_{xx} = 0, \ (x,t) \in \Omega_T, \\ \delta u(x,0) = 0, \qquad \delta u_t(x,0) = 0, \qquad\qquad x \in (0,\ell), \\ \delta u(0,t) \ = 0, \qquad \delta u_x(0,t) = 0, \qquad\qquad t \in [0,T], \\ [r(x)\delta u_{xx} + \kappa(x)\delta u_{xxt}]_{x=\ell} \ = 0, \\ \qquad - \left[\left(r(x)\delta u_{xx} + \kappa(x)\delta u_{xxt}\right)_x\right]_{x=\ell} = \delta g(t), \ t \in [0,T], \end{cases} \tag{3.52}$$

where $\delta g(t) = g_1(t) - g_2(t)$. By the definition of input-output operator Ψ,

$$\|\Psi(g_1) - \Psi(g_2)\|^2_{L^2(0,T)} = \|r(0)\delta u_{xx}(0,.) + \kappa(0)\delta u_{xxt}(0,.)\|^2_{L^2(0,T)}.$$

Using estimate (3.50) to the solution $\delta u(x,t)$ of problem (3.52) we arrive at the desired estimate (3.48). Hence the proof. □

The compactness of input-output operators Φ and Ψ means to the ill-posedness of both inverse problems IBVP1 and IBVP2 (see,[40] and also, [50], Lemma 1.3.1).

3.2.2 Existence and uniqueness of solutions to the minimization problems

Using the Lipschitz continuity of the input-output operators Φ and Ψ, we show that the existence and uniqueness of minimizer for the functionals $\mathcal{J}_{1\alpha}$ and $\mathcal{J}_{2\alpha}$ corresponding to IBVP-1 and IBVP-2 respectively.

Theorem 3.4. *Suppose the conditions (3.14) hold true. Then, both minimization problems (3.6) and (3.11) have a solution on the admissible source of inputs $\mathcal{G}_1$.*

Proof. We only prove the existence of minimizer for the functional $\mathcal{J}_2$. By the same arguments, it can be proved for $\mathcal{J}_1$ as well.
Let $u(x,t;g_1)$, $u(x,t;g_2)$ be the solutions of (3.1) corresponding to the inputs $g_1, g_2 \in \mathcal{G}_1$ respectively. Then the function $\delta u(x,t)$ solves the problem (3.52) with input

$\delta g(t) = g_1(t) - g_2(t)$. Since

$$\Big|\mathcal{J}_2(g_1) - \mathcal{J}_2(g_2)\Big|^2 = \Big|\sqrt{\mathcal{J}_2(g_1)} + \sqrt{\mathcal{J}_2(g_2)}\Big|^2 \Big|\sqrt{\mathcal{J}_2(g_1)} - \sqrt{\mathcal{J}_2(g_2)}\Big|^2, \qquad (3.53)$$

and appealing to the estimate (3.48), we get

$$\begin{aligned}\Big|\sqrt{\mathcal{J}_2(g_1)} - \sqrt{\mathcal{J}_2(g_2)}\Big|^2 &= \frac{1}{2}\Big|\|\Psi(g_1) - \omega\|_{L^2(0,T)} - \|\Psi(g_2) - \omega\|_{L^2(0,T)}\Big|^2 \\ &\le \frac{1}{2}\|\Psi(g_1) - \Psi(g_2)\|^2_{L^2(0,T)} \le \frac{L_1^2}{2}\,\|g_1 - g_2\|^2_{H^1(0,T)}, \qquad (3.54)\end{aligned}$$

where L_1^2 is defined in Proposition 3.1. Using the same step done for (3.50), we can show from (3.18) and (3.19) that

$$\begin{aligned}\|\Psi(g_m)\|^2_{L^2(0,T)} \;\le\; & C_7^2\Big(\|g_m\|^2_{L^2(0,T)} + \|u_{tt}(.,.;g_m)\|^2_{L^2(0,T;L^2(0,\ell))} \\ & + \|u_t(.,.;g_m)\|^2_{L^2(0,T;L^2(0,\ell))}\Big) \le L_1^2\,\|g_m\|^2_{H^1(0,T)}, \qquad (3.55)\end{aligned}$$

$m = 1, 2$. Applying triangle inequality and (3.55) , we obtain

$$\begin{aligned}\Big|\sqrt{\mathcal{J}_2(g_1)} + \sqrt{\mathcal{J}_2(g_2)}\Big|^2 \;\le\;& 2\left(\|\Psi(g_1)\|^2_{L^2(0,T)} + \|\Psi(g_2)\|^2_{L^2(0,T)} + 2\|\omega\|^2_{L^2(0,T)}\right) \\ \le\;& 4\left(L_1^2\,\mathfrak{K} + \|\omega\|^2_{L^2(0,T)}\right), \qquad (3.56)\end{aligned}$$

since $\|g_m\|^2_{H^1(0,T)} \le \mathfrak{K}$, for $m = 1, 2$. Consequently, (3.53), (3.54) and (3.56) will lead to the estimate

$$\Big|\mathcal{J}_2(g_1) - \mathcal{J}_2(g_2)\Big|^2 \le 2L_1^2\left(L_1^2\,\mathfrak{K} + \|\omega\|^2_{L^2(0,T)}\right)\|g_1 - g_2\|^2_{H^1(0,T)},$$

whence the functional $\mathcal{J}_2$ is weakly lower-semi continuous on a nonempty closed convex set $\mathcal{G}_1$ ([101], Section 2.5, Lemma 5). Hence by the generalized Weierstrass theorem (see, Theorem 1.8) the functional $\mathcal{J}_2(g)$ has a minimizer $g \in \mathcal{G}_1$. □

Remark 3.2. By a careful inspection, we note that the existence of solutions to IBVP-2 in Theorem 3.4 is proved with inputs only in $\mathcal{G}_1$, but not with $\mathcal{G}_3$ defined in (3.9). Further, the following Corollary 3.1 can also be proved for the Tikhonov functionals $\mathcal{J}_{1\alpha}$ and $\mathcal{J}_{2\alpha}$ with regularizer $\|g\|^2_{L^2(0,T)}$ instead of $\|g'\|^2_{L^2(0,T)}$ and $\|g'''\|^2_{L^2(0,T)}$, and also can be justified that both IBVP-1, IBVP-2 have unique solutions on $\mathcal{G}_1$.

Corollary 3.1. *Assume that the conditions (3.14) hold true. Then the regularized Tikhonov* *functionals* $\mathcal{J}_{1\alpha}(g),\ \mathcal{J}_{2\alpha}(g)$ *defined by (3.7) and (3.12) have a unique minimizer on* $\mathcal{G}_1$ *and* $\mathcal{G}_3$, *respectively.*

Proof. By Theorem 3.4, the functional $\mathcal{J}_1(g)$ is lower semi-continuous. Further, the regularized Tikhonov functional $\mathcal{J}_{1\alpha}(g_n)$ corresponding to g_n defined in (3.7) satisfy $\mathcal{J}_{1\alpha}(g) \leq \liminf_{n\to\infty} \mathcal{J}_{1\alpha,n}(g)$, as $g_n \rightharpoonup g$ in $\mathcal{G}_1$, whence $\mathcal{J}_{1\alpha}(g)$ is lower semi-continuous. By the linearity of direct problem (3.1), we have

$$u(x,t;\nu g_1 + (1-\nu)g_2) = \nu u(x,t;g_1) + (1-\nu)u(x,t;g_2),\ \nu \in (0,1),$$

and hence, one can get that

$$\begin{aligned} \mathcal{J}_{1\alpha}(\nu g_1 + (1-\nu)g_2) &= \mathcal{J}_1(\nu g_1 + (1-\nu)g_2) + \frac{\alpha}{2}\|\nu g_1' + (1-\nu)g_2'\|^2_{L^2(0,T)} \\ &< \nu\mathcal{J}_{1\alpha}(g_1) + (1-\nu)\mathcal{J}_{1\alpha}(g_2), \forall\, g_1, g_2 \in \mathcal{G}_1, \nu \in (0,1). \end{aligned}$$

It shows that the functional $\mathcal{J}_{1\alpha}(g)$ is strictly convex on $\mathcal{G}_1$. By combining the above arguments and using the generalized Weierstrass theorem, we conclude that the functional $\mathcal{J}_{1\alpha}(g)$ has a unique minimizer. By the similar arguments, we can prove that the regularized functional $\mathcal{J}_{2\alpha}$ defined by (3.12) has a unique minimizer in admissible source inputs $\mathcal{G}_3$. □

Remark 3.3. We can directly prove the uniqueness result for the IBVP-1 as follows. Let $g_1,\ g_2 \in \mathcal{G}_1$ be two arbitrary given functions and $u_k(x,t) := u(x,t;g_k),\ k = 1,2$, be the corresponding solutions of direct problem (3.1). Suppose that there exist two functions $g_1,\ g_2 \in \mathcal{G}_1$, which are not identically zero such that $g_1(t) \neq g_2(t)$, but the measurements $\nu_1(t) = \nu_2(t),\ \forall\, t \in [0,T]$, where we recall that $\nu_1(t) = u_1(\ell,t), \nu_2(t) = u_2(\ell,t)$. Then the function $z(x,t) = u_1(x,t) - u_2(x,t)$ solves the initial boundary value problem (3.1) with $g(t)$ replaced by $\tilde{g}(t) = g_1(t) - g_2(t)$. Now consider the equation

$$\rho(x)z_{tt} + \mu(x)z_t + (r(x)z_{xx})_{xx} + (\kappa(x)z_{xxt})_{xx} = 0. \tag{3.57}$$

Multiply (3.57) by $2z_t(x,t)$, use the formal identities (3.20), then integrating by parts, using the initial and boundary conditions, we obtain the following integral identity

$$\begin{aligned} &\int_0^\ell \Big(\rho(x)z_t(t)^2 + r(x)z_{xx}^2(t)\Big)dx + 2\int_0^t\int_0^\ell \mu(x)(z_t)^2 dx d\tau \\ &\quad +2\int_0^t\int_0^\ell \kappa(x)(z_{xxt})^2 dx d\tau = 2\tilde{g}(t)z(\ell,t) - 2\int_0^t \tilde{g}'(\tau)z(\ell,\tau). \end{aligned} \tag{3.58}$$

By the above assumption $z(\ell,t) = u_1(\ell,t) - u_2(\ell,t) = 0, t \in [0,T]$, which implies that right-hand side in (3.58) is zero. Since the initial and boundary conditions are homogeneous, $z(x,t) \equiv 0,\ (x,t) \in \Omega_T$. This contradiction completes the uniqueness of IBVP-1.

Since for the IBVP-2, we consider measured data as bending moment $\omega(t)$ (see, (3.8)) the identity (3.58) cannot be used for the uniqueness of the solution. Hence, it seems that one cannot directly get the uniqueness of the solution to the IBVP-2.

3.3 Fréchet differentiability of the Tikhonov functionals

In this section, we show that the functionals $\mathcal{J}_1(g)$ and $\mathcal{J}_2(g)$ are Fréchet differentiable on the admissible sources $\mathcal{G}_1$ and $\mathcal{G}_3$, respectively. The Fréchet derivatives are expressed in terms of the weak solutions of associated adjoint problems with boundary data given in terms of measured data (3.2) and (3.10).

For any $g, \delta g \in \mathcal{G}_1$ and $g, \delta g \in \mathcal{G}_3$ the increments of the respective functionals $\mathcal{J}_1(g)$ and $\mathcal{J}_2(g)$ denoted by $\delta\mathcal{J}_m(g) = \mathcal{J}_m(g+\delta g) - \mathcal{J}_m(g),\ m = 1,2$ satisfies the following identities

$$\begin{aligned}
\delta\mathcal{J}_1(g) &= \int_0^T \left[u(\ell,t;g) - \nu(t)\right]\delta u(\ell,t)dt + \frac{1}{2}\int_0^T \delta u(\ell,t)^2 dt,\\
\delta\mathcal{J}_2(g) &= \int_0^T \Big(r(0)u_{xx}(0,t;g) + \kappa(0)u_{xxt}(0,t;g) + \omega(t)\Big)\Big(r(0)\delta u_{xx}(0,t)\\
&\quad +\kappa(0)\delta u_{xxt}(0,t)\Big)dt\\
&\quad +\frac{1}{2}\int_0^T \Big(r(0)\delta u_{xx}(0,t) + \kappa(0)\delta u_{xxt}(0,t)\Big)^2 dt,
\end{aligned}$$

where $\delta u(x,t) := u(x,t,g+\delta g) - u(x,t,g)$ solves the problem (3.52).

The following lemma shows the representation of first integral of $\delta\mathcal{J}_1(g)$ and $\delta\mathcal{J}_2(g)$ in terms of solution of associated adjoint problems.

Lemma 3.1. *Let assumptions (3.14) hold true. Then the following integral relationships between the direct and adjoint problems hold:*

(i1) *For any $g,\ \delta g \in \mathcal{G}_1$ of IBVP-1, we have*

$$\int_0^T \delta u(\ell,t)\xi(t)dt = \int_0^T \phi(\ell,t)\delta g(t)dt, \tag{3.59}$$

where $\phi(x,t)$ is the solution of adjoint problem

$$\begin{cases} \rho(x)\phi_{tt} - \mu(x)\phi_t + (r(x)\phi_{xx})_{xx} - (\kappa(x)\phi_{xxt})_{xx} = 0, \ (x,t) \in \Omega_T, \\ \phi(x,T) = 0, \ \phi_t(x,T) = 0, \qquad\qquad x \in (0,\ell), \\ \phi(0,t) \ = 0, \ \phi_x(0,t) = 0, \qquad\qquad t \in [0,T], \\ [r(x)\phi_{xx} - \kappa(x)\phi_{xxt}]_{x=\ell} \quad = 0, \\ \qquad \big[\big(-r(x)\phi_{xx} + \kappa(x)\phi_{xxt}\big)_x\big]_{x=\ell} = \xi(t), \quad t \in [0,T], \end{cases} \tag{3.60}$$

with Neumann input $\xi \in L^2(0,T)$ and δu is the solution of (3.52).

(i2) *For any $g, \ \delta g \in \mathcal{G}_3$ of IBVP-2, we get*

$$\int_0^T \Big(r(0)\delta u_{xx}(0,t) + \kappa(0)\delta u_{xxt}(0,t)\Big)\Theta(t)dt = \int_0^T \varphi(\ell,t)\delta g(t)dt, \tag{3.61}$$

where $\varphi(x,t)$ is the solution of the adjoint problem

$$\begin{cases} \rho(x)\varphi_{tt} - \mu(x)\varphi_t + (r(x)\varphi_{xx})_{xx} - (\kappa(x)\varphi_{xxt})_{xx} = 0, \ (x,t) \in \Omega_T, \\ \varphi(x,T) = 0, \ \varphi_t(x,T) = 0, \qquad\qquad x \in (0,\ell), \\ \varphi(0,t) \ = 0, \ \varphi_x(0,t) = \Theta(t), \qquad\qquad t \in [0,T], \\ [r(x)\varphi_{xx} - \kappa(x)\varphi_{xxt}]_{x=\ell} \quad = 0, \\ \qquad \big[\big(-r(x)\varphi_{xx} + \kappa(x)\varphi_{xxt}\big)_x\big]_{x=\ell} = 0, \quad t \in [0,T], \end{cases} \tag{3.62}$$

with Dirichlet input $\Theta \in L^2(0,T)$.

Proof. Multiplying the adjoint equation (3.60) by $\delta u(x,t)$, integrating over $(0,T) \times (0,\ell)$, integrating by parts and applying data values of (3.52) and (3.60), one may get

$$\int_0^T \int_0^\ell \Big(\rho(x)\delta u_{tt} + \mu(x)\delta u_t + (r(x)\delta u_{xx})_{xx} + (\kappa(x)\delta u_{xxt})_{xx}\Big)\phi(x,t)\,dxdt \\ - \int_0^T \xi(t)\delta u(\ell,t)dt + \int_0^T \delta g(t)\phi(\ell,t)dt = 0. \tag{3.63}$$

On the other hand, multiplying (3.52) by ϕ and integrating over $(0,T) \times (0,\ell)$, we notice that the first integral of (3.63) is zero, which gives (3.59).

Next, multiplying the first equation of (3.62) by $\delta u(x,t)$, integrate over $(0,T) \times (0,\ell)$, and utilizing the initial and boundary conditions of (3.62) and (3.52), we get

$$\int_0^T \int_0^\ell \Big(\rho(x)\delta u_{tt} + \mu(x)\delta u_t + (r(x)\delta u_{xx})_{xx} + (\kappa(x)\delta u_{xxt})_{xx}\Big)\varphi(x,t)\,dxdt \\ - \int_0^T \Theta(t)\Big(r(0)\delta u_{xx}(0,t) + \kappa(0)\delta u_{xxt}(0,t)\Big)dt + \int_0^T \delta g(t)\varphi(\ell,t)dt = 0. \quad (3.64)$$

Again from equation (3.52), we can conclude that the first integral of (3.64) becomes zero, which leads to (3.61). Hence the proof. □

In virtue of Theorem 3.1, if the arbitrary Neumann input $\xi(t)$ of adjoint problem (3.60) satisfy the regularity and consistency condition $\xi \in H^1(0,T)$, $\xi(T) = 0$, then this adjoint problem admits a unique weak solution $\phi \in H^1(0,T;\mathcal{V}_1^2(0,\ell))$, $\phi_{tt} \in L^2(0,T;L^2(0,\ell))$ as the change of variable t with $\tau = T - t$ shows. Also, $\phi(x,t)$ satisfies the estimates (3.16)-(3.19). Indeed, we have

$$\begin{aligned}
\|\phi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} &\leq \frac{4C_0^2}{3r_0^2}\,(1+T)\,\ell^3\|\xi'\|^2_{L^2(0,T)}, &(3.65)\\
\|\phi_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))} &\leq \frac{(C_0^2+1)}{3\kappa_0\, r_0}(1+T)\ell^3\,\|\xi'\|^2_{L^2(0,T)}, &(3.66)\\
\|\phi_t\|^2_{L^2(0,T;\mathcal{V}_1^2(0,\ell))} &\leq \frac{C^*(C_0^2+1)}{3\kappa_0 r_0}\,(1+T)\ell^3\,\|\xi'\|^2_{L^2(0,T)},\\
\|\phi_{tt}\|^2_{L^2(0,T;L^2(0,\ell)} &\leq \frac{C_1^2}{2\rho_0}(1+T)\ell^3\|\xi'\|^2_{L^2(0,T)},
\end{aligned}$$

where the constants $C_0^2, C_1^2, C^*, r_0, \kappa_0$ and ρ_0 are defined in Theorem 3.1.

The following theorem shows the necessary estimates for the weak solution $\varphi(x,t)$ to the adjoint problem (3.62).

Theorem 3.5. *Suppose that conditions (3.14) hold true and the Dirichlet input $\Theta(t)$ satisfies the regularity condition $\Theta \in H^2(0,T)$. Then there exists a weak solution $\varphi(x,t)$ of (3.62) satisfying the estimates*

$$\begin{aligned}
\|\varphi_t\|^2_{L^2(0,T;L^2(0,\ell))} &\leq 2\left(C_6^2(\exp(T/\rho_0)-1) + \frac{\ell^3}{3}\right) G(\Theta), &(3.67)\\
\|\varphi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} &\leq \frac{TC_6^2}{r_0}\exp(T/\rho_0)\,G(\Theta), &(3.68)\\
\|\varphi_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))} &\leq \frac{C_6^2}{2\kappa_0}\exp(T/\rho_0)\,G(\Theta), &(3.69)
\end{aligned}$$

where the constants

$$G(\Theta) := \|\Theta'\|^2_{L^2(0,T)} + \|\Theta''\|^2_{L^2(0,T)},$$

$$C_6^2 = \frac{2\ell^3}{3}\left(\max(\mu_1^2,\rho_1^2) + \rho_1 \max(1/T,\, T/3)\right).$$

Proof. In order to transform the adjoint problem (3.62) into a problem with homogeneous boundary condition, we use the transformation (see, Appendix C.3, [51] and also Section 3, [85]) $\psi(x,t) = \varphi(x,t) - x\Theta(t),\ \ (x,t) \in (0,\ell)\times[0,T)$. Then the function $\psi(x,t)$ solves the following problem

$$\begin{cases} \rho(x)\psi_{tt} - \mu(x)\psi_t + (r(x)\psi_{xx})_{xx} - (\kappa(x)\psi_{xxt})_{xx} \\ \qquad\qquad = x\mu(x)\Theta'(t) - x\rho(x)\Theta''(t),\ (x,t)\in\Omega_T, \\ \psi(x,T) = -x\Theta(T),\ \psi_t(x,T) = -x\Theta'(T),\ \ x\in(0,\ell), \\ \psi(0,t)\ = 0,\ \psi_x(0,t) = 0, \qquad\qquad t\in[0,T], \\ \left[r(x)\psi_{xx} - \kappa(x)\psi_{xxt}\right]_{x=\ell} \quad = 0, \\ \qquad \left[\left(-r(x)\psi_{xx} + \kappa(x)\psi_{xxt}\right)_x\right]_{x=\ell} = 0,\ \ t\in[0,T]. \end{cases} \tag{3.70}$$

Multiply both sides of equation (3.70) by $-2\psi_t(x,t)$, apply the identities

$$-2\left(r(x)\psi_{xx}\right)_{xx}\psi_t \equiv -2\left[(r(x)\psi_{xx})_x\psi_t - r(x)\psi_{xx}\psi_{xt}\right]_x - r(x)(\psi_{xx}^2)_t,$$
$$2\left(\kappa(x)\psi_{xxt}\right)_{xx}\psi_t \equiv 2\left[(\kappa(x)\psi_{xxt})_x\psi_t + \kappa(x)\psi_{xxt}\psi_{xt}\right]_x + 2\kappa(x)(\psi_{xxt}^2),$$

and integrating by parts using the initial and boundary conditions of (3.70), we obtain the following energy inequality

$$\begin{aligned} &\int_0^\ell \left(\rho(x)\psi_t(t)^2 + r(x)\psi_{xx}(t)^2\right)dx + 2\int_t^T\int_0^\ell \mu(x)\psi_\tau^2\,dxd\tau \\ &\quad +2\int_t^T\int_0^\ell \kappa(x)(\psi_{xx\tau})^2\,dxd\tau \\ &\quad = 2\int_t^T\int_0^\ell \Big(x\rho(x)\Theta''(\tau) - x\mu(x)\Theta'(\tau)\Big)\psi_\tau\,dxd\tau + \int_0^\ell \rho(x)\left(x\Theta'(T)\right)^2dx \\ &\quad \le \int_t^T\int_0^\ell \psi_\tau^2\,dxd\tau + \tilde{C}_6^2\int_t^T[\Theta'(\tau)^2 + \Theta''(\tau)^2]d\tau + \rho_1\frac{\ell^3}{3}\Theta'(T)^2, \end{aligned} \tag{3.71}$$

where $\tilde{C}_6^2 = \frac{2\ell^3}{3}\max(\mu_1^2,\rho_1^2)$. Using the identity $\Theta'(T) = \frac{1}{T}\int_0^T (t\Theta'(t))_t\,dt$, we get

$$\Theta'(T)^2 \le 2\left(\frac{1}{T}\|\Theta'\|^2_{L^2(0,T)} + \frac{T}{3}\|\Theta''\|^2_{L^2(0,T)}\right). \tag{3.72}$$

We employ the inequality (3.72) in (3.71) to deduce that

$$\int_0^\ell \Big(\rho(x)\psi_t(t)^2 \quad + \quad r(x)\psi_{xx}(t)^2\Big)dx + 2\int_t^T\int_0^\ell \mu(x)\psi_\tau^2\, dxd\tau + 2\int_t^T\int_0^\ell \kappa(x)\psi_{xx\tau}^2 dxd\tau$$
$$\leq \int_t^T\int_0^\ell \psi_\tau^2\, dxd\tau + C_6^2\left[\|\Theta'\|^2_{L^2(0,T)} + \|\Theta''\|^2_{L^2(0,T)}\right], \qquad (3.73)$$

where $C_6^2 = \tilde{C}_6^2 + \frac{2}{3}\rho_1\ell^3\max(1/T, T/3)$. Applying Gronwall's inequality, we obtain the first consequence of inequality (3.73) as follows

$$\int_0^\ell \psi_t(t)^2 dx \leq \frac{C_6^2}{\rho_0}\exp((T-t)/\rho_0)\Big[\|\Theta'\|^2_{L^2(0,T)} + \|\Theta''\|^2_{L^2(0,T)}\Big], \qquad (3.74)$$

and integrate (3.74) over $(0,T)$ to further obtain

$$\|\psi_t\|^2_{L^2(0,T;L^2(0,\ell))} \leq C_6^2(\exp(T/\rho_0) - 1)\, G(\Theta), \qquad (3.75)$$

where $G(\Theta) := \|\Theta'\|^2_{L^2(0,T)} + \|\Theta''\|^2_{L^2(0,T)}$. Substituting (3.75) in (3.73), we get

$$\|\psi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} \quad \leq \quad \frac{TC_6^2}{r_0}\exp(T/\rho_0)\, G(\Theta), \qquad (3.76)$$
$$\|\psi_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))} \quad \leq \quad \frac{C_6^2}{2\kappa_0}\exp(T/\rho_0)\, G(\Theta). \qquad (3.77)$$

Using the estimates (3.76), (3.77) and (1.16), we get $\psi,\ \psi_t \ \in L^2(0,T;\mathcal{V}_1^2(0,\ell))$. Since, $\psi_t + x\Theta'(t) = \varphi_t(x,t)$, we obtain

$$\|\varphi_t\|^2_{L^2(0,T;L^2(0,\ell))} \quad \leq \quad 2\|\psi_t\|^2_{L^2(0,T;L^2(0,\ell))} + \frac{2\ell^3}{3}\|\Theta'\|^2_{L^2(0,T)}, \qquad (3.78)$$

and

$$\psi_{xx}(x,t) = \varphi_{xx}(x,t), \psi_{xxt}(x,t) = \varphi_{xxt}(x,t).$$

The estimates (3.67)-(3.69) follow from (3.75)-(3.77) and (3.78). Hence the proof. □

By applying the formal Lagrange multiplier method (see, [93], Section 3.1) for IBVP-2, we obtain the actual input $\Theta(t)$ in (3.62) as follows

$$\Theta(t) = r(0)u_{xx}(0,t;g) + \kappa(0)u_{xxt}(0,t;g) + \omega(t), \ \ t \in [0,T]. \qquad (3.79)$$

Hence as a result of integral identity (3.61), the following input-output relationship arises:

$$\int_0^T \Big(r(0)u_{xx}(0,t) \quad + \quad \kappa(0)u_{xxt}(0,t)+\omega(t)\Big)\Big(r(0)\delta u_{xx}(0,t)+\kappa(0)\delta u_{xxt}(0,t)\Big)dt$$
$$= \int_0^T \varphi(\ell,t)\delta g(t)dt.$$

Thus the variation of the functional $\delta\mathcal{J}_2$ can be derived through the solution $\varphi(x,t)$ of the adjoint problem (3.62) with the input (3.79) as follows:

$$\delta\mathcal{J}_2(g)=\int_0^T \varphi(\ell,t)\delta g(t)dt+\frac{1}{2}\int_0^T \left(r(0)\delta u_{xx}(0,t)+\kappa(0)\delta u_{xxt}(0,t)\right)^2 dt. \tag{3.80}$$

Similarly, we consider the adjoint problem (3.60) with Dirichlet input

$$\xi(t)=u(\ell,t;g)-\nu(t),\ t\in[0,T]. \tag{3.81}$$

As in the case of IBVP-2, we also obtain that

$$\int_0^T \left[u(\ell,t;g)-\nu(t)\right]\delta u(\ell,t)=\int_0^T \phi(\ell,t)\delta g(t)dt,$$

and

$$\delta\mathcal{J}_1(g)=\int_0^T \phi(\ell,t)\delta g(t)dt+\frac{1}{2}\int_0^T \delta u(\ell,t)^2dt. \tag{3.82}$$

Let us now justify the substitutions (3.81) and (3.79) in the context of solutions of (3.60) and (3.62), respectively. By invoking the theory developed in ([9]) to the backward problem (3.70) and Theorem 3.5, it follows that the weak solution $\varphi\in H^1([0,T];\mathcal{V}_1^2(0,\ell))$ of problem (3.62) exists, if the input $\Theta(t)$ belongs to $H^2(0,T)$. Furthermore, by Theorem 3.1, for the existence of the weak solution $\phi\in H^1([0,T];\mathcal{V}_1^2(0,\ell))$ of problem (3.60), the Neumann input $\xi(t)$ should satisfy the regularity condition $\xi\in H^1(0,T)$. In view of the substitutions (3.81) and (3.79), we infer that the measured outputs $\nu(t)$, $\omega(t)$ and the outputs $u(\ell,t;g)$, $-r(0)u_{xx}(0,t;g)-\kappa(0)u_{xxt}(0,t;g)$ should obey the following regularity conditions:

$$\nu\in H^1(0,T),\ \omega\in H^2(0,T), \tag{3.83}$$

$$u(\ell,\cdot)\in H^1(0,T),\ -r(0)u_{xx}(0,\cdot)-\kappa(0)u_{xxt}(0,\cdot)\in H^2(0,T). \tag{3.84}$$

The conditions (3.83) mean that the measured outputs $\nu(t)$ and $\omega(t)$ should be more regular, although both of them originally belong to the class $L^2(0,T)$. In particular, the Neumann measured output $\omega(t)$ requires more regularity than that of the Dirichlet measured output $\nu(t)$. Furthermore, evidently for the weak solution of the direct problem (3.1), the first condition of (3.84) holds. The identity

$$\begin{aligned} r(0)u_{xxtt}(0,t) \quad + \quad & \kappa(0)u_{xxttt}(0,t) \\ & = \ell g''(t) + \int_0^\ell x\left[\rho(x)u_{tttt} + \mu(x)u_{ttt}\right]dx, \ \forall t \in [0,T], \quad (3.85) \end{aligned}$$

for the Neumann output and Theorem 3.3 show that the second condition of (3.84) is also satisfied for the regular weak solution with improved regularity.

Theorem 3.6. *Let the conditions (3.14) hold true.*
(i1) *Assume that the Dirichlet measured output $\nu(t)$ satisfies the regularity condition $\nu \in H^1(0,T)$. Then the Tikhonov functional $\mathcal{J}_1(g)$ corresponding to the problem IBVP-1 is Fréchet differentiable on the set of admissible sources $\mathcal{G}_1$. Furthermore, for the Fréchet derivative of this functional at $g \in \mathcal{G}_1$, the following gradient formula holds:*

$$\mathcal{J}_1'(g)(t) = \phi(\ell,t;g,\xi),\ t \in (0,T), \quad (3.86)$$

where $\phi(x,t;g)$ is the weak solutions of the adjoint problem (3.60) with the input

$$\xi(t) = u(\ell,t;g) - \nu(t).$$

(i2) *Assume that the conditions of Theorem 3.3 hold and $g \in \mathcal{G}_3$. Suppose in addition, the Neumann measured output $\omega(t)$ satisfies the regularity condition $\omega \in H^2(0,T)$. Then the Tikhonov functional $\mathcal{J}_2(g)$ corresponding to the problem IBVP-2 is Fréchet differentiable on $\mathcal{G}_3$. Moreover, the gradient formula*

$$\mathcal{J}_2'(g)(t) = \varphi(\ell,t;g,\Theta),\ t \in (0,T) \quad (3.87)$$

holds through the weak solution $\varphi(x,t;g,\Theta)$ of the adjoint problem (3.62) with the input

$$\Theta(t) = r(0)u_{xx}(0,t;g) + \kappa(0)u_{xxt}(0,t;g) + \omega(t).$$

Proof. By employing the inequality (3.21) and estimate (3.25) to the solution $\delta u(x,t)$ of problem (3.52), we obtain from (3.82) that

$$
\begin{aligned}
\left|\delta\mathcal{J}_1(g) - \int_0^T \phi(\ell,t)\delta g(t)dt\right| &= \frac{1}{2}\|\delta u(\ell,.)\|^2_{L^2(0,T)} \\
&\leq \frac{\ell^3}{6}\|\delta u_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} \\
&\leq \frac{2\ell^6(1+T)C_0^2}{9r_0^2}\|\delta g\|^2_{H^1(0,T)}, \qquad (3.88)
\end{aligned}
$$

where $\delta g(t) = g_1(t) - g_2(t)$. As a consequence, we have

$$
\frac{\left|\delta\mathcal{J}_1(g) - \int_0^T \phi(\ell,t)\delta g(t)dt\right|}{\|\delta g\|_{H^1(0,T)}} \to 0 \quad \text{as} \quad \|\delta g\|_{H^1(0,T)} \to 0^+.
$$

This means the Fréchet differentiability of the functional $\mathcal{J}_1$.

Next, we consider formula (3.80) for the variation of the functional $\mathcal{J}_2$. Using estimate (3.50) to the solution $\delta u(x,t)$ of (3.70), and then estimates (3.18) and (3.19), we get

$$
\begin{aligned}
&\left|\delta\mathcal{J}_2(g) - \int_0^T \varphi(\ell,t)\delta g(t)dt\right| = \frac{1}{2}\|r(0)\delta u_{xx}(0,.) + \kappa(0)\delta u_{xxt}(0,.)\|^2_{L^2(0,T)} \\
&\leq \frac{C_7^2}{2}\Big(\|\delta u_{tt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|\delta u_t\|^2_{L^2(0,T;L^2(0,\ell))} + \|\delta g\|^2_{L^2(0,T)}\Big) \\
&\leq C_8^2\|\delta g\|^2_{H^1(0,T)}, \qquad (3.89)
\end{aligned}
$$

where $C_8^2 = \frac{C_7^2}{2}\left[1 + \left(\frac{C_1^2}{2\rho_0} + \frac{C^*\left(C_0^2+1\right)}{3r_0\kappa_0}\right)(1+T)\ell^3\right]$. This shows that

$$
\frac{\left|\delta\mathcal{J}_2(g) - \int_0^T \varphi(\ell,t)\delta g(t)dt\right|}{\|\delta g\|_{H^1(0,T)}} \to 0 \quad \text{as} \quad \|\delta g\|_{H^1(0,T)} \to 0^+.
$$

Hence, the definition of the Fréchet derivative gives the formulas (3.86) and (3.87). □

Remark 3.4. The gradient formulas (3.86) and (3.87) show that there is no need for the weak solutions $\phi(x,t)$ and $\varphi(x,t)$ of adjoint problems (3.60) and (3.62) with inputs (3.81) and (3.79), respectively. Namely, these solutions need to satisfy only the conditions $\phi_x, \varphi_x \in L^2(0,T;L^2(0,\ell))$. By introducing a weaker solution, as in ([41]) and ([47]), the above conditions (3.83) and (3.84) can be weakened.

Corollary 3.2. *Suppose the conditions of Theorem 3.6 hold true. Then the regularized Tikhonov functionals $\mathcal{J}_{1\alpha}(g)$, $\mathcal{J}_{2\alpha}(g)$ defined in (3.7) and (3.12) are Fréchet differentiable* *on $\mathcal{G}_1$ and $\mathcal{G}_3$, respectively. The Fréchet derivatives are given by*

$$\mathcal{J}'_{1\alpha}(g)(t) = \phi(\ell, t; g, \xi) + \alpha g'(t), \ \forall g \in \mathcal{G}_1 \tag{3.90}$$

$$\mathcal{J}'_{2\alpha}(g)(t) = \varphi(\ell, t; g, \Theta) + \alpha g'''(t), \ \forall g \in \mathcal{G}_3, \tag{3.91}$$

where $\phi,\ \varphi \in H^1(0,T;\mathcal{V}_1^2(0,\ell))$ are the weak solutions of the adjoint problems (3.60) and (3.62) with boundary data ξ and Θ as in Theorem 3.6, respectively.

Proof. For any $g,\ \delta g \in \mathcal{G}_1$ and $g,\ \delta g \in \mathcal{G}_3$, the increment corresponding to the functionals $\mathcal{J}_{m\alpha}(g),\ m = 1,2$ are given by

$$\begin{aligned}
\delta\mathcal{J}_{1\alpha}(g) &= \int_0^T \phi(\ell,t)\delta g(t)dt + \alpha\int_0^T g'(t)\delta g'(t)dt + \frac{\alpha}{2}\int_0^T (\delta g'(t))^2 dt \\
&\quad + \frac{1}{2}\int_0^T \delta u(\ell,t)^2 dt, \ \ \forall g, \delta g \in \mathcal{G}_1, \\
\delta\mathcal{J}_{2\alpha}(g) &= \int_0^T \varphi(\ell,t)\delta g(t)dt + \alpha\int_0^T g'''(t)\delta g'''(t)dt + \frac{\alpha}{2}\int_0^T (\delta g'''(t))^2 dt \\
&\quad + \frac{1}{2}\int_0^T \Big(r(0)\delta u_{xx}(0,t) + \kappa(0)\delta u_{xxt}(0,t)\Big)^2 dt, \ \ \forall g, \delta g \in \mathcal{G}_3.
\end{aligned}$$

By doing calculations similar to (3.88) and (3.89) of Theorem 3.6, we can show that the last two integrals of $\delta\mathcal{J}_{m\alpha},\ m = 1,2$ are of the orders $\mathcal{O}\left(\|\delta g\|^2_{H^1(0,T)}\right)$ and $\mathcal{O}\left(\|\delta g\|^2_{H^3(0,T)}\right)$, respectively. From the definition of Fréchet derivative, we obtain the desired results (3.90) and (3.91). □

3.4 Monotonicity of the gradient algorithm

The Lipschitz continuity of the Fréchet derivatives of functionals $\mathcal{J}_1, \mathcal{J}_2$ has an important advantage when applying gradient-based methods to solve an inverse problem. In particular, in the case of gradient type algorithms such as Landweber iteration algorithm $g^{(n+1)}(x) = g^{(n)}(x) - \gamma_n \mathcal{J}'(g^{(n)}(x)),\ n = 0, 1, 2, ...$, or conjugate gradient algorithm applied to solve inverse problems, we may have trouble in predicting the relaxation parameter $\gamma_n > 0$. Using the Lipschitz constants associated with the Lipschitz continuity of $\mathcal{J}'_1, \mathcal{J}'_2$, the relaxation parameter can be calculated and that can be used to discuss the convergence of the iterative scheme as well. The following result shows that $\mathcal{J}'_1$, $\mathcal{J}'_2$ are Lipschitz contionus on $\mathcal{G}_1$ and $\mathcal{G}_3$ respectively.

Proposition 3.2. *Let the conditions of Theorem 3.6 hold true. Then the Fréchet gradients of the functionals* $\mathcal{J}_i'(g),\ i=1,2$, *defined by (3.86) and (3.87) are Lipschitz continuous. Moreover,*

$$\begin{aligned}
&\|\mathcal{J}_1'(g+\delta g)-\mathcal{J}_1'(g)\|_{L^2(0,T)} \le L_2\|\delta g\|_{H^1(0,T)},\ g,\delta g\in\mathcal{G}_1,\\
&\|\mathcal{J}_2'(g+\delta g)-\mathcal{J}_2'(g)\|_{L^2(0,T)} \le L_3\|\delta g\|_{H^3(0,T)},\ g,\delta g\in\mathcal{G}_3,
\end{aligned} \tag{3.92}$$

where the Lipschitz constants

$$\begin{aligned}
L_2^2 &= \frac{C_0^2}{\kappa_0 r_0}(C_0^2+1)\left(\frac{2\ell^6}{9r_0}\,(1+T)\right)^2,\\
L_3^2 &= \left(\frac{C_7^2\ell^3 T C_6^2}{3r_0}\exp(T/\rho_0)\right)\left(1+\frac{1}{2\rho_0}\left[3C_5^2\exp(C_5^2T)+C_1^2\ell^3(1+T)\right]\right),
\end{aligned}$$

where $C_5, C_6, C_7>0$ *are the constants introduced in Theorem 3.2, Theorem 3.5 and Proposition 3.1, respectively.*

Proof. For any $g,\delta g\in\mathcal{G}_1$, from the Fréchet derivative (3.86) it is clear that

$$\|\mathcal{J}_1'(g+\delta g)-\mathcal{J}_1'(g)\|^2_{L^2(0,T)} = \int_0^T \delta\phi(\ell,t)^2dt,$$

where $\delta\phi$ is the solution of (3.60) with data

$$\delta\xi(t)=\delta u(\ell,t)=u(\ell,t;g+\delta g)-u(\ell,t;g).$$

Applying the trace estimate (3.21) which holds for $\delta\phi(\ell,t)$, and the estimates (3.65), (3.28), we obtain

$$\begin{aligned}
\|\mathcal{J}_1'(g+\delta g)-\mathcal{J}_1'(g)\|^2_{L^2(0,T)} &\le \frac{\ell^3}{3}\|\delta\phi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} \le \frac{4\ell^6}{9r_0^2}C_0^2(1+T)\|\delta\xi'\|^2_{L^2(0,T)}\\
&\le \frac{4\ell^9}{27r_0^2}C_0^2(1+T)\|\delta u_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))}\\
&\le L_2^2\|\delta g\|^2_{H^1(0,T)},
\end{aligned}$$

where $L_2^2=\frac{C_0^2}{\kappa_0 r_0}(C_0^2+1)\left(\frac{2\ell^6}{9r_0}\,(1+T)\right)^2$. Next we prove that $\mathcal{J}_2'(g)$ is Lipschitz continuous. The gradient formula (3.87) and the trace inequality (3.21) lead to

$$\|\mathcal{J}_2'(g+\delta g)-\mathcal{J}_2'(g)\|^2_{L^2(0,T)} = \|\delta\varphi(\ell,.)\|^2_{L^2(0,T)} \le \frac{\ell^3}{3}\|\delta\varphi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))},$$

where $\delta\varphi(x,t)$ is the solution of (3.62) with the boundary data $\delta\varphi_x(0,t) = \delta\Theta(t) = (r(0)\delta u_{xx}(0,t) + \kappa(0)\delta u_{xxt}(0,t))$. By employing the estimate (3.68), one can get

$$\|\mathcal{J}_2'(g+\delta g) - \mathcal{J}_2'(g)\|^2_{L^2(0,T)} \leq \frac{\ell^3 T C_6^2}{3r_0} \exp(T/\rho_0)\Big(\|r(0)\delta u_{xxt}(0,.)$$
$$+\kappa(0)\delta u_{xxtt}(0,.)\|^2_{L^2(0,T)} + \|r(0)\delta u_{xxtt}(0,.) + \kappa(0)\delta u_{xxttt}(0,.)\|^2_{L^2(0,T)}\Big) \ .$$

From the identity (3.85), it holds that

$$\begin{aligned}
&\| \left(r(0)\delta u_{xx}(0,.;g) + \kappa(0)\delta u_{xxt}(0,.;g)\right)_{tt} \|^2_{L^2(0,T)} \\
&\leq \ C_7^2 \left(\|\delta u_{tttt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|\delta u_{ttt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|\delta g''\|^2_{L^2(0,T)}\right), \qquad (3.93)
\end{aligned}$$

where C_7 is the constant defined in 3.50, and coupling with (3.51), we arrive at

$$\begin{aligned}
&\|\mathcal{J}_2'(g+\delta g) - \mathcal{J}_2'(g)\|^2_{L^2(0,T)} \\
&\leq \frac{C_7^2 \ell^3 T C_6^2}{3r_0} \exp(T/\rho_0)\Big(\|\delta g''\|^2_{L^2(0,T)} + \|\delta g'\|^2_{L^2(0,T)} + \|\delta u_{tttt}\|^2_{L^2(0,T;L^2(0,\ell))} \\
&+2\|\delta u_{ttt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|\delta u_{tt}\|^2_{L^2(0,T;L^2(0,\ell))}\Big).
\end{aligned}$$

By the estimates (3.19), (3.47) and (3.33), we obtain the desired result (3.92). □

Next, we discuss the convergence of Landweber iterative scheme. The sequence of iterations $\{g^{(n)}\} \subset \mathcal{G}_1$ of IBVP-1 and $\{g^{(n)}\} \subset \mathcal{G}_3$ of IBVP-2 are defined by

$$g^{(n+1)}(t) = g^{(n)}(t) - \gamma_n \mathcal{J}_m'(g^{(n)}(t)),\ m = 1,2,\ n = 0,1,2,..., \qquad (3.94)$$

where iteration parameter γ_n is given by the minimum problem

$$f_n(\gamma_n) := \inf_{\gamma \geq 0} f_n(\gamma),\ \ f_n(\gamma) := \mathcal{J}(g^{(n)} - \gamma \mathcal{J}_m'(g^{(n)}))(t).$$

Proposition 3.3. *Assume that the conditions of Proposition 3.2 hold true and let* $g^{(n)}$ *be the iteration defined by (3.94) with* $\gamma_n = \gamma > 0$. *Then the following inequalities hold*

$$\begin{aligned}
\mathcal{J}_1(g^{(n)}) - \mathcal{J}_1(g^{(n+1)}) \ &\geq \ \frac{1}{2L_2}\|\mathcal{J}_1'(g^{(n)})\|^2_{L^2(0,T)}, \\
\mathcal{J}_2(g^{(n)}) - \mathcal{J}_2(g^{(n+1)}) \ &\geq \ \frac{1}{2L_3}\|\mathcal{J}_2'(g^{(n)})\|^2_{L^2(0,T)},\ \ \forall\, n = 0,1,2,...,
\end{aligned}$$

where L_2 and L_3 are the Lipschitz constants defined in Proposition 3.2. Moreover, the sequence $\mathcal{J}_m(g^{(n)}),\ m = 1, 2$ is monotone decreasing convergent sequence with $\lim_{n\to\infty} \|\mathcal{J}'_m(g^{(n)})\|_{L^2(0,T)} = 0,\ m = 1, 2.$

The proof follows from the similar arguments of Lemma 4.3 and Corollary 4.1 of [42]. Consequently, let us set $\mathcal{J}_1^* = \mathcal{J}_1(g^*) = \lim_{n\to\infty} \mathcal{J}_1(g^{(n)})$ be the limit of the sequence $\mathcal{J}_1(g^{(n)})$. It is evident that the sequence of iterations $\{g^{(n)}\} \subset \mathcal{G}_1$ of IBVP-1 weakly converges to g^* in $L^2(0, T)$. Similar, conclusions hold for IBVP-2 as well.

3.5 Stability estimates by variational methods

This section establishes a variational inequality, which has to be satisfied by an optimal solution of the minimization problems (3.7) and (3.12). This variational inequality is the key ingredient in deriving the stability estimates for the inverse problems. The stability estimates for IBVP-1 and IBVP-2 are obtained through the regular solutions established in Theorems 3.1,3.2 and 3.3 under a suitable smoothness of the boundary data $g(t)$. This forces us to introduce the more regularized Tikhonov functionals $\mathcal{J}_{1\alpha}(g)$ and $\mathcal{J}_{2\alpha}(g)$ as in (3.7) and (3.12) with inputs in $\mathcal{G}_1$ and $\mathcal{G}_3$, respectively.

Proposition 3.4. *Let $(\bar{u}(\ell, .), \bar{g})$ and $(-(r(0)\bar{u}_{xx}(0, .) + \kappa(0)\bar{u}_{xxt}(0, .)), \bar{g})$ be the solutions of IBVP-1 and IBVP-2 respectively. Then for the problem IBVP-1, the following variational inequality holds*

$$\int_0^T (g_\alpha(t) - \bar{g}(t))\, \phi(\ell, t; \bar{g})dt + \alpha \int_0^T \bar{g}'(t)\, (g'_\alpha(t) - \bar{g}'(t))\, dt \geq 0, \tag{3.95}$$

$\forall g_\alpha \in \mathcal{G}_1$, *while for the case of IBVP-2, it holds that*

$$\int_0^T (g_\alpha(t) - \bar{g}(t))\, \varphi(\ell, t; \bar{g})dt + \alpha \int_0^T \bar{g}'''(t)\, (g'''_\alpha(t) - \bar{g}'''(t))\, dt \geq 0, \tag{3.96}$$

$\forall g_\alpha \in \mathcal{G}_3$, *where ϕ and φ are the weak solutions of the adjoint problems (3.60) and (3.62) with data (3.81) and (3.79), respectively.*

Proof. For any $0 \leq \gamma \leq 1$, we choose an arbitrary element $g_\alpha \in \mathcal{G}_1$ such that $g_\gamma = \bar{g} + \gamma(g_\alpha - \bar{g}) \in \mathcal{G}_1$. The regularized Tikhonov functional corresponding to $(u_\gamma(\ell, .), g_\gamma)$ is given by

$$\mathcal{J}_{1\alpha}(g_\gamma) = \frac{1}{2} \int_0^T (u_\gamma(\ell, t; g_\gamma) - \nu(t))^2\, dt + \frac{\alpha}{2} \int_0^T \left(g'_\gamma(t)\right)^2 dt.$$

Since the functional $J_{1\alpha}(g_\gamma)$ is Fréchet differentiable at g_γ, we have

$$\begin{aligned}\frac{d}{d\gamma}\Big(\mathcal{J}_{1\alpha}\left(\bar{g}+\gamma(g_\alpha-\bar{g})\right)\Big)\Big|_{\gamma=0} &= \int_0^T \Big(u_\gamma(\ell,t;g_\gamma)-\nu(t)\Big)\frac{\partial u_\gamma}{\partial \gamma}|_{\gamma=0}\,dt \\ &\quad +\alpha\int_0^T \bar{g}'(t)\left(g_\alpha'(t)-\bar{g}'(t)\right)dt. \end{aligned} \tag{3.97}$$

Considering the system (3.1) corresponding to the data g_γ and setting

$$\eta = \frac{\partial u_\gamma}{\partial \gamma}|_{\gamma=0},$$

we see that $\eta(x,t)$ satisfies the system

$$\begin{cases} \rho(x)\eta_{tt}+\mu(x)\eta_t+(r(x)\eta_{xx})_{xx}+(\kappa(x)\eta_{xxt})_{xx}=0,\ (x,t)\in\Omega_T, \\ \eta(x,0)=0,\ \eta_t(x,0)=0, \qquad x\in(0,\ell), \\ \eta(0,t)\ =0,\ \ \eta_x(0,t)=0, \qquad t\in[0,T], \\ \left[r(x)\eta_{xx}+\kappa(x)\eta_{xxt}\right]_{x=\ell}\ =0, \\ \quad -\left[\left(r(x)\eta_{xx}+\kappa(x)\eta_{xxt}\right)_x\right]_{x=\ell}=g_\alpha(t)-\bar{g}(t),\ \ t\in[0,T]. \end{cases} \tag{3.98}$$

Since $\bar{g}$ is the optimal solution, we obtain

$$\frac{d}{d\gamma}\Big(\mathcal{J}_{1\alpha}(\bar{g}+\gamma(g_\alpha-\bar{g}))\Big)\Big|_{\gamma=0}\geq 0,\ \ \forall\, g_\alpha\in\mathcal{G}_1.$$

Therefore, from (3.97) we have

$$\int_0^T\left[\bar{u}(\ell,t;\bar{g})-\nu(t)\right]\eta(\ell,t)dt+\alpha\int_0^T \bar{g}'(t)\left(g_\alpha'(t)-\bar{g}'(t)\right)dt\geq 0,\ \ \forall\, g_\alpha\in\mathcal{G}_1.$$

In virtue of the relationship

$$\left[\bar{u}(\ell,t;\bar{g})-\nu(t)\right]=\left(-r(x)\phi_{xx}+\kappa(x)\phi_{xxt}\right)_x|_{x=\ell},$$

between the measured data and the adjoint solution ϕ of (3.60), one can rewrite

$$\int_0^T\left(-r(x)\phi_{xx}+\kappa(x)\phi_{xxt}\right)_x|_{x=\ell}\,\eta(\ell,t)dt+\alpha\int_0^T \bar{g}'(t)\left(g_\alpha'(t)-\bar{g}'(t)\right)dt\geq 0. \tag{3.99}$$

In order to express the first integral of (3.99) solely interms of solution of adjoint system, we multiply equation (3.60) by $\eta(x,t)$, integrating by parts and apply initial and boundary conditions, we get

$$
\begin{aligned}
- & \int_0^T \left(-r(x)\phi_{xx} + \kappa(x)\phi_{xxt}\right)_x |_{x=\ell}\, \eta(\ell,t)dt \\
+ & \int_0^\ell \int_0^T \Big(\rho(x)\eta_{tt} + \mu(x)\eta_t + (r(x)\eta_{xx})_{xx} + (\kappa(x)\eta_{xxt})_{xx}\Big)\phi(x,t)dtdx \\
& + \int_0^T \left(g_\alpha(t) - \bar{g}(t)\right)\phi(\ell,t)dt = 0.
\end{aligned}
$$

Multiplying (3.98) by $\phi(x,t)$, integrating over $(0,\ell)\times(0,T)$, and using it in the previous equation, we get

$$
\int_0^T \left(-r(x)\phi_{xx} + \kappa(x)\phi_{xxt}\right)_x |_{x=\ell}\, \eta(\ell,t)\; dt = \int_0^T \left(g_\alpha(t) - \bar{g}(t)\right)\phi(\ell,t)dt.
$$

Substitution of this identity in (3.99) leads to the desired inequality (3.95).

By repeating the calculation for $\mathcal{J}_{2\alpha}(g_\gamma)$ with the adjoint problem (3.62) and the input (3.79), one can obtain the variational inequality (3.96) for IBVP-2. Hence the proof. □

Next, we have the following stability estimate for the IBVP-1 in terms of the measured data. We obtain a lower bound for the internal damping coefficient $\kappa(x)$ which is sufficient to obtain a Lipschitz type stability estimate for the shear force $g(t)$.

Theorem 3.7. *Suppose the assumptions (3.14) hold true. Let $g_\alpha,\ \widehat{g}_\alpha \in \mathcal{G}_1$ are unique minimizers of the regularized Tikhonov functional $\mathcal{J}_{1\alpha}$ defined by (3.7) corresponding to the measured outputs $\nu,\ \widehat{\nu} \in H^1(0,T)$, respectively. Suppose the internal damping coefficient $\kappa(x)$ satisfies the condition*

$$
\kappa(x) \geq \Big(\frac{\sqrt{2}T^2\ell^6\exp(T)(1+T)}{9r_0}\Big)\alpha^{-1} := \kappa_0. \tag{3.100}
$$

Then the following stability estimate holds:

$$
\|\widehat{g}_\alpha - g_\alpha\|^2_{L^2(0,T)} \leq C_{ST}\|\widehat{\nu}' - \nu'\|^2_{L^2(0,T)}, \tag{3.101}
$$

where $C_{ST} = \frac{9r_0\kappa_0 T^2}{\ell^6\exp(T)(1+T)}$.

Proof. Take $\bar{g}(t) = \widehat{g}_\alpha(t)$ in the variational inequality (3.95) to get

$$
\int_0^T \left(g_\alpha(t) - \widehat{g}_\alpha(t)\right)\phi(\ell,t;\widehat{g}_\alpha)\; dt\; + \alpha\int_0^T \widehat{g}'_\alpha(t)\left(g'_\alpha(t) - \widehat{g}'_\alpha(t)\right)\; dt \geq 0. \tag{3.102}
$$

Similarly, we replace g_α with $\widehat{g}_\alpha(t)$ and $\bar{g}$ with g_α in (3.95) we get,

$$\int_0^T \left(\widehat{g}_\alpha(t) - g_\alpha(t)\right)\phi(\ell, t; g_\alpha)\, dt \ + \alpha \int_0^T g'_\alpha(t)\left(\widehat{g}'_\alpha(t) - g'_\alpha(t)\right)\ dt \geq 0. \tag{3.103}$$

We deduce from (3.102) and (3.103) that

$$\alpha \int_0^T \left(\widehat{g}'_\alpha(t) - g'_\alpha(t)\right)^2 dt \leq \int_0^T \left(\widehat{g}_\alpha(t) - g_\alpha(t)\right)\delta\phi(\ell, t)dt, \tag{3.104}$$

where $\delta\phi(\ell, t) = \phi(\ell, t; g_\alpha) - \phi(\ell, t; \widehat{g}_\alpha)$ is the solution of the adjoint problem (3.60) with data $\delta\xi(t) = \delta u(\ell, t) - \delta\nu(t)$, $\delta u(\ell, t) = u(\ell, t; g_\alpha) - u(\ell, t; \widehat{g}_\alpha)$ and $\delta\nu(t) = \nu(t) - \widehat{\nu}(t)$. Applying Hölder's inequality on the right-hand side of (3.104) and squaring on both sides, we obtain

$$\alpha^2 \|\widehat{g}'_\alpha - g'_\alpha\|^4_{L^2(0,T)} \leq \|\widehat{g}_\alpha - g_\alpha\|^2_{L^2(0,T)} \|\delta\phi(\ell, .)\|^2_{L^2(0,T)}. \tag{3.105}$$

The inequalities (3.21) and (3.22) further lead to the following estimates

$$\begin{aligned} \|\widehat{g}_\alpha - g_\alpha\|^2_{L^2(0,T)} &\leq T^2 \|\widehat{g}'_\alpha - g'_\alpha\|^2_{L^2(0,T)} \\ \|\delta\phi(\ell, .)\|^2_{L^2(0,T)} &\leq \frac{T^2\ell^3}{6}\|\delta\phi_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))}, \end{aligned} \tag{3.106}$$

so that making use of (3.66), the estimate (3.105) becomes as follows

$$\begin{aligned} \alpha^2 \|\widehat{g}'_\alpha - g'_\alpha\|^2_{L^2(0,T)} &\leq \frac{T^4\ell^6}{18 r_0 \kappa_0} \exp(T)(1+T)\|\delta\xi'\|^2_{L^2(0,T)} \\ &\leq \frac{T^4\ell^6}{9 r_0 \kappa_0} \exp(T)(1+T)\Big[\|\delta u_t(\ell, .)\|^2_{L^2(0,T)} + \|\delta\nu'\|^2_{L^2(0,T)}\Big]. \end{aligned}$$

By using the trace estimate $\|\delta u_t(\ell, .)\|^2_{L^2(0,T)} \leq \frac{\ell^3}{3}\|\delta u_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))}$ and appealing to (3.28), which also holds for δu, we get

$$\alpha^2 \|\widehat{g}'_\alpha - g'_\alpha\|^2_{L^2(0,T)} \leq C_\alpha(\kappa_0)\|\widehat{g}'_\alpha - g'_\alpha\|^2_{L^2(0,T)} + \frac{T^4\ell^6}{9 r_0\kappa_0}\exp(T)(1+T)\|\delta\nu'\|^2_{L^2(0,T)},$$

where $C_\alpha(\kappa_0) = \frac{T^4\ell^{12}}{81\kappa_0^2 r_0^2}\exp(2T)(1+T)^2$ and $\delta g'(t) = g_\alpha(t) - \widehat{g}'_\alpha(t)$.

Suppose the lower bound of the internal damping coefficient $\kappa(x)$ be chosen so that $C_\alpha(\kappa_0) = \frac{\alpha^2}{2}$, that is choosing κ_0 as in (3.100) and invoking (3.106), we obtain the stability estimate (3.101) with the stability constant C_{ST}. Hence the proof. □

Table 3.1: Stability constant C_{ST} corresponding to T, α and κ_0.

T	α	$\kappa_0 = .0025\left(\frac{T^2\exp(T)(1+T)}{\alpha}\right)$	$C_{ST} = 576\left(\frac{\kappa_0 T^2}{\exp(T)(1+T)}\right)$
.1	10^{-3}	.0303	.143
.5	10^{-2}	.155	9.02
	10^{-3}	1.55	90.25
	10^{-4}	15.5	902.51
.75	10^{-3}	5.21	455.64
1	10^{-3}	13.6	1440.91

We infer from Theorem 3.7 that the stability estimate (3.101) and the lower bound of the internal damping coefficient $\kappa(x)$ (with units kgm^3/s) are valid for all non-negative values of the external damping coefficient $\mu(x)$, including the critical case, $\mu(x) = 0\ kg/ms$. The following example illustrates the specific values of lower bound for internal damping coefficient corresponding to four different final times T, which are obtained by choosing $\ell = .5\ m$, and $r_0 = 1$ (see, [6]). Using the specific values of κ_0, we can analyze the stability constants C_{ST}. In a real application, the value of the parameter of regularization ranges between 10^{-2} to 10^{-4}. The formula (3.100) shows that the lower bound κ_0 of the internal damping coefficient $\kappa(x)$ is in the order of α^{-1}. Hence, a smaller value of α increases the value of the lower bound for the internal damping coefficient, and it drastically increases the stability constant C_{ST}. This is evidently clear from the second to fourth rows of the table for the fixed time $T = .5\ s$. Also, one may notice that the increase in final time T increases the lower bound κ_0, and the stability constant C_{ST}, which indicates that the stability estimates hold only for small intervals of time T. The rows corresponding to the fixed $\alpha = 10^{-3}$ show that the stability constant C_{ST} increases drastically when T increases. Hence, to get the consistent stability estimate for the IBVP-1, the value of the final time $T > 0$ must be small, which is reasonable in terms of applications.

Remark 3.5. The assumption (3.100) on the Kelvin-Voigt damping coefficient $\kappa(x) = c_d I(x)$ can be justified by fixing the specific values of various coefficients in (3.1) and utilizing the estimations of the damping coefficients derived from the dynamic experiments (see, [6]) of real applications. For a beam of length $l = 1$, moment of inertia $I(x) = 1.64\times10^{-9}$, mass density $\rho(x) = 1.02$ and Young's modulus $E(x) = 2.68 \times 10^{10}$, the damping coefficients are estimated in ([6]) as $\mu = 1.7561$ and $c_d = 2.05 \times 10^5$. Thus, for the choice of $\alpha = 10^{-2}$ and $T = 0.0133$, we can see from (3.100) that $c_d \geq \kappa_0 / I(x) = 39597.48$.

In the case of IBVP-2, we obtain a stability estimate under a sufficient condition on the parameter of regularization $\alpha > 0$.

Theorem 3.8. *Assume the conditions given in (3.14) and Theorem 3.3 hold true. Let $g_\alpha,\ \widehat{g}_\alpha \in \mathcal{G}_3$ are unique minimizers of the regularized Tikhonov functional $\mathcal{J}_{2\alpha}$ defined by (3.12) corresponding to the measured outputs $\omega,\ \widehat{\omega} \in H^2(0,T)$, respectively. If the parameter of regularization α satisfies the condition*

$$\alpha^2 > C_9^2 C_{10}^2, \tag{3.107}$$

where C_9^2 and C_{10}^2 are defined in the proof. Then the following stability estimate holds:

$$\|\widehat{g}_\alpha - g_\alpha\|^2_{L^2(0,T)} \;\; \leq \;\; \widetilde{C}_{ST} \|\widehat{\omega} - \omega\|^2_{H^2(0,T)}, \tag{3.108}$$

where $\widetilde{C}_{ST} = \frac{4T^{13}\ell^3 C_6^2 \exp(T/\rho_0)}{\left(\alpha^2 - C_{10}^2 C_9^2\right) 3r_0}$.

Proof. By repeating the similar steps done in Theorem 3.7 for the variational inequality (3.96), we get

$$\alpha \int_0^T \left(\widehat{g}_\alpha'''(t) - g_\alpha'''(t)\right)^2 dt \leq \int_0^T \left(\widehat{g}_\alpha(t) - g_\alpha(t)\right) \delta\varphi(\ell, t) dt, \tag{3.109}$$

where $\delta\varphi(\ell,t) = \varphi(\ell,t;g_\alpha) - \varphi(\ell,t;\widehat{g}_\alpha)$ is the solution of the adjoint problem (3.62) with

$$\delta\Theta(t) = r(0)\delta u_{xx}(0,t) + \kappa(0)\delta u_{xxt}(0,t) + \delta\omega(t).$$

The repeated application of Hölder's inequality gives

$$\|\widehat{g}_\alpha - g_\alpha\|^2_{L^2(0,T)} \leq T^6 \|\widehat{g}_\alpha''' - g_\alpha'''\|^2_{L^2(0,T)}, \text{ for } g_\alpha,\ \widehat{g}_\alpha \in \mathcal{G}_3, \tag{3.110}$$

and using $\|\delta\varphi(\ell,.)\|^2_{L^2(0,T)} \leq \frac{\ell^3}{3} \|\delta\varphi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))}$, we deduce from (3.109) that

$$\begin{aligned} \alpha^2 \|\widehat{g}_\alpha''' - g_\alpha'''\|^4_{L^2(0,T)} &\leq \|\widehat{g}_\alpha - g_\alpha\|^2_{L^2(0,T)} \|\delta\varphi(\ell,.)\|^2_{L^2(0,T)} \\ &\leq \frac{T^6\ell^3}{3} \|\delta\varphi_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} \|\widehat{g}_\alpha''' - g_\alpha'''\|^2_{L^2(0,T)}. \end{aligned}$$

In virtue of the estimates (3.68), (3.51) and (3.93), one can get

$$\alpha^2\|\widehat{g}_\alpha''' - g_\alpha'''\|^2_{L^2(0,T)} \le \frac{T^7\ell^3 C_6^2}{3r_0}\exp(T/\rho_0)\left[\|\delta\Theta'\|^2_{L^2(0,T)} + \|\delta\Theta''\|^2_{L^2(0,T)}\right]$$

$$\le \frac{2T^7\ell^3C_6^2}{3r_0}\exp(T/\rho_0)\Big[C_7^2\Big(\|\delta u_{tttt}\|^2_{L^2(0,T;L^2(0,\ell))} + 2\|\delta u_{ttt}\|^2_{L^2(0,T;L^2(0,\ell))}$$

$$+\|\delta u_{tt}\|^2_{L^2(0,T;L^2(0,\ell))} + \|\delta g''\|^2_{L^2(0,T)} + \|\delta g'\|^2_{L^2(0,T)}\Big) + 2\|\delta\omega\|^2_{H^2(0,T)}\Big],$$

where $\delta g(t) = g_\alpha(t) - \widehat{g}_\alpha(t)$, the constants $C_6^2 = \frac{2\ell^3}{3}\Big(\max(\mu_1^2, \rho_1^2) + \rho_1\max(1/T,\, T/3)\Big)$, and $C_7^2 = 2\ell^2\max\left(1, \frac{2\ell}{3}(\rho_1^2+\mu_1^2)\right)$. By employing the regularity estimates (3.19), (3.33), (3.47) for $\delta u_{tt},\ \delta u_{ttt}$ and δu_{tttt} respectively, we deduce that

$$\alpha^2\|\widehat{g}_\alpha''' - g_\alpha'''\|^2_{L^2(0,T)} \le \frac{2T^7\ell^3C_6^2}{3r_0}\exp(T/\rho_0)C_7^2\Big[\frac{1}{2\rho_0}\Big(3C_5^2\exp(C_5^2T) + C_1^2\ell^3(1+T)\Big) + 1\Big]\|\delta g\|^2_{H^3(0,T)} + \frac{4T^7\ell^3}{3r_0}C_6^2\exp(T/\rho_0)\|\delta\omega\|^2_{H^2(0,T)}, \quad (3.111)$$

where C_5^2 is the constant defined in Theorem 3.2. For any $g \in \mathcal{G}_3$, we obtain that

$$\|g\|^2_{H^3(0,T)} \le C_9^2\|g'''\|^2_{L^2(0,T)}, \quad (3.112)$$

where $C_9^2 = (1 + T^2 + T^4 + T^6)$. Substituting (3.112) in (3.111), we arrive at

$$\alpha^2\|\widehat{g}_\alpha''' - g_\alpha'''\|^2_{L^2(0,T)} \le C_{10}^2C_9^2\|\delta g'''\|^2_{L^2(0,T)} + \frac{4T^7\ell^3}{3r_0}C_6^2\exp(T/\rho_0)\|\delta\omega\|^2_{H^2(0,T)},$$

where $C_{10}^2 = \frac{2T^7\ell^3C_6^2}{3r_0}\exp(T/\rho_0)C_7^2\Big[\frac{1}{2\rho_0}\Big(3C_5^2\exp(C_5^2T) + C_1^2\ell^3(1+T)\Big) + 1\Big]$.
Choosing $\alpha^2 > C_9^2C_{10}^2$, we conclude the stability result (3.108) through (3.110). □

The condition (3.107) is simple to test and implement, and it does not impose a significant constraint. For instance, when $T = .04,\ \ell = .4,\ \rho_1 = 1,\ \mu_1 = 1,\ r = 20$ and $\kappa_0 = 1$, we obtain $\alpha^2 > 9.37 \times 10^{-9}$. In most of the physical experiments, α varies from 10^{-2} to 10^{-4}, and so to validate the condition $\alpha^2 > 9.37 \times 10^{-9}$, one can choose α as 10^{-2}. In this case, we obtain the stability constant $\widetilde{C}_{ST} = 3.27 \times 10^{-17}$, which is comparatively small. Therefore, Theorem 3.8 provides a significant stability estimate for the determination of the shear force in terms of a feasible condition on the regularization parameter α.

Chapter 4

Determination of a spatial load in a damped Kirchhoff-Love plate equation

In this chapter, we consider the two-dimensional generalization of the Euler-Bernoulli beam governed by the simply supported rectangular Kirchhoff plate with viscous damping. We assume that the flexural rigidity or bending stiffness coefficient D depends on the spatial variable, which makes the model much more complex than the classical plate equation [69] (see, also [39]). The methodology discussed in the previous chapters for the inverse problems related to the damped Euler-Bernoulli beam equation, such as the SVD and Tikhonov regularization methods, are developed for the inverse source problem of identifying the unknown spatial load $F(x)$ in the damped Kirchhoff-Love plate equation. The ideas presented here will be able to be successfully used to solve other inverse problems related to the Kirchhoff plate equation.

Now consider an inverse problem of recovering the unknown spatial load $F(x)$ in a general Kirchhoff plate equation

$$\begin{cases} \rho_h(x)u_{tt} + \mu(x)u_t + (D(x)(u_{x_1x_1} + \nu u_{x_2x_2}))_{x_1x_1} \\ \qquad + (D(x)(u_{x_2x_2} + \nu u_{x_1x_1}))_{x_2x_2} + 2(1-\nu)\left(D(x)u_{x_1x_2}\right)_{x_1x_2} \\ \qquad\qquad = F(x)G(t),\ (x,t) \in \Omega_T := \Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ x \in \Omega, \\ u(x,t) = 0,\ u_{x_2x_2}(x,t) = 0,\ (x,t) \in (\Gamma_1 \cap \Gamma_3) \times [0,T], \\ u(x,t) = 0,\ u_{x_1x_1}(x,t) = 0,\ (x,t) \in (\Gamma_2 \cap \Gamma_4) \times [0,T], \end{cases} \tag{4.1}$$

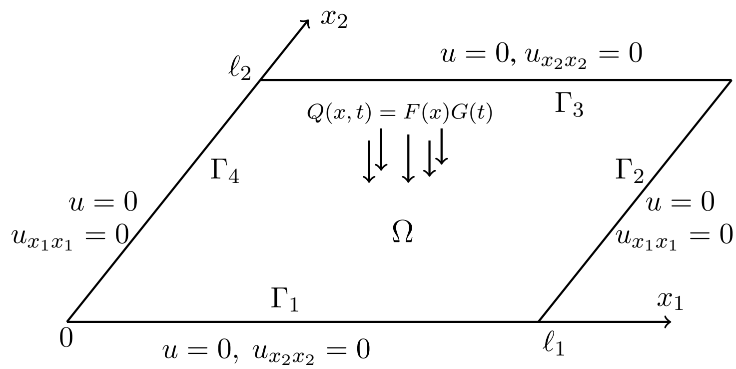

Figure 4.1: A simply supported Kirchhoff plate under transverse loading

from the final time measured displacement

$$u_T(x) := u(x,T),\ x \in \Omega, \tag{4.2}$$

where Ω is defined by 1.11. The functions $F(x)$ and $G(t)$ represent the spatial and temporal components of the transverse load $Q(x,t) := F(x)G(t)$ applied on the plate (Fig. 4.1). The function $u(x,t)$ is the normal component of the displacement vector at position $x \in \Omega$ and at time $t \in (0,T)$, called the deflection. Further, $\rho_h(x) = \rho(x)h(x) > 0$, while $\rho(x)$ is the density and $h(x) > 0$ is the thickness of a plate,

$$D(x) = \frac{E(x)h^3(x)}{12(1-\nu^2)}$$

is the flexural rigidity of the plate, where $E(x)$ is the Young modulus and $\nu \in (0,1)$ is the Poisson ratio. The parameter $D(x)$ plays the same role as the flexural rigidity $r(x) = E(x)I(x)$ in beam bending. However, it should be noted at this point that $D(x) > I(x)$, which means a plate is always stiffer than a beam of the same span and thickness ([92]).

In the considered inverse problem, the geometry of the plate does not play a principal role. Therefore, for the sake of simplicity and clear interpretation of the results, we assume that the plate has a rectangular form with edges parallel to the coordinate axes Ox_1 and Ox_2, as shown in Fig.4.1. Although in the considered mathematical model governed by (4.1), the plate is assumed to be simply supported, the results obtained in this chapter remain valid also for clamped as well as clamped-supported plates. Note that the deflection and bending moment are both zero on these simply-supported edges according to the physical meaning. Hence, physically, the boundary conditions in (4.1) should be defined as follows:

$$\begin{aligned} &u(x,t)=0,\ M_2:=D(x)\left(\nu u_{x_1x_1}+u_{x_2x_2}\right)=0,\ \text{on } \Gamma_1\times[0,T],\\ &u(x,t)=0,\ M_1:=-D(x)\left(u_{x_1x_1}+\nu u_{x_2x_2}\right)=0,\ \text{on } \Gamma_2\times[0,T],\\ &u(x,t)=0,\ M_2:=-D(x)\left(\nu u_{x_1x_1}+u_{x_2x_2}\right)=0,\ \text{on } \Gamma_3\times[0,T],\\ &u(x,t)=0,\ M_1:=D(x)\left(u_{x_1x_1}+\nu u_{x_2x_2}\right)=0,\ \text{on } \Gamma_4\times[0,T]. \end{aligned} \tag{4.3}$$

Let us consider, for example, the boundary conditions on the simply supported edge Γ_1 defined in (1.12). From the first condition $u(x_1,0,t)=0$, it follows that along the edge Γ_1, all the partial derivatives of $u(x,t)$ with respect to x_1 are zero. In particular, $u_{x_1x_1}(x,t)=0$, $(x,t)\in\Gamma_1\times[0,T]$. Taking this into account in the second condition given by $M_2:=D(x)\left(\nu u_{x_1x_1}+u_{x_2x_2}\right)=0$ on Γ_1, we deduce that $u_{x_2x_2}(x,t)=0$, $(x,t)\in\Gamma_1\times[0,T]$. Therefore, the simply supported boundary condition on Γ_1 can also appear in the following equivalent form:

$$u(x,t)=0,\ u_{x_2x_2}(x,t)=0,\ (x,t)\in\Gamma_1\times[0,T].$$

Similar equivalent boundary conditions are obtained for other simply supported edges $\Gamma_2,\Gamma_3,\Gamma_4$. These boundary conditions are set in the direct problem (4.1).

Our main contributions of this chapter are summarized as follows:

1. We proved the existence and uniqueness of weak solution of (4.1) when the data $u_0\in H^2(\Omega)\cap H_0^1(\Omega),\ v_0\in L^2(\Omega)$ and the coefficients satisfying Assumption 4.1.

2. Inverse problem is reformulated as a quasi-solution or least square problem and coupled with weak solution of the direct problem, we solved the minimization problem.

3. We computed the Fréchet derivative of the Tikhonov functional and proved Lipschitz continuity of the Fréchet derivative, which plays a vital role in the gradient-based numerical algorithm for the inverse problem.

4. Another significant result is a stability estimate for the inverse source problem (4.1)-(4.2). Since the inverse problem is posed in the context of a minimization problem, using a first-order necessary optimality condition satisfied by an optimal pair $(u(x,t;F_{*,\alpha}),F_{*,\alpha})$, we establish a local stability estimate for the unknown source term $F_{*,\alpha}\in L^2(\Omega)$ under some conditions on final time T and damping coefficient μ.

5. Using the spectral properties of an input-output operator, we find the SVD of the regularized solution $F_\alpha \in L^2(\Omega)$ as well as establish a stability estimate with the help of regularity assumption on temporal load $G(t)$. A comparison study of the representation formulas for the source function F_α obtained from Tikhonov regularization and SVD has been done at the end of this chapter. Indeed, this study helps one to devise a better numerical scheme by combining the CGA and TSVD. The study of such a comparison for the case of heat and wave equations has been carried out in [49].

The chapter is organized as follows: In section 4.1, we discuss the well-posedness of the direct problem in detail. In section 4.2, we give the formulation of the inverse problem and study the compactness, and the Lipschitz continuity of the input-output operator. The main results, Fréchet derivative of the Tikhonov functional, Lipschitz continuity of the Frechét derivative, and the existence of a minimizer for Tikhonov functional are also given in this section. Stability estimates for the source term are obtained in this section by deriving suitable conditions on the final time and damping parameter. In section 4.3, we determine the SVD of regularized solution, and establish the relationship between this method and Tikhonov regularization. In section 4.4, we establish an another stability estimate for the inverse problem, using spectral properties of the input-output operator.

4.1 Solvability of direct problem

Since the well-posedness of the direct problem is fundamental for the study of inverse problem, we discuss results related to the existence and uniqueness of the direct problem by deriving appropriate an *a priori* estimates using the Faedo-Galerkin finite dimensional approximation.

The following assumption plays a fundamental role in the analysis of direct and inverse problems.

Assumption 4.1.

$$\begin{cases} \rho, D, \mu \in L^\infty(\Omega), h \in C(\Omega), \\ 0 < D_0 \leq D(x) \leq D_1, \ \ 0 \leq \mu_0 \leq \mu(x) \leq \mu_1, \\ 0 < \rho_0 \leq \rho(x) \leq \rho_1, \ with\ \nabla\rho, \ \Delta\rho \in L^\infty(\Omega), \\ 0 < h_0 \leq h(x) \leq h_1, \ with\ \nabla h, \Delta h \in L^\infty(\Omega), \\ \|\nabla(\rho h)\|_{L^\infty(\Omega)} \leq \rho_2 \ and\ \|\Delta(\rho h)\|_{L^\infty(\Omega)} \leq \rho_3, \\ F \in L^2(\Omega), \ G \in L^2(0,T), \ G(t) \not\equiv 0. \end{cases} \tag{4.4}$$

Definition 4.1 (Weak solution). Let $0 < T < \infty,\ u_0 \in \mathcal{V}^2(\Omega),\ v_0 \in L^2(\Omega),\ F \in L^2(\Omega)$ and $G \in L^2(0,T)$ be given. A function $u \in L^2(0,T;\mathcal{V}^2(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ and $u_{tt} \in L^2(0,T;\mathcal{V}^2(\Omega)')$ is called a weak solution of (4.1) if

$$
\begin{aligned}
i)\qquad & \langle \rho_h u_{tt}(t), v\rangle + (\mu u_t(t), v) + (D(u_{x_1x_1}(t) + \nu u_{x_2x_2}(t)), v_{x_1x_1}) \\
& +(D(u_{x_2x_2}(t) + \nu u_{x_1x_1}(t)), v_{x_2x_2}) + 2(1-\nu)(Du_{x_1x_2}(t), v_{x_1x_2}) \\
& \quad = (FG(t), v),\ \forall v \in \mathcal{V}^2(\Omega), \quad \text{a.e. } t \in [0,T], \\
ii)\qquad & u(0) = u_0,\ u_t(0) = v_0,
\end{aligned}
$$

where $\mathcal{V}^2(\Omega)$ is defined by (1.13).

Remark 4.1. If we have $u \in L^2(0,T;\mathcal{V}^2(\Omega)),\ u_t \in L^2(0,T;L^2(\Omega))$ and $u_{tt} \in L^2(0,T;\mathcal{V}^2(\Omega)')$, then Theorem 1.7 shows that $u \in C([0,T];H_0^1(\Omega))$ and $u_t \in C([0,T];\mathcal{V}^2(\Omega)')$. Hence the initial conditions $u(0) = u_0$ and $u_t(0) = v_0$ are well defined.

We want to show that the problem (4.1) has unique weak solution which continuously depends on the data in the appropriate norms. For this well-posedness of the direct problem, we adapt the Faedo-Galerkin method.

First, we define an n dimensional subspace $W_n := span\{\xi_1, \xi_2, ..., \xi_n\}$ of $\mathcal{V}^2(\Omega)$ and look for Faedo-Galerkin approximation $u_n(t) := u_n(x,t)$ of the form

$$
u_n(t) = \sum_{i=1}^{n} r_{i,n}(t)\xi_i, \quad u_{0,n} = \sum_{i=1}^{n} p_{i,n}\xi_i, \quad v_{0,n} = \sum_{i=1}^{n} q_{i,n}\xi_i,
$$

where ξ_i form an orthogonal basis for $\mathcal{V}^2(\Omega)$ and orthonormal basis for $L^2(\Omega)$. Here the coefficients $r_{i,n}, p_{i,n}$ and $q_{i,n}$ are choosen so the $u_n(t)$ satisfies

$$
\begin{cases}
(\rho_h u_n''(t), v) + (\mu\, u_n'(t), v) + (D(u_{n,x_1x_1} + \nu u_{n,x_2x_2}), v_{x_1x_1}) \\
+(D(u_{n,x_2x_2} + \nu u_{n,x_1x_1}), v_{x_2x_2}) + 2(1-\nu)(Du_{n,x_1x_2}, v_{x_1x_2}) \\
\qquad\qquad = (FG(t), v)\ \ \forall\, v \in W_n, \ \ t \in [0,T], \\
u_n(0) = u_{0,n}, \ \ u_n'(0) = v_{0,n}.
\end{cases}
\tag{4.5}
$$

It easy to see that the problem (4.5) is equivalent to the following system of ODE:

$$
\begin{cases}
MR_n''(t) + NR_n'(t) + \Big[P + Q \ + \ 2(1-\nu)S\Big] R_n(t) \\
\qquad\qquad\qquad\qquad = \ \mathrm{g}_n(t), \ \ \text{for } \ t \in [0,T], \\
\qquad R_n(0) = \bar{U}_n, \ \ R_n'(0) \ = \ \bar{V}_n,
\end{cases}
$$

where $R_n(t) = (r_{1,n}(t), r_{2,n}(t), ..., r_{n,n}(t))^T$, the entries of the matrix M, N, P, Q, S are

$$\begin{aligned} M &= [(\rho_h \xi_i, \xi_j)]^T_{n\times n}, \quad N = [(\mu\xi_i, \xi_j)]^{\mathrm{T}}_{n\times n}, \\ P &= [(D(\xi_{i,x_1x_1} + \nu\xi_{i,x_2x_2}), \xi_{j,x_1x_1})]^{\mathrm{T}}_{n\times n}, \\ Q &= [(D(\xi_{i,x_2x_2} + \nu\xi_{i,x_1x_1}), \xi_{j,x_2x_2})]^{\mathrm{T}}_{n\times n}, \quad S = [(D\xi_{i,x_1x_2}, \xi_{j,x_1x_2}]^{\mathrm{T}}_{n\times n}, \end{aligned}$$

and $\mathrm{g}_j(t) = (FG(t), \xi_j), \quad \mathrm{g}_n(t) = (\mathrm{g}_1(t), \mathrm{g}_2(t), ..., \mathrm{g}_n(t))^{\mathrm{T}}$,

$U_j = (u_0, \xi_j), \quad V_j = (v_0, \xi_j), \quad \bar{U}_n = (U_{1,}, U_2, ..., U_n)^{\mathrm{T}}, \;\; \bar{V}_n = (V_1, V_2, ..., V_n)^{\mathrm{T}}$.

By using the standard theory of linear ODE, for every $n \geq 1$ there exists a unique solution $u_n \in C^1([0,T]; W_n)$ with $u''_n \in L^2(0,T; W_n)$ of problem (4.5).

Theorem 4.1. *Suppose Assumption 4.1 holds true. Then there exists a unique weak solution u of the problem (4.1) in the sense of Definition 4.1. Moreover,*

$$\begin{aligned} \|u_t\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq \frac{C_e+1}{C_0} M(F,G,u_0,v_0), && (4.6)\\ \|u_t\|^2_{L^2(0,T;L^2(\Omega))} &\leq C_e\, M(F,G,u_0,v_0), && (4.7)\\ \|u\|^2_{L^2(0,T;\mathcal{V}^2(\Omega))} &\leq \frac{C_0 C' C_e}{D_0(1-\nu)} M(F,G,u_0,v_0), && (4.8)\\ \|u_{tt}\|^2_{L^2(0,T;\mathcal{V}^2(\Omega)')} &\leq \tilde{C}_0 M(F,G,u_0,v_0), && (4.9) \end{aligned}$$

and

$$\sum_{i,j=1}^{2} \|u_{x_ix_j}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \frac{C_e+1}{D_0(1-\nu)} M(F,G,u_0,v_0), \qquad (4.10)$$

where $M(F,G,u_0,v_0) := \Big[\|F\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} + D_1(1+\nu)\sum_{i,j=1}^2 \|u_{0,x_ix_j}\|^2_{L^2(\Omega)}$ $+C_1\|v_0\|^2_{L^2(\Omega)}\Big]$, $C_e = \exp(T/C_0) - 1$, $C_0 = \rho_0 h_0$, $C_1 = \rho_1 h_1$, $\rho_0, \rho_1, D_0, D_1, h_0, h_1 > 0$ *are the constants introduced in (4.4) and the constant $\tilde{C}_0$ is given in the proof.*

Proof. Since $u_n \in H^2(0,T;\mathcal{V}^2(\Omega))$, we may choose $v = u'_n$ as a test function in (4.5) integrate it over $\Omega_t := \Omega \times (0,t)$ and then apply the integration by parts formula. Taking into account the non-homogeneous initial and homogeneous boundary conditions in (4.1) and doing elementary transformations, we obtain the following energy identity:

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \rho_h(x)u_n'(t)^2dx + \int_0^t\int_\Omega \mu(x)u_n'(\tau)^2dxd\tau \\
&+\quad \frac{1}{2}\int_\Omega D(x)\left[u_{n,x_1x_1}^2 + 2u_{n,x_1x_2}^2 + u_{n,x_2x_2}^2 + 2\nu(u_{n,x_1x_1}u_{n,x_2x_2} - u_{n,x_1x_2}^2)\right]dx \\
&\quad = \int_0^t\int_\Omega G(\tau)F(x)u_n'(\tau)dxd\tau + \frac{1}{2}\int_\Omega \rho_h(x)u_n'(0^+)^2dx \\
&\qquad +\frac{1}{2}\int_\Omega D(x)\left[u_{n,x_1x_1}^2 + 2u_{n,x_1x_2}^2 + u_{n,x_2x_2}^2\right. \\
&\qquad \left. + 2\nu(u_{n,x_1x_1}u_{n,x_2x_2} - u_{n,x_1x_2}^2)\right]_{t=0^+}dx, \qquad (4.11)
\end{aligned}
$$

for all $t \in [0,T]$. To evaluate the terms in the third left-hand side and the last right-hand side integrals, we use the inequality

$$
\begin{aligned}
&\left|\int_\Omega 2\nu D(x)\left[u_{n,x_1x_1}u_{n,x_2x_2} - u_{n,x_1x_2}^2\right]dx\right| \\
&\leq \int_\Omega \nu D(x)\left[u_{n,x_1x_1}^2 + 2u_{n,x_1x_2}^2 + u_{n,x_2x_2}^2\right]dx,\ \forall u_n \in L^2(0,T;\mathcal{V}^2(\Omega)). \quad (4.12)
\end{aligned}
$$

Since $D(x) \geq D_0 > 0$ and $\nu \in (0,1)$, making use (4.12) in (4.11), we obtain

$$
\begin{aligned}
&C_0\int_\Omega u_n'(t)^2dx + D_0(1-\nu)\int_\Omega\left[u_{n,x_1x_1}^2 + 2u_{n,x_1x_2}^2 + u_{n,x_2x_2}^2\right]dx \\
&+\quad 2\int_0^t\int_\Omega \mu(x)u_n'(\tau)^2dxd\tau \\
&\leq \quad \int_0^t\int_\Omega u_n'(\tau)^2dxd\tau + \|F\|_{L^2(\Omega)}^2\|G\|_{L^2(0,T)}^2 \qquad (4.13) \\
&\qquad +D_1(1+\nu)\sum_{i,j=1}^2\|u_{0,n,x_ix_j}\|_{L^2(\Omega)}^2 + C_1\|v_{0,n}\|_{L^2(\Omega)}^2,
\end{aligned}
$$

where $C_0 = \rho_0 h_0$ and $C_1 = \rho_1 h_1$. From integral inequality (4.13), we deduce that

$$
\begin{aligned}
C_0\int_\Omega u_n'(t)^2dx \quad \leq \quad & \int_0^t\int_\Omega u_n'(\tau)^2dxd\tau + \|F\|_{L^2(\Omega)}^2\|G\|_{L^2(0,T)}^2 \\
& +D_1(1+\nu)\sum_{i,j=1}^2\|u_{0,n,x_ix_j}\|_{L^2(\Omega)}^2 + C_1\|v_{0,n}\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Applying Gronwall's inequality, we get

$$\|u_n'(t)\|^2_{L^2(\Omega)} \leq \frac{\exp(t/C_0)}{C_0} M(F,G,u_0,v_0), \tag{4.14}$$

where $M(F,G,u_0,v_0)$ is the constant defined in (4.6). This implies that

$$\max_{t\in[0,T]} \|u_n'(t)\|^2_{L^2(\Omega)} \leq \frac{C_e+1}{C_0} M(F,G,u_0,v_0), \tag{4.15}$$

$$\|u_n'\|^2_{L^2(0,T;L^2(\Omega))} \leq C_e\, M(F,G,u_0,v_0) \tag{4.16}$$

where $C_e = \exp(T/C_0) - 1, C_0 = \rho_0 h_0$, $C_1 = \rho_1 h_1$. The second consequence of energy estimate (4.13) is the inequality

$$\begin{aligned} &D_0(1-\nu)\int_\Omega \left[u^2_{n,x_1x_1} + 2u^2_{n,x_1x_2} + u^2_{n,x_2x_2}\right] dx \\ &\leq \int_0^t \int_\Omega u_n'(\tau)^2 dx d\tau + \|F\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} \\ &\quad + D_1(1+\nu)\sum_{i,j=1}^2 \|u_{0,n,x_ix_j}\|^2_{L^2(\Omega)} + C_1\|v_{0,n}\|^2_{L^2(\Omega)}. \end{aligned}$$

Using the estimate (4.14), we deduce that

$$\sum_{i,j=1}^2 \|u_{n,x_ix_j}(t)\|^2_{L^2(\Omega)} \leq \frac{\exp(t/C_0)}{D_0(1-\nu)} M(F,G,u_0,v_0). \tag{4.17}$$

Taking maximum over $t \in [0,T]$, we get

$$\max_{t\in[0,T]} \sum_{i,j=1}^2 \|u_{n,x_ix_j}(t)\|^2_{L^2(\Omega)} \leq \frac{C_e+1}{D_0(1-\nu)} M(F,G,u_0,v_0). \tag{4.18}$$

Integrating (4.17) with respect to time, we obtain

$$\sum_{i,j=1}^2 \|u_{n,x_ix_j}\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{C_0 C_e}{D_0(1-\nu)} M(F,G,u_0,v_0). \tag{4.19}$$

We use the identity

$$\int_\Omega \left(u_{n,x_1x_1}u_{n,x_2x_2} - u^2_{n,x_1x_2}\right) dx = 0,$$

which holds for clamped and simply supported plates [91], and that can be easily proven by transforming the integrals and definition of Δu_n to obtain

$$\int_\Omega \left[u_{n,x_1x_1}^2 + 2u_{n,x_1x_2}^2 + u_{n,x_2x_2}^2\right] dx = \int_\Omega |\Delta u_n|^2 dx,$$

whence $\|\Delta u_n\|^2_{L^2(0,T;L^2(\Omega))} = \sum_{i,j=1}^2 \|u_{n,x_ix_j}\|^2_{L^2(0,T;L^2(\Omega))}$.
Since $\|u_n(t)\|^2_{\mathcal{V}^2(\Omega)} \leq C'\|\Delta u_n(t)\|^2_{L^2(\Omega)}$, using the estimate (4.19), we get

$$\|u_n\|^2_{L^2(0,T;\mathcal{V}^2(\Omega))} \;\; \leq \;\; \frac{C_0C'C_e}{D_0(1-\nu)}\, M(F,G,u_0,v_0). \tag{4.20}$$

In order to estimate $\|u_n''\|_{L^2(0,T;\mathcal{V}^2(\Omega)')}$, we proceed as follows. Instead of (4.5), we consider the weak form

$$\begin{aligned}(u_n''(t),v) \;\; = \;\; & -(\mu u_n'(t), v/\rho_h) - \big(D(u_{n,x_1x_1}+\nu u_{n,x_2x_2}), (v/\rho_h)_{x_1x_1}\big) \\ & - \big(D(u_{n,x_2x_2}+\nu u_{n,x_1x_1}), (v/\rho_h)_{x_2x_2}\big) \\ & -2(1-\nu)\big(Du_{n,x_1x_2}, (v/\rho_h)_{x_1x_2}\big) + (FG(t), v/\rho_h)\,, \;\; \forall\, v\in W_n.\end{aligned} \tag{4.21}$$

Consider $v \in \mathcal{V}^2(\Omega)$ with $\|v\|_{\mathcal{V}^2(\Omega)} \leq 1$ and represent v as $v = \sigma + \beta$, where $\sigma \in W_n$ and $\beta \in W_n^\perp,\; \|\sigma\|_{\mathcal{V}^2(\Omega)} \leq \|v\|_{\mathcal{V}^2(\Omega)} \leq 1$. Since (4.21) holds for all $v \in W_n$, which also holds for $\sigma \in W_n$, by applying Cauchy's inequality, we estimate the right-hand as follows

$$\begin{aligned}|(u_n''(t),v)| \;\; \leq \;\; & \frac{1}{C_0}\mu_1\|u_n'(t)\|_{L^2(\Omega)}\|\sigma\|_{L^2(\Omega)} \\ & +D_1\left(\|u_{n,x_1x_1}\|_{L^2(\Omega)} + \nu\|u_{n,x_2x_2}\|_{L^2(\Omega)}\right)\|\,(\sigma/\rho_h)_{x_1x_1}\,\|_{L^2(\Omega)} \\ & +D_1\left(\|u_{n,x_2x_2}\|_{L^2(\Omega)} + \nu\|u_{n,x_1x_1}\|_{L^2(\Omega)}\right)\|\,(\sigma/\rho_h)_{x_2x_2}\,\|_{L^2(\Omega)} \\ & +D_1 2(1-\nu)\|u_{n,x_1x_2}\|_{L^2(\Omega)}\|\,(\sigma/\rho_h)_{x_1x_2}\,\|_{L^2(\Omega)} \\ & +\frac{1}{C_0}\|F\|_{L^2(\Omega)}\|G\|_{L^2(0,T)}\|\sigma\|_{L^2(\Omega)}.\end{aligned} \tag{4.22}$$

Next, we need to estimate $\|\Delta\,(\sigma/\rho_h)\,\|_{L^2(\Omega)}$. It is easy to see that

$$\Delta\left(\frac{\sigma}{\rho_h}\right) = \frac{\Delta\sigma}{\rho_h} - \frac{2}{\rho_h}\left(\nabla\,(\sigma/\rho_h)\cdot\nabla(\rho_h)\right) - \frac{\sigma\Delta(\rho_h)}{(\rho_h)^2}.$$

By Assumption 4.1, there exists a constant C_2 depending on $\rho_i, h_i, i = 0,1,2$ such that $\|\Delta\,(\sigma/\rho_h)\,\|^2_{L^2(\Omega)} \leq C_2/C_0$. It shows that $\|\,(\sigma/\rho_h)_{x_1x_1}\,\|^2_{L^2(\Omega)} \leq C_2/C_0,\; \|\,(\sigma/\rho_h)_{x_2x_2}\,\|^2_{L^2(\Omega)} \leq C_2/C_0$. Similarly, we can also find a constant C_3 depending on $\rho_i, h_i, i = 0,1,2$ such that $\|\,(\sigma/\rho_h)_{x_1x_2}\,\|^2_{L^2(\Omega)} \leq C_3/C_0$. Choosing $C_4 = \max\{1, \mu_1, D_1\,(1+\nu)\,C_2, 2D_1(1-\nu)C_3\}$, and taking supremum over $v \in \mathcal{V}^2(\Omega)$ with $\|v\|_{\mathcal{V}^2(\Omega)} \leq 1$ on (4.22), we obtain

$$\begin{aligned} \|u_n''(t)\|_{\mathcal{V}^2(\Omega)'} &\leq \frac{C_4}{C_0}\left[\|u_n'(t)\|_{L^2(\Omega)} + \|u_{n,x_1x_1}\|_{L^2(\Omega)} + \|u_{n,x_2x_2}\|_{L^2(\Omega)}\right. \\ &\quad \left. + \|u_{n,x_1x_2}\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}\|G\|_{L^2(0,T)}\right]. \end{aligned}$$

Squaring on both sides, integrating over $(0,T)$ and using the estimates (4.16),(4.19), we get

$$\|u_n''\|^2_{L^2(0,T;\mathcal{V}^2(\Omega)')} \leq \tilde{C}_0 M(F,G,u_0,v_0), \tag{4.23}$$

where $\tilde{C_0} = 5(C_4/C_0)^2(T + C_e(1 + 3C_0/D_0(1-\nu)))$.

By invoking the estimates (4.20), (4.19), (4.16) and (4.23), we see that the sequences $\{u_n\}, \{u_{n,x_1x_2}\}, \{u_n'\}$ and $\{u_n''\}$ are bounded in $L^2(0,T;\mathcal{V}^2(\Omega)), L^2(0,T;L^2(\Omega))$, $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;\mathcal{V}^2(\Omega)')$ respectively. Then by the Banach-Alaoglu weak compactness theorem (see [15], Theorem 3.16), there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and functions $u \in L^2(0,T;\mathcal{V}^2(\Omega))$, $u_{x_1x_2} \in L^2(0,T;L^2(\Omega))$, $u' \in L^2(0,T;L^2(\Omega))$, and $u'' \in L^2(0,T;\mathcal{V}^2(\Omega)')$ such that

$$\begin{cases} u_{n_k} \rightharpoonup u & \text{weakly in} \quad L^2(0,T;\mathcal{V}^2(\Omega)) \\ u_{n_k,x_1x_2} \rightharpoonup u_{x_1x_2} & \text{weakly in} \quad L^2(0,T;L^2(\Omega)) \\ u_{n_k}' \rightharpoonup u' & \text{weakly in} \quad L^2(0,T;L^2(\Omega)) \\ u_{n_k}'' \rightharpoonup u'' & \text{weakly in} \quad L^2(0,T;\mathcal{V}^2(\Omega)'). \end{cases} \tag{4.24}$$

By passing these limits on (4.5), we should be able to get the weak solution u of the direct problem (4.1), which satisfies (4.6)-(4.10) in view of (4.15),(4.16),(4.20),(4.23),(4.18) respectively.

The direct outcome of the estimate (4.10) is the uniqueness of the direct problem (4.1). Let $u_1,\ u_2 \in \mathcal{V}^2(\Omega)$ be the two weak solutions of the direct problem (4.1). Then $\mathcal{U}(x,t) := u_1(x,t) - u_2(x,t) \in \mathcal{V}^2(\Omega)$ is the solution of (4.1) with homogeneous initial data $u_0(x) = v_0(x) = 0$ and source function $F(x)G(t) = 0$. Using (1.17) and (4.10) , we conclude that

$$\|\mathcal{U}\|_{L^\infty(0,T;\mathcal{V}^2(\Omega))} = 0, \text{ whence } \mathcal{U}(x,t) = 0,\ \forall\ (x,t) \in \Omega_T.$$

It remains to check that $u(t)$ satisfies initial conditions $u(0) = u_0$ and $u_t(0) = v_0$. It is clear from the existence theory that (see Remark 4.1) $u \in C([0,T];H^1(\Omega))$ and $u' \in C([0,T];\mathcal{V}^2(\Omega)')$.

Consider a test function $v \in C^2([0,T];\mathcal{V}^2(\Omega))$ with $v(T)=0$ and $v'(T)=0$. Taking the weak form of Definition 4.1-(i) satisfied by u, integrating over $(0,T)$ and integrating by parts twice with respect to time in the first term, we arrive at

$$I = \int_0^T (FG(t), v(t))dt \ + (\rho_h v(0), u'(0)) - (\rho_h v'(0), u(0)), \tag{4.25}$$

where

$$\begin{aligned} I \ := \ & \int_0^T \Big((u(t), \rho_h v''(t)) + (\mu u(t), v'(t)) + (D(u_{x_1x_1}(t) \\ & + \nu u_{x_2x_2}(t)), v_{x_1x_1}(t)) + (D(u_{x_2x_2}(t) + \nu u_{x_1x_1}(t)), v_{x_2x_2}(t)) \\ & + 2(1-\nu)(Du_{x_1x_2}(t), v_{x_1x_2}(t)) \Big) dt. \end{aligned}$$

On the other hand, time integrating by parts twice in the first term of (4.5) and passing the weak limit (4.24), we deduce that

$$I = \int_0^T (FG(t), v(t))dt \ + (\rho_h v(0), v_0) - (\rho_h v'(0), u_0). \tag{4.26}$$

Comparing (4.25) and (4.26), we obtain the desired result. This completes the proof. $\square$

4.2 Inverse problem

In this section, first, the inverse problem is formulated as an operator equation which includes the input–output operator introduced below. Then we reformulate it as a minimum problem for the regularized Tikhonov functional and discuss the well-posedness of the inverse problem. Using the estimates of solutions of the direct problem, we prove the compactness and Lipschitz continuity of the input–output operator, which allow us to prove the existence of a minimizer for the regularized Tikhonov functional. We also compute the Frechet derivative of the Tikhonov functional and establish the Lipschitz continuity of the Fréchet derivative.

4.2.1 Mathematical formulation of inverse problem

Let us define a set of admissible sources:

$$\mathcal{F} = \{F \in L^2(\Omega) : \ \|F\|_{L^2(\Omega)} \le \gamma, \ \gamma > 0\}.$$

For a given $F \in \mathcal{F},\ u(x,t;F)$ is the corresponding weak solution of the direct problem (4.1). Now we define input-output operator as follows:

$$\begin{aligned} \Phi \quad &: \quad \mathcal{F} \subset L^2(\Omega) \mapsto H_0^1(\Omega) \subset L^2(\Omega), \\ (\Phi F)(x) \quad &:= \quad u(x,t;F)|_{t=T}. \end{aligned} \tag{4.27}$$

We shall write this inverse problem in terms of the functional equation as

$$\Phi F = u_T, \quad F \in \mathcal{F}, \quad u_T \in L^2(\Omega). \tag{4.28}$$

We notice that in the case of noiseless measured output data u_T, the solution of the inverse problem is the solution of the functional equation (4.28). However, one may observe that due to measurement error in the measured output $u_T(x)$, the exact equality in the functional equation (4.28) is not possible in practice. We formulate this inverse problem as a minimization problem defined by a Tikhonov functional as follows:

$$\mathcal{J}(F) := \frac{1}{2}\|\Phi F - u_T\|^2_{L^2(\Omega)}. \tag{4.29}$$

The regularized form of the Tikhonov functional is written as

$$\mathcal{J}_\alpha(F) := \frac{1}{2}\|\Phi F - u_T\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|F\|^2_{L^2(\Omega)}, \tag{4.30}$$

where $\alpha > 0$ is the parameter of regularization. We may now consider the inverse problem which we are interested as a minimum problem for the functional $\mathcal{J}_\alpha(F)$ on the set $\mathcal{F}$. More precisely, we need to solve the following problem:

Minimization Problem: find $F \in \mathcal{F}$ which minimizes the functional $\mathcal{J}_\alpha(F)$ subject to $u(x,t;F)$ solves the problem (4.1).

4.2.2 Compactness and Lipschitz continuity of the input-output operator

In this subsection, we prove that the input-output operator Φ is compact and Lipschitz continuous by using an *a priori* estimates for weak solution of the direct problem (4.1). By the compactness of the operator, we conclude that the inverse problem is ill-possed.

Proposition 4.1. *Suppose Assumption 4.1 holds. Then the input-output operator* $\Phi : \mathcal{F} \subset L^2(\Omega) \mapsto H_0^1(\Omega) \subset L^2(\Omega)$ *defined by (4.27) is compact. Moreover, the operator* Φ *is*

Lipschitz continuous:

$$\|\Phi(F_1) - \Phi(F_2)\|_{L^2(\Omega)} \leq L_0 \|F_1 - F_2\|_{L^2(\Omega)} \ \forall \ F_1, F_2 \in \mathcal{F},$$

where the Lipschitz constant $L_0 = (TC_e)^{1/2} \|G\|_{L^2(0,T)}$.

Proof. Let $\{F_m\} \subset \mathcal{F}, m = 1, 2, ...,$ be a bounded sequence of sources in $L^2(\Omega)$ and $u(x, T; F_m)$ denoted by $\{u_{Tm}\}$ is the corresponding output. The estimate (4.10) and Remark 4.1 show that the sequence of output $\{u_{Tm}\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the sequence of outputs $\{u_{Tm}\}$ is precompact in $L^2(\Omega)$, that is, the input-output operator Φ maps bounded sequence of inputs to a precompact sequence of outputs. Hence Φ is compact.

Next, we show that Φ is Lipschitz continuous. Let $u(x, t; F_1)$ and $u(x, t; F_2)$ be the solutions of the direct problem (4.1) corresponding to the source terms F_1 and F_2 respectively for a common initial data u_0 and v_0. Then $\delta u(x, t) = u(x, t; F_1) - u(x, t; F_2)$ solves the following problem

$$\begin{cases} \rho_h(x)\delta u_{tt} + \mu(x)\delta u_t + (D(x)(\delta u_{x_1x_1} + \nu \delta u_{x_2x_2}))_{x_1x_1} \\ \qquad + (D(x)(\delta u_{x_2x_2} + \nu \delta u_{x_1x_1}))_{x_2x_2} \\ \qquad +2(1-\nu)\left(D(x)\delta u_{x_1x_2}\right)_{x_1x_2} = \delta F(x)G(t), \ (x,t) \in \Omega_T, \\ \delta u(x, 0) = 0, \ \delta u_t(x, 0) = 0, \ x \in \Omega, \\ \delta u(x, t) = 0, \ \delta u_{x_2x_2}(x, t) = 0, \ (x, t) \in (\Gamma_1 \cap \Gamma_3) \times [0, T], \\ \delta u(x, t) = 0, \ \delta u_{x_1x_1}(x, t) = 0, \ (x, t) \in (\Gamma_2 \cap \Gamma_4) \times [0, T], \end{cases} \tag{4.31}$$

where $\delta F(x) = F_1(x) - F_2(x)$. From the definition of the operator Φ, we have

$$\begin{aligned} \|\Phi(F_1) - \Phi(F_2)\|_{L^2(\Omega)} &= \|u(x, T; F_1) - u(x, T; F_2)\|_{L^2(\Omega)} \\ &= \|\delta u(., T)\|_{L^2(\Omega)}. \end{aligned}$$

Now we use the inequality

$$\|\delta u(., T)\|^2_{L^2(\Omega)} \leq T \|\delta u_t\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.32}$$

since the estimate (4.7) holds for $\delta u_t(x, t)$, we get

$$\|\Phi(F_1) - \Phi(F_2)\|^2_{L^2(\Omega)} \leq TC_e \|G\|^2_{L^2(0,T)} \|F_1 - F_2\|^2_{L^2(\Omega)}.$$

This completes the proof. □

Remark 4.2. Since the input-output operator Φ is compact, the inverse source problem (4.1)-(4.2) is ill-posed in the sense of Hadamard (see, [40] and also, [50], Lemma 1.3.1).

4.2.3 Lipschitz continuity of the Fréchet derivative of the functional

To derive the Fréchet derivative of the functional, we introduce an adjoint system with final data in terms of the measured data, derive an *a priori* estimates, and discuss the well-posedness of the problem. Note that the increment $\delta\mathcal{J}(F) := \mathcal{J}(F+\delta F) - \mathcal{J}(F)$ of the Tikhonov functional $\mathcal{J}(F)$ satisfies the following identity

$$\delta\mathcal{J}(F) = \int_\Omega [u(x,T;F) - u_T(x)]\,\delta u(x,T)dx + \frac{1}{2}\int_\Omega (\delta u(x,T))^2\,dx, \tag{4.33}$$

where $\delta u(x,t) = u(x,t;F+\delta F) - u(x,t;F)$ solves the problem (4.31). We express the first integral of $\delta\mathcal{J}$, which consists of both $u(x,t;F+\delta F)$ and $u(x,t;F)$, in terms of an integral with the solution of an adjoint system alone.

Lemma 4.1. *Let Assumption 4.1 holds true and $u_T \in L^2(\Omega)$. Then the following integral relationship between the inputs and outputs holds:*

$$-\int_\Omega \rho_h(x)q(x)\delta u(x,T)dx = \int_\Omega \left(\int_0^T G(t)\psi(x,t;q)dt\right)\delta F(x)dx, \tag{4.34}$$

for all $F \in L^2(\Omega)$, where $\psi \in L^2(0,T;\mathcal{V}^2(\Omega))$ is the weak solution of the following backward problem

$$\begin{cases} \rho_h(x)\psi_{tt} - \mu(x)\psi_t + \left(D(x)(\psi_{x_1x_1} + \nu\psi_{x_2x_2})\right)_{x_1x_1} \\ \qquad + \left(D(x)(\psi_{x_2x_2} + \nu\psi_{x_1x_1})\right)_{x_2x_2} \\ \qquad +2(1-\nu)\left(D(x)\psi_{x_1x_2}\right)_{x_1x_2} = 0,\ (x,t)\in\Omega_T, \\ \psi(x,T) = 0,\ \psi_t(x,T) = q(x),\ x\in\Omega, \\ \psi(x,t) = 0,\ \psi_{x_2x_2}(x,t) = 0,\ (x,t)\in(\Gamma_1\cap\Gamma_3)\times[0,T], \\ \psi(x,t) = 0,\ \psi_{x_1x_1}(x,t) = 0,\ (x,t)\in(\Gamma_2\cap\Gamma_4)\times[0,T], \end{cases} \tag{4.35}$$

with the (final velocity) input $q(x)$ at $t = T$, and $\delta u \in L^2(0,T;\mathcal{V}^2(\Omega))$ is the weak solution of the problem (4.31).

Proof. Multiply both sides of equation (4.31) by an arbitrary function $\psi(x,t)$, integrate over Ω_T. We have:

$$
\begin{aligned}
&\int_{\Omega_T} \left[\rho_h(x)\psi_{tt} - \mu(x)\psi_t\right] \delta u \, dtdx \\
&+ \int_{\Omega} \left[\rho_h(x)\delta u_t \psi - \rho_h(x)\delta u \psi_t + \mu \delta u \psi\right]_{t=0}^{t=T} dx \\
&+ \int_{\Omega_T} \big[\left(D(x)(\delta u_{x_1x_1} + \nu \delta u_{x_2x_2})\right)_{x_1x_1} + \left(D(x)(\delta u_{x_2x_2} + \nu \delta u_{x_1x_1})\right)_{x_2x_2} \\
&\quad +2(1-\nu)\left(D(x)\delta u_{x_1x_2}\right)_{x_1x_2} \big] \psi \, dxdt = \int_{\Omega_T} \delta F(x) G(t) \psi(x,t) dxdt.
\end{aligned}
$$

Now we use the fact that the arbitrary function $\psi(x,t)$ solves the backward problem (4.35). Applying the integration by parts formula multiple times to the third left-hand side integral and taking into account the homogeneous boundary and initial/final conditions in (4.31) and (4.35), we arrive at the following integral identity:

$$
-\int_{\Omega} \rho_h(x)\delta u(x,T)\psi_t(x,T)dx = \int_{\Omega_T} \delta F(x) G(t) \psi(x,t) dxdt.
$$

This leads to the required relationship (4.34). □

Note that the backward problem (4.35) is a well-posed as the change of the variable t with $\tau = T - t$ shows. Hence all the estimates (4.6)-(4.9) derived in Theorem 4.1 can also be applied to the solution of this problem.

Lemma 4.2. *Suppose Assumption 4.1 holds true and let $q \in L^2(\Omega)$. Then the weak solution ψ of (4.35) satisfies the following estimates*

$$
\begin{aligned}
\|\psi_t\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq \frac{C_e+1}{C_0} C_1 \|q\|^2_{L^2(\Omega)}, \\
\|\psi_t\|^2_{L^2(0,T;L^2(\Omega))} &\leq C_e C_1 \|q\|^2_{L^2(\Omega)}, \\
\sum_{i,j=1}^{2} \|\psi_{x_ix_j}\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq \frac{C_e+1}{D_0(1-\nu)} C_1 \|q\|^2_{L^2(\Omega)}, \\
\|\psi_{tt}\|^2_{L^2(0,T;\mathcal{V}^2(\Omega)')} &\leq \tilde{C}_0 C_1 \|q\|^2_{L^2(\Omega)},
\end{aligned}
\tag{4.36}
$$

where the constants C_e, C_0, D_0, C_1 and $\tilde{C}_0$ are defined in Theorem 4.1.

Assume now that the arbitrary final time input $q(x)$ in the backward problem (4.35) is specified as follows:

$$q(x) = -\frac{1}{\rho_h(x)}[u(x,T;F) - u_T(x)],\ x \in \Omega. \tag{4.37}$$

The backward problem (4.35) with the final time (velocity) input (4.37) is defined as an *adjoint problem* corresponding to the inverse problem (4.1) and (4.2). Substituting the input (4.37) in the integral relationship (4.34), we obtain the main input-output relationship:

$$\begin{aligned}&\int_\Omega [u(x,T;F) - u_T(x)]\delta u(x,T)dx \\ &\qquad = \int_\Omega \left(\int_0^T G(t)\psi(x,t;F)dt\right)\delta F(x)dx,\end{aligned} \tag{4.38}$$

where $\psi(x,t;F)$ is the solution of the backward problem (4.35) with the input defined in (4.37) which, in turn, depends on the given input $F \in \mathcal{F}$. In view of the input-output relationship (4.38), the increment formula (4.33) takes the following new form:

$$\delta\mathcal{J}(F) = \int_\Omega \left(\int_0^T G(t)\psi(x,t;F)dt\right)\delta F(x)dx + \frac{1}{2}\int_\Omega (\delta u(x,T))^2\,dx. \tag{4.39}$$

Proposition 4.2. *Let the conditions of Lemma 4.2 hold true and $\psi(x,t;F)$ be the solution of the adjoint problem (4.35). Then the Tikhonov functional $\mathcal{J}(F)$ is Fréchet differentiable at $F \in \mathcal{F}$ and the derivative is given by,*

$$\mathcal{J}'(F)(x) = \int_0^T \psi(x,t;F)G(t)dt, \quad F \in L^2(\Omega). \tag{4.40}$$

Moreover, the Fréchet gradient $\mathcal{J}'(F)$ is Lipschitz continuous:

$$\|\mathcal{J}'(F+\delta F) - \mathcal{J}'(F)\|_{L^2(\Omega)} \le L_1\,\|\delta F\|_{L^2(\Omega)}, \tag{4.41}$$

where the Lipschitz constant $L_1 = \frac{\sqrt{TC_1}}{\sqrt{2}C_0}TC_e\|G\|^2_{L^2(0,T)}$, and the constants C_0, C_e, C_1 are defined in Theorem 4.1.

Proof. Using the integral identity (4.39), we need to show that the following holds to get the Fréchet derivative of $\mathcal{J}(F)$:

$$\left|\delta\mathcal{J}(F) - \int_\Omega \left(\int_0^T \psi(x,t;F)G(t)dt\right)\delta F(x)dx\right| = o\left(\|\delta F\|_{L^2(\Omega)}\right).$$

As a consequence of estimates (4.7) and (4.32) , we can show that the last integral of (4.39) is $o\left(\|\delta F\|_{L^2(\Omega)}\right)$:

$$\frac{\|\delta u(.,T)\|^2_{L^2(\Omega)}}{\|\delta F\|_{L^2(\Omega)}} \;\leq\; TC_e\|\delta F\|_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} \to 0 \;\text{ as }\; \|\delta F\|_{L^2(\Omega)} \to 0^+.$$

By the definition of the Fréchet derivative, we arrive at the formula (4.40).

Next, we show that $\mathcal{J}'(F)$ is Lipschitz continuous. From the computation of the Fréchet derivative, we have

$$\|\mathcal{J}'(F+\delta F) - \mathcal{J}'(F)\|^2_{L^2(\Omega)} = \int_\Omega \left(\int_0^T \delta\psi(x,t;\delta F)G(t)dt\right)^2 dx, \tag{4.42}$$

where $\delta\psi(x,t) = \psi(x,t;F+\delta F) - \psi(x,t;F)$ is the solution of the following adjoint problem:

$$\begin{cases} \rho_h(x)\delta\psi_{tt} - \mu(x)\delta\psi_t + \left(D(x)(\delta\psi_{x_1x_1} + \nu\delta\psi_{x_2x_2})\right)_{x_1x_1} \\ \qquad + \left(D(x)(\delta\psi_{x_2x_2} + \nu\delta\psi_{x_1x_1})\right)_{x_2x_2} \\ \qquad\qquad +2(1-\nu)\left(D(x)\delta\psi_{x_1x_2}\right)_{x_1x_2} = 0,\ (x,t) \in \Omega_T, \\ \delta\psi(x,T) = 0,\ \delta\psi_t(x,T) = -\frac{1}{\rho_h(x)}[\delta u(x,T)],\ x \in \Omega, \\ \delta\psi(x,t) = 0,\ \delta\psi_{x_2x_2}(x,t) = 0,\ (x,t) \in (\Gamma_1 \cap \Gamma_3) \times [0,T], \\ \delta\psi(x,t) = 0,\ \delta\psi_{x_1x_1}(x,t) = 0,\ (x,t) \in (\Gamma_2 \cap \Gamma_4) \times [0,T], \end{cases} \tag{4.43}$$

and $\delta u(x,T)$ is the solution of the equation (4.31). By applying Hölder's inequality in (4.42), we obtain

$$\|\mathcal{J}'(F+\delta F) - \mathcal{J}'(F)\|^2_{L^2(\Omega)} \leq \|G\|^2_{L^2(0,T)}\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))}. \tag{4.44}$$

Using the estimate $\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} \leq (T^2/2)\|\delta\psi_t\|^2_{L^2(0,T,L^2(\Omega))}$ and (4.36), which also hold for $\delta\psi(x,t)$, we get

$$\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; \frac{T^2C_1C_e}{2C_0^2}\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} \leq \frac{T^3C_1C_e}{2C_0^2}\|\delta u_t\|^2_{L^2(0,T;L^2(\Omega))},$$

since $\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} \leq T\|\delta u_t\|^2_{L^2(0,T,L^2(\Omega))}$. Hence, using the estimate (4.7), we get

$$\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{T^3C_1C_e^2}{2C_0^2}\|\delta F\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)}. \tag{4.45}$$

Substituting (4.45) in (4.44), we obtain the required result (4.41). Hence the proof. □

It is worth noticing that in the determination of source terms (or) parameters through the numerical implementation of gradient type algorithm such as the Landweber iteration algorithm

$$F^{(n+1)}(x) = F^{(n)}(x) - w_n \mathcal{J}'(F^{(n)}(x)),\ n = 0, 1, 2, ...,$$

or CGA, the Lipschitz continuity of the Fréchet gradient plays a pivotal role (see, [42], [50]). In particular, when we apply these gradient type algorithm for linear and nonlinear inverse problems, we may face the difficulty in estimating the relaxation parameter $w_n > 0$. But in the case of Lipschitz continuity of Fréchet gradient, the relaxation parameter can be obtained via Lipschitz constant $L_1 > 0$. Moreover, when $w_n \in (0, \frac{2}{L_1})$, the Lipschitz continuity of the gradient of the Tikhonov functional implies the monotonicity of numerical sequence $\{\mathcal{J}(F^n)\}$, where $F^{(n)}(x)$ is the nth iteration of CGA. It leads to the convergence analysis of this numerical sequence.

4.2.4 Existence of a minimizer

In this subsection, we prove the existence of a unique minimizer $F \in \mathcal{F}$ for the regularized Tikhonov functional $\mathcal{J}_\alpha(F)$ using the classical arguments. One can also use the compactness and Lipschitz continuity of the input-output operator Φ to obtain the result.

Theorem 4.2. *Suppose Assumption 4.1 holds true. Then for any* $\alpha > 0$, *there exists a unique admissible source function* $F_{*,\alpha} \in \mathcal{F}$ *which minimizes the regularized Tikhonov functional* $\mathcal{J}_\alpha(F)$.

Proof. As the functional $\mathcal{J}_\alpha(F)$ is bounded from below, one can argue that there exists a minimizing sequence $\{F_n\} \in \mathcal{F}$ converges weakly to an admissible source $F \in \mathcal{F}$. Note that the following identity holds for the Tikhonov functional $\mathcal{J}(F)$:

$$\begin{aligned} \mathcal{J}(F_n) - \mathcal{J}(F) &= \frac{1}{2}\int_\Omega [u(x,T;F_n) - u(x,T;F)]^2\, dx \\ &\quad + \int_\Omega [u(x,T;F) - u_T(x)]\, \delta u(x,T) dx, \end{aligned} \tag{4.46}$$

where $\delta u(x,T) = u(x,T;F_n) - u(x,T;F)$. Replacing the last integral of (4.46) by the identity

$$\int_\Omega [u(x,T;F) - u_T(x)]\delta u(x,T) dx = \int_\Omega \left(\int_0^T G(t)\psi(x,t;F) dt \right) \delta F_n(x) dx,$$

where $\delta F_n(x) = F_n(x) - F(x)$, we have

$$\begin{aligned} \mathcal{J}(F_n) - \mathcal{J}(F) &= \frac{1}{2}\int_\Omega [u(x,T;F_n) - u(x,T;F)]^2\,dx \\ &\quad + \int_\Omega \int_0^T \psi(x,t;F)G(t)\delta F_n(x)\,dt\,dx, \end{aligned} \tag{4.47}$$

where $\psi(x,t;F)$ is the solution of (4.35). Applying Hölder's inequality, we get

$$\begin{aligned} \int_\Omega \left| \int_0^T \psi(x,t;F)G(t)dt \right|^2 dx &\le \int_\Omega \|\psi(x)\|^2_{L^2(0,T)} \|G\|^2_{L^2(0,T)}\,dx \\ &= \|\psi\|^2_{L^2(0,T;L^2(\Omega))} \|G\|^2_{L^2(0,T)} < +\infty, \end{aligned}$$

since $\psi \in L^2(0,T;L^2(\Omega))$ and $G \in L^2(0,T)$. It shows that

$$\int_0^T \psi(x,t;F)G(t)dt \in L^2(\Omega).$$

Since $F_n \rightharpoonup F$ weakly in $L^2(\Omega)$, by the definition of weak convergence, we get

$$\int_\Omega \left(\int_0^T \psi(x,t;F)G(t)dt \right) F_n(x)dx \to \int_\Omega \left(\int_0^T \psi(x,t;F)G(t)dt \right) F(x)dx,$$

as $n \to \infty$.

To prove the convergence of $\frac{1}{2}\int_\Omega [u(x,T;F_n) - u(x,T;F)]^2\,dx$, we proceed as follows. Let F_n be the minimizing sequence for the functional $\mathcal{J}(F)$ and $\{u(x,t;F_n)\}$ is the corresponding sequence of weak solution to the direct problem. Now assume that $F_n \rightharpoonup F$ in $\mathcal{F}$. The estimate (4.8) show that $\{u(x,t;F_n)\}$ is bounded in $L^2(0,T;\mathcal{V}^2(\Omega))$. Then there exists a subsequence of $\{u(x,t;F_n)\}$ which is denoted again by $u(x,t;F_n)$ such that $u(x,t;F_n) \rightharpoonup u^*(x,t)$ in $L^2(0,T;\mathcal{V}^2(\Omega))$. By choosing the test function as $v \in C^2(0,T;\mathcal{V}^2(\Omega))$ with $v(T) = v'(T) = 0$ for the weak form of Definition 4.1-i, integrate over $(0,T)$ and passing the limit $n \to \infty$, we get $u^*(x,t) = u(x,t;F)$. Moreover, the estimate (4.10) and Remark 4.1 show that the sequence $\{u(x,T;F_n)\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the sequence $\{u(x,T;F_n)\}$ is precompact in $L^2(\Omega)$. Hence there exists a subsequence $u(x,T;F_n)$ such that $u(x,t;F_n) \to u(x,T;F)$ in $L^2(\Omega)$. Hence, taking limit $n \to \infty$ in (4.47), we conclude that $\mathcal{J}(F) = \lim_{n\to\infty} \mathcal{J}(F_n)$.

Besides, since $F_n \rightharpoonup F$ in $L^2(\Omega)$, it is lower semi-continuous, that is, $\|F\|_{L^2(\Omega)} \le$

$\liminf_{n\to\infty} \|F_n\|_{L^2(\Omega)}$, and so we arrive at

$$\mathcal{J}_\alpha(F) \leq \liminf_{n\to\infty} \mathcal{J}_\alpha(F_n) \text{ as } F_n \rightharpoonup F \text{ in } \mathcal{F},$$

whence $\mathcal{J}_\alpha(F)$ is lower semi-continuous on $\mathcal{F}$.

Moreover, the functional $\mathcal{J}_\alpha(F)$ is strictly convex. Indeed, by the linearity of the inverse problem, we have

$$u(x,t;\nu F_1 + (1-\nu)F_2) = \nu u(x,t;F_1) + (1-\nu)u(x,t;F_2),\ \nu \in (0,1),$$

and hence the functional $\mathcal{J}_\alpha(F)$ is strictly convex.

By combining these arguments and using generalized Weierstrass theorem (see, Theorem 1.8), we conclude that the regularized functional $\mathcal{J}_\alpha(F)$ has a unique minimizer $F \in \mathcal{F}$. The proof is thus completed. □

We can deduce the representation of the source term $F \in \mathcal{F}$ using the solution of the adjoint problem (4.35) as follows. As in the case of the Fréchet gradient (4.40) of the functional $\mathcal{J}(F)$, the Fréchet gradient of the regularized Tikhonov functional $\mathcal{J}_\alpha(F)$ is given by

$$\mathcal{J}'_\alpha(F) = \int_0^T \psi(x,t;F)G(t)dt + \alpha F. \tag{4.48}$$

Moreover, by Theorem 4.2, the regularized functional $\mathcal{J}_\alpha(F)$ has a unique minimizer and this minimizer $F_{*,\alpha} \in L^2(\Omega)$ can be represented in terms of solution of adjoint problem as follows

$$F_{*,\alpha}(x) = -\frac{1}{\alpha}\int_0^T \psi(x,t;F_{*,\alpha})G(t)dt, \tag{4.49}$$

since $F_{*,\alpha}$ satisfy the necessary condition $\mathcal{J}'_\alpha(F_{*,\alpha}) = 0$.

4.3 Stability estimates for the regularized solution: variational approach

In this subsection, we first establish a first-order necessary optimality condition, which has to be satisfied by an optimal solution of the minimization problem. This optimality condition is the crucial ingredient in obtaining the stability estimate for the inverse source problem. As a direct consequence of this theorem 1.9, we have:

Corollary 4.1. *For the considered final data inverse source problem (4.1)-(4.2) with the gradient formula (4.40), for any $F_* \in \mathcal{F}_*$ the following variational inequality holds:*

$$\int_\Omega \int_0^T [F(x) - F_*(x)]\, G(t)\psi(x,t;F_*)dtdx \geq 0, \;\; \textit{for all } F \in \mathcal{F}, \tag{4.50}$$

where

$$\mathcal{F}_* := \{F \in \mathcal{F} \,:\, \mathcal{J}(F) = J_* := \inf_{\bar{F}\in\mathcal{F}} \mathcal{J}(\bar{F})\} \tag{4.51}$$

and $\mathcal{J}(F)$ is the Tikhonov functional defined in (4.29).

Proof. Note that the Tikhonov functional $\mathcal{J}$ defined by (4.29) with gradient formula (4.40) is continuously differentiable function on $\mathcal{F}$ and $\mathcal{F} \subset L^2(\Omega)$ is convex. By Theorem 1.9, for any $F_* \in \mathcal{F}_*$, the following integral inequality holds:

$$\int_\Omega \int_0^T [F(x) - F_*(x)]\, G(t)\psi(x,t;F_*)dtdx \geq 0, \;\; \text{for all } F \in \mathcal{F},$$

where the set $\mathcal{F}_*$ is defined in (4.51). □

Moreover, in the case of regularized Tikhonov functional $\mathcal{J}_\alpha(F)$, we have:

Corollary 4.2. *Let the conditions of Theorem 1.9 hold. Then for the unique minimizer $F_{*,\alpha} \in \mathcal{F}$ of the regularized Tikhonov functional (4.30) with the gradient formula (4.48), the following variational inequality holds:*

$$\begin{aligned}\int_\Omega \int_0^T [F(x) - F_{*,\alpha}(x)]\, G(t)\psi(x,t;F_{*,\alpha})dxdt \\ +\alpha \int_\Omega F_{*,\alpha}(x)\,[F(x) - F_{*,\alpha}(x)]\,dx \geq 0, \quad \textit{for all } F \in \mathcal{F}.\end{aligned} \tag{4.52}$$

Note that the connection of the final time inverse source problem for heat equation with a variational inequality of type (4.50) was first revealed in [42]. Next, we establish sufficient conditions on the final time and the damping parameter, which give the stability estimates for the source term in terms of the measured data.

Theorem 4.3. *Suppose Assumption 4.1 holds true. Let $F_{*,\alpha}, \widehat{F}_{*,\alpha} \in \mathcal{F}$ are (unique) minimizers of the regularized Tikhonov functional (4.30), corresponding to the measured outputs $u_T, \widehat{u}_T \in L^2(\Omega)$, respectively. Then, the following hold true:*

(a) If the final time satisfies the condition

$$0 < T \le \left(\frac{C_0^4}{2C_1}\right)^{1/5} \left(\frac{\alpha}{\|G\|^2_{L^2(0,T)}}\right)^{2/5} := T^*, \tag{4.53}$$

then the following Lipschitz stability estimate holds:

$$\|F_{*,\alpha} - \widehat{F}_{*,\alpha}\|_{L^2(\Omega)} \le C_{ST}\|u_T - \widehat{u}_T\|_{L^2(\Omega)}, \tag{4.54}$$

with the stability constant

$$C_{ST} = \frac{C_0^{1/2}}{T\|G\|_{L^2(0,T)}}. \tag{4.55}$$

(b) Suppose the damping coefficient $\mu(x)$ satisfies the condition

$$\mu(x) \ge \left(\frac{2C_1}{C_0^2}\right)^{1/3} \left(\frac{\|G\|^2_{L^2(0,T)}}{\alpha}\right)^{2/3} T := \mu_0 > 0 \tag{4.56}$$

for any given $T > 0$. Then the following Lipschitz stability estimate holds:

$$\|F_{*,\alpha} - \widehat{F}_{*,\alpha}\|_{L^2(\Omega)} \le \widetilde{C}_{ST}\|u_T - \widehat{u}_T\|_{L^2(\Omega)} \tag{4.57}$$

where

$$\widetilde{C}_{ST} = \frac{\mu_0}{\sqrt{T}\,\|G\|_{L^2(0,T)}}, \tag{4.58}$$

is the stability constant and C_0, C_1 are the constants defined in Theorem 4.1.

Proof. In the variational inequality (4.52) for the solution $F_{*,\alpha} \in \mathcal{F}$, we replace $F(x)$ with $\widehat{F}_{*,\alpha} \in \mathcal{F}$ to get

$$\int_\Omega \int_0^T \left[\widehat{F}_{*,\alpha}(x) - F_{*,\alpha}(x)\right] G(t)\psi(x,t;F_{*,\alpha})dxdt \\ +\alpha \int_\Omega F_{*,\alpha}(x)\left[\widehat{F}_{*,\alpha}(x) - F_{*,\alpha}(x)\right] dx \ge 0. \tag{4.59}$$

Similarly, we replace $F_{*,\alpha}$ with $\widehat{F}_{*,\alpha} \in \mathcal{F}$ and F with $F_{*,\alpha}$ in the variational inequality

$$\int_\Omega \int_0^T \left[F_{*,\alpha}(x) - \widehat{F}_{*,\alpha}(x)\right] G(t)\psi(x,t;\widehat{F}_{*,\alpha})dxdt \\ +\alpha \int_\Omega \widehat{F}_{*,\alpha}(x)\left[F_{*,\alpha}(x) - \widehat{F}_{*,\alpha}(x)\right] dx \geq 0. \tag{4.60}$$

From inequalities (4.59) and (4.60) we deduce that

$$\alpha \int_\Omega \left[\widehat{F}_{*,\alpha}(x) - F_{*,\alpha}(x)\right]^2 dx \\ \leq \int_\Omega \int_0^T \left[\widehat{F}_{*,\alpha}(x) - F_{*,\alpha}(x)\right] G(t)\delta\psi(x,t)dxdt, \tag{4.61}$$

where $\delta\psi(x,t) = \psi(x,t;F_{*,\alpha}) - \psi(x,t;\widehat{F}_{*,\alpha})$ is the solution of the backward problem (4.43) with $\delta\psi_t(x,T) = -\dfrac{1}{\rho_h(x)}[\delta u(x,T) - \delta u_T(x)]$, $\delta u(x,T) = u(x,T;F_{*,\alpha}) - u(x,T;\widehat{F}_{*,\alpha})$ and $\delta u_T(x) = u_T(x) - \widehat{u}_T(x)$.

Applying Hölder's inequality in the integral on the right-hand side of (4.61), we obtain

$$\alpha\|F_{*,\alpha} - \widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \leq \|F_{*,\alpha} - \widehat{F}_{*,\alpha}\|_{L^2(\Omega)} \int_0^T |G(t)|\|\delta\psi(t)\|_{L^2(\Omega)}dt.$$

It leads to the estimate

$$\alpha^2\|F_{*,\alpha} - \widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \leq \|G\|^2_{L^2(0,T)}\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))}. \tag{4.62}$$

To estimate the norm $\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))}$, we use the inequality

$$\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} \leq (T^2/2)\|\delta\psi_t\|^2_{L^2(0,T,L^2(\Omega))}$$

and then apply the estimate (4.36), which holds also for $\delta\psi(x,t)$. We have:

$$\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{T^2}{C_0^2}C_1C_e\left(\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} + \|\delta u_T\|^2_{L^2(\Omega)}\right). \tag{4.63}$$

Now, we use the inequality $\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} \leq T\,\|\delta u_t\|^2_{L^2(0;T,L^2(\Omega))}$ with the estimate

$$\|\delta u_t\|^2_{L^2(0,T;L^2(\Omega))} \leq C_e\|\delta F\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)},$$

which is a consequence of the second estimate (4.7) in Theorem 4.1 applied to the weak solution of problem (4.31), to deduce that

$$\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} \le TC_e\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)}. \tag{4.64}$$

In view of (4.64), inequality (4.63) takes the following form:

$$\begin{aligned}\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} \;\le\; & \frac{T^3C_1C_e^2}{C_0^2}\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} \\ & +\frac{T^2C_1C_e}{C_0^2}\|u_T-\widehat{u}_T\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.65}$$

Substituting (4.65) in (4.62), we get

$$\begin{aligned}\alpha^2\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \;\le\; & \frac{T^3C_1C_e^2}{C_0^2}\|G\|^4_{L^2(0,T)}\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \\ & +\frac{T^2C_1C_e}{C_0^2}\|G\|^2_{L^2(0,T)}\,\|u_T-\widehat{u}_T\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.66}$$

Let the value $T>0$ of the final time is chosen so that the following condition holds:

$$\frac{C_1C_e^2}{C_0^2}\,T^3\|G\|^4_{L^2(0,T)} = \frac{\alpha^2}{2}. \tag{4.67}$$

Then (4.66) implies that $C_eT\|G\|^2_{L^2(0,T)}\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \le \|u_T-\widehat{u}_T\|^2_{L^2(\Omega)}$. Hence,

$$\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \le \frac{1}{C_eT\|G\|^2_{L^2(0,T)}}\,\|\delta u_T\|^2_{L^2(\Omega)},$$

which leads to the stability estimate (4.54) with the stability constant C_{ST} defined in (4.55), since $C_e=\exp(T/C_0)-1\ge T/C_0$. Using the latter inequality in (4.67), we deduce that $T\le\left(\frac{C_0^4}{2C_1\|G\|^4_{L^2(0,T)}}\right)^{1/5}\alpha^{2/5}$, which yields the condition (4.53).

In order to prove (b), use the new assumption $\mu(x)\ge\mu_0>0$ in the energy identity (4.11) and applying (4.12), we may obtain

$$\begin{aligned}2\int_0^t\int_\Omega \mu(x)u_\tau(\tau)^2dxd\tau \;\le\; & \mu_0\int_0^t\int_\Omega u_\tau(\tau)^2dxd\tau+\frac{1}{\mu_0}\|F\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} \\ & +D_1(1+\nu)\|u_0\|^2_{\mathcal{V}^2(\Omega)}+C_1\|v_0\|^2_{L^2(\Omega)}.\end{aligned}$$

It leads to the estimate

$$\begin{aligned}\|u_t\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{1}{\mu_0^2}\|F\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} \\ &+\frac{1}{\mu_0}\left(D_1(1+\nu)\|u_0\|^2_{\mathcal{V}^2(\Omega)} + C_1\|v_0\|^2_{L^2(\Omega)}\right). \qquad (4.68)\end{aligned}$$

Since an estimate similar to (4.68) also hold for $\|\delta\psi_t\|_{L^2(0,T;L^2(\Omega))}$, we have

$$\begin{aligned}\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{1}{2}T^2\|\delta\psi_t\|^2_{L^2(0,T;L^2(\Omega))} \\ &\le \frac{T^2C_1}{\mu_0C_0^2}\left(\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} + \|\delta u_T\|^2_{L^2(\Omega)}\right). \qquad (4.69)\end{aligned}$$

As in the previous part (a), we obtain

$$\|\delta u(\cdot,T)\|^2_{L^2(\Omega)} \le T\,\|\delta u_t\|^2_{L^2(0;T,L^2(\Omega))} \le \frac{T}{\mu_0^2}\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)}.$$

Using this estimate in (4.69), one may obtain

$$\begin{aligned}\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{T^3C_1}{\mu_0^3\,C_0^2}\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)}\|G\|^2_{L^2(0,T)} \\ &+\frac{T^2C_1}{\mu_0C_0^2}\|u_T-\widehat{u}_T\|^2_{L^2(\Omega)}. \qquad (4.70)\end{aligned}$$

Substituting (4.70) in (4.62), we get

$$\begin{aligned}\alpha^2\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} &\le C_\alpha(\mu_0)\|F_{*,\alpha}-\widehat{F}_{*,\alpha}\|^2_{L^2(\Omega)} \\ &+ \frac{T^2C_1}{\mu_0C_0^2}\|G\|^2_{L^2(0,T)}\,\|u_T-\widehat{u}_T\|^2_{L^2(\Omega)},\end{aligned}$$

where $C_\alpha(\mu_0)=\frac{T^3C_1}{\mu_0^3\,C_0^2}\|G\|^4_{L^2(0,T)}$. Choosing μ_0 such that $C_\alpha(\mu_0)=\frac{\alpha^2}{2}$, that is, $\mu(x)$ satisfy the condition given by (4.56), we can obtain the stability estimate (4.57) with constant $\widetilde{C}_{ST}$ as in (4.58). Hence the proof. □

Remark 4.3. The first series of results given in part (a) of the above theorem that contains not only a stability estimate but also an estimate (4.53) for the upper limit $T^* > 0$ of the final time is true for all non-negative values $\mu(x) \ge 0$ of the damping coefficient including also the undamped case $\mu(x) = 0$. However, the stability constant C_{ST} defined in (4.55) does not contain information on the degree of influence of the damping coefficient, which is analyzed in the results of [51] is very significant in the unique determination of an unknown spatial load. In this sense, the results in part (b) of the theorem that holds if the damping coefficient $\mu(x)$ is strictly positive can be interpreted as complementary. Namely, in this part, the lower limit $\mu_0 > 0$ of this coefficient is established, and then a similar to (4.54), the stability estimate (4.57) is proved.

Furthermore, the new stability constant $\widetilde{C}_{\mathrm{ST}}$ defined in (4.58) depends on this lower limit $\mu_0 > 0$ as well as on the upper limit $T^* > 0$ of the final time found in part (a). In fact, the dependence of the new constant $\widetilde{C}_{\mathrm{ST}}$ on the previous one C_{ST} is easily deduced from formulas (4.55) and (4.58): $\widetilde{C}_{\mathrm{ST}} = \frac{\sqrt{T}\,\mu_0}{\sqrt{C_0}}\, C_{\mathrm{ST}}$. □

The following example gives an idea of the specific values of the upper limit $T^* > 0$ of the final time and the lower limit $\mu_0 > 0$ of the damping coefficient, also the stability constants $\widetilde{C}_{\mathrm{ST}}$ and C_{ST}, depending on values of the parameter of regularization $\alpha > 0$ and the norm $\|G\|_{L^2(0,T)}$ of the temporal load $G(t)$.

Example 4.1. Some permissible values of the upper limit of the final time and the lower limit of the damping coefficient depending on values of the parameter of regularization $\alpha > 0$ and the norm $\|G\|_{L^2(0,T)}$ of the temporal load. For clarity of explanations and comments, it is assumed that $C_0 = C_1 = 1$, which means $\rho(x) = h(x) = 1$.

Formulas (4.53) and (4.56) for the upper limit $T^* > 0$ of the final time and the lower limit $\mu_0 > 0$ of the damping coefficient show that these limits are mainly determined by the ratio $\alpha/\|G\|^2_{L^2(0,T)}$ and the inverse ratio $\|G\|^2_{L^2(0,T)}/\alpha$, respectively. In particular, this means that a decrease in one of these values $T^* > 0$ or $\mu_0 > 0$ leads to an increase in the other. As a consequence of this, it is reasonable to use the same order of magnitude for the values of α and $\|G\|^2_{L^2(0,T)}$. Since in real applications the value of the regularization parameter α varies in the range from 10^{-2} to 10^{-4}, the upper limits $T^* > 0$ of the final time cannot be too large, as the formula (4.53) suggests.

Even if we take the value of the norm $\|G\|_{L^2(0,T)}$ to be very small, say 10^{-1}, the upper limit $T^* > 0$ will not even be more than one. Furthermore, if the value of this norm is in the order of 1, then $T^* > 0$ will be in the order of $\mathcal{O}\left(\alpha^{2/5}\right)$, i.e., very small, whereas $\mu_0 > 0$ will be in the order of $\mathcal{O}\left(\alpha^{-2/3}\right)$ but it is controlled by T^*. It is evident that the increase in value of the regularization parameter α decreases both the stability constants C_{ST} and $\widetilde{C}_{\mathrm{ST}}$. Some specific values of the upper limit $T^* > 0$ of the final time and the lower limit $\mu_0 > 0$ of the damping coefficient as well as of the stability constants defined in (4.55) and (4.58) are given in Table 1.

Table 4.1: Admissible upper limits $T^* > 0$ and the lower limits $\mu_0 > 0$ depending on $\alpha > 0$ and $\|G\|_{L^2(0,T)}$.

α	$T_* = \left[\frac{\alpha}{\sqrt{2}\|G\|^2_{L^2(0,T)}}\right]^{2/5}$	$C_{\text{ST}} = \frac{1}{T\,\|G\|_{L^2(0,T)}}$	$\mu_0 = T\left[\frac{\alpha}{\sqrt{2}\|G\|^2_{L^2(0,T)}}\right]^{-2/3}$	$\widetilde{C}_{\text{ST}} = \frac{\mu_0}{\sqrt{T}\,\|G\|_{L^2(0,T)}}$
$10^{-5/2}$	0.54	18.51	1.46	19.94
10^{-3}	0.34	29.41	1.98	34.09
10^{-4}	0.13	76.9	3.52	97.86

Taking into account the fact that the value of the damping coefficient is in the range from 10^{-2} to 10 (see, for instance [81]), we deduce from the first and second lines of the table that when $\mu_0 < 2$, the stability estimates (4.54) and (4.57) hold for acceptable values of $T^* > 0$ and C_{ST}, as well. The last row of the table suggests that the increase of the lower limit $\mu_0 > 2$ of the damping coefficient causes the value of the stability constant $\widetilde{C}_{\text{ST}}$ to increase drastically and thus, the value of the upper limit $T^* > 0$ of the final time to decrease. $\square$

4.4 Singular value decomposition of the input-output operator

This section is devoted to analyzing the relationship between the two widely used methods, the Tikhonov regularization method and singular value decomposition, in determining the source term.

First, let us derive a unique representation for the unknown source $F \in \mathcal{F}$ using the singular system of the input-output operator Φ. Assume that $D \in H^2(\Omega)$.

Consider the operator $\mathcal{L} : D(\mathcal{L}) \subset L^2(\Omega) \mapsto L^2(\Omega)$ defined by

$$\begin{aligned}(\mathcal{L}v)(x) := & \left(D(x)(v_{x_1x_1} + \nu v_{x_2x_2})\right)_{x_1x_1} + \left(D(x)(v_{x_2x_2} + \nu v_{x_1x_1})\right)_{x_2x_2} \\ & + 2(1-\nu)\left(D(x)v_{x_1x_2}\right)_{x_1x_2}, \quad x \in \Omega\end{aligned} \tag{4.71}$$

where $D(\mathcal{L}) = \{v \in \mathcal{V}^2(\Omega) \cap H^4(\Omega) : v_{x_2x_2}(x) = 0,\ x \in (\Gamma_1 \cap \Gamma_3),\ \ v_{x_1x_1}(x) = 0,\ x \in (\Gamma_2 \cap \Gamma_4)\}$. We can easily see that the operator $\mathcal{L}$ is self-adjoint and positive definite. Furthermore, there exist eigenfunctions $\{v_{kl}\}_{k,l=1}^{\infty}$ corresponding to the eigenvalues $\{\lambda_{kl}\}_{k,l=1}^{\infty}$

such that

$$\begin{cases} (\mathcal{L}v_{kl})(x) = \lambda_{kl}v_{kl}, & x \in \Omega \\ v_{kl}(x) = 0,\ v_{kl,x_2x_2}(x) = 0,\ x \in (\Gamma_1 \cap \Gamma_3) \\ v_{kl}(x) = 0,\ v_{kl,x_1x_1}(x) = 0,\ x \in (\Gamma_2 \cap \Gamma_4)\,. \end{cases} \tag{4.72}$$

In addition, the eigenvectors $\{v_{kl}\}_{k,l=1}^{\infty}$ form an orthonormal basis for $L^2(\Omega)$ (see, [34]).

Remark 4.4. In the case of $D(x) = 1$ and $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (0,\pi), x_2 \in (0,\pi)\}$, the eigenvalues of the operator $\mathcal{L} : D(\mathcal{L}) \subset L^2(\Omega) \mapsto L^2(\Omega)$ defined in (4.71) subject to the boundary conditions (4.72) are as follows (see, [79], section 14.4)

$$\begin{aligned} \lambda_{kl} &= (k^2 + l^2)^2,\ \ k = 1, 2, ...,\ l = 1, 2, ..., \\ v_{kl}(x) &= \frac{2}{\pi}(\sin(kx_1)\sin(lx_2)),\ \ x \in \Omega. \end{aligned} \tag{4.73}$$

We analyze the singular system of input-output operator Φ by considering the inverse source problem (4.1)-(4.2) with $\rho_h(x) = 1$ and $u_0(x) = v_0(x) = 0$. The following proposition gives spectral properties of the operator Φ.

Theorem 4.4. *The input-output operator $\Phi : \mathcal{F} \subset L^2(\Omega) \mapsto \mathcal{V}^2(\Omega) \subset L^2(\Omega)$ defined by $(\Phi F)(x) := u(x, T; F)$ corresponding to the inverse source problem (4.1) and (4.2) is self-adjoint. Furthermore, $\{\kappa_{kl}, v_{kl}\}$ is the eigensystem of the operator Φ, that is,*

$$(\Phi v_{kl})(x) = \kappa_{kl}v_{kl}(x), \tag{4.74}$$

where κ_{kl} has three possible values:

$$\kappa_{kl} = \begin{cases} \dfrac{1}{\omega_{kl}} \displaystyle\int_0^T e^{-\mu(T-t)/2} \sin(\omega_{kl}(T-t))\, G(t)dt, \\ \qquad\qquad \omega_{kl} = \frac{1}{2}\sqrt{4\lambda_{kl} - \mu^2}, \quad \textit{if}\ 0 < \mu < 2\sqrt{\lambda_{kl}} \\ \displaystyle\int_0^T (T-t)e^{-\mu(T-t)/2} G(t)dt, \quad \textit{if}\ \mu = 2\sqrt{\lambda_{kl}} \\ \dfrac{1}{2\widehat{\omega}_{kl}} \displaystyle\int_0^T e^{-\mu(T-t)/2} \left[e^{\widehat{\omega}_{kl}(T-t)} - e^{-\widehat{\omega}_{kl}(T-t)}\right] G(t)dt, \\ \qquad\qquad \widehat{\omega}_{kl} = \frac{1}{2}\sqrt{\mu^2 - 4\lambda_{kl}},\ \ \textit{if}\ \mu > 2\sqrt{\lambda_{kl}}, \end{cases} \tag{4.75}$$

and $\{v_{kl}\}_{k,l=1}^{\infty}$ are orthonormal eigenvectors corresponding to eigenvalues $\{\lambda_{kl}\}_{k,l=1}^{\infty}$ of the operator $\mathcal{L} : D(\mathcal{L}) \subset L^2(\Omega) \mapsto L^2(\Omega)$ of the system (4.72).

Proof. Let us use the eigenfunctions $\{v_{kl}\}_{k,l=1}^{\infty}$, which forms an orthonormal basis for $L^2(\Omega)$, to write the Fourier series expansion for the solution to the initial boundary value problem (4.1) as follows

$$u(x,t) = \sum_{k,l=1}^{\infty}(u(t), v_{kl})v_{kl}(x) = \sum_{k,l=1}^{\infty} u_{kl}(t)v_{kl}(x).$$

Now take inner product between (4.1) and $v_{kl}(x)$ to get

$$\begin{cases} (u_{tt}(t), v_{kl}) + \mu(u_t(t), v_{kl}) + \lambda_{kl}(u(t), v_{kl}) = (F, v_{kl})G(t) \\ (u(0), v_{kl}) = (u_t(0), v_{kl}) = 0. \end{cases}$$

This leads to the ODE:

$$\begin{cases} u''_{kl}(t) + \mu u'_{kl}(t) + \lambda_{kl}u_{kl}(t) = F_{kl}G(t) \\ u_{kl}(0) = u'_{kl}(0) = 0 \end{cases} \tag{4.76}$$

for each $k, l = 1, 2, 3, ...$ and $F_{kl} := (F, v_{kl})$ is the Fourier coefficient of the function $F \in L^2(\Omega)$.

The roots of the characteristic equation in relation to (4.76) is given by $m = \frac{-\mu \pm \sqrt{\mu^2 - 4\lambda_{kl}}}{2}$. We get three solutions corresponding to the sign of the discriminant which are determined by $\mu < 2\sqrt{\lambda_{kl}}$, $\mu = 2\sqrt{\lambda_{kl}}$ and $\mu > 2\sqrt{\lambda_{kl}}$. The possible solutions of the Cauchy problem (4.76) corresponding to these cases are given as follows:

$$u_{kl}(t) = \begin{cases} \dfrac{F_{kl}}{\omega_{kl}} \displaystyle\int_0^t e^{-\mu(t-\tau)/2} \sin(\omega_{kl}(t-\tau))\, G(\tau)d\tau, \\ \qquad \omega_{kl} = \frac{1}{2}\sqrt{4\lambda_{kl} - \mu^2}, \qquad \text{if } 0 < \mu < 2\sqrt{\lambda_{kl}} \\ F_{kl} \displaystyle\int_0^t (t-\tau)e^{-\mu(t-\tau)/2} G(\tau)d\tau, \quad \text{if } \mu = 2\sqrt{\lambda_{kl}} \\ \dfrac{F_{kl}}{2\widehat{\omega}_{kl}} \displaystyle\int_0^t e^{-\mu(t-\tau)/2}\left[e^{\widehat{\omega}_{kl}(t-\tau)} - e^{-\widehat{\omega}_{kl}(t-\tau)}\right] G(\tau)d\tau, \\ \qquad \widehat{\omega}_{kl} = \frac{1}{2}\sqrt{\mu^2 - 4\lambda_{kl}}, \qquad \text{if } \mu > 2\sqrt{\lambda_{kl}}. \end{cases}$$

Hence, the Fourier series expansion of the input-output operator $(\Phi F)(x) := u(x, T; F)$ is given by $(\Phi F)(x) = \sum_{k,l=1}^{\infty} \kappa_{kl} F_{kl} v_{kl}(x)$, where κ_{kl} takes three possible values as defined in (4.75).

Note that $\{v_{kl}\}_{k,l=1}^{\infty}$ are eigenvectors of the input-output operator Φ corresponding to the eigenvalues $\{\kappa_{kl}\}_{k,l=1}^{\infty}$. Indeed,

$$(\Phi v_{k'l'})(x) = \sum_{k,l=1}^{\infty} \kappa_{kl}(v_{k'l'}(x), v_{kl})v_{kl}(x) = \kappa_{k'l'}v_{k'l'}(x).$$

This implies (4.74), and the operator Φ is also self-adjoint. This completes the proof. □

Using the spectral properties of Φ, the regularized solution F_α can be represented by a series with measured data u_T and the singular values κ_{kl} of Φ. The following theorem gives the series representation of F_α.

Theorem 4.5. *Let Assumption 4.1 hold and $u_T \in L^2(\Omega)$. Suppose that temporal load $G(t) > 0$ is such that $\kappa_{kl} > 0$ for all $k, l = 1, 2, 3, ...$ Then the unique minimum $F_{*,\alpha} \in F$ of the regularized Tikhonov functional $\mathcal{J}_\alpha(F)$ can be represented as follows*

$$F_{*,\alpha}(x) = \sum_{k,l=1}^{\infty} \frac{q(\alpha; \kappa_{kl})}{\kappa_{kl}} u_{T,kl}v_{kl}(x), \;\; x \in \Omega, \;\alpha \geq 0 \tag{4.77}$$

where $q(\alpha; \kappa_{kl}) = \frac{\kappa_{kl}^2}{\alpha + \kappa_{kl}^2}$ is the filter function and $u_{T,kl}$ is the kl-th Fourier coefficient of the data u_T.

Proof. Proof is similar to that of Theorem 3.1.3 given in [50]. □

In view of Theorem 4.5 and Remark 4.4, we have the following analysis.

Below we assume, without loss of generality, that $0 < \mu < 2\sqrt{\lambda_{11}}$ and regularization parameter $\alpha = 0$, where $\lambda_{11} > 0$ is the smallest eigenvalue . The cases $\mu = 2\sqrt{\lambda_{k_*l_*}}$ and $\mu > 2\sqrt{\lambda_{k_*l_*}}$ can be investigated in a similar way.

Evidently, only the positivity $\kappa_{kl} > 0$ of the singular values for all $k, l = 1, 2, 3, ...$ can not guarantee the convergence of the singular value expansion

$$F_*(x) = \sum_{k,l=1}^{\infty} \frac{1}{\kappa_{kl}} \, u_{T,kl} \, v_{kl}(x), \; x \in \Omega, \tag{4.78}$$

as the Picard criterion shows (see, [50])

$$\sum_{k,l=1}^{\infty} \frac{u_{T,kl}^2}{\kappa_{kl}^2} < \infty. \tag{4.79}$$

To understand what this criterion means for the considered inverse problem, we use the first formula in (4.75) to estimate the singular values as follows:

$$
\begin{aligned}
0 < \kappa_{kl} \ \ &< \ \ \frac{1}{\omega_{kl}} \left(\int_0^T \sin^2(\omega_{kl} t) dt \right)^{1/2} \left(\int_0^T G^2(T-t)\, dt \right)^{1/2} \\
&\le \frac{\sqrt{T}}{\sqrt{\lambda_{kl}} \sqrt{1-\mu^2/(4\lambda_{kl})}} \, \|G\|_{L^2(0,T)}.
\end{aligned}
$$

The right-hand-side of this inequality shows that the singular values κ_{kl}, $k, l = 1, 2, 3, ...$ have the asymptotic property $\mathcal{O}\left(k^{-2} + l^{-2}\right)$, since by (4.73) $\lambda_{kl} \sim \mathcal{O}\left(k^4 + l^4\right)$. As a consequence of this property and the Picard criterion (4.79), we deduce that the series (4.78) converges if and only if

$$
\sum_{k,l=1}^{\infty} \left(k^2 + l^2\right)^2 u_{T,kl}^2 < \infty. \tag{4.80}
$$

Based on characterization of Sobolev spaces by Fourier transform [27], we conclude from (4.80) that, if the measured output $u_T(x)$ satisfies the following regularity and consistency conditions:

$$
\begin{aligned}
u_T \in \ \mathcal{M}^2(\Omega) : \ &= \ \{ v \in H^2(\Omega) : v(x) = 0, \ x \in \Gamma \}, \qquad (4.81) \\
& \quad u_{T,x_2x_2}(x) = 0, \ x \in (\Gamma_1 \cap \Gamma_3)\,, \ u_{T,x_1x_1}(x) = 0, \ x \in (\Gamma_2 \cap \Gamma_4) \},
\end{aligned}
$$

then the condition (4.80) is satisfied. Thus, we have:

Theorem 4.6. *Let conditions of Theorem 4.5 hold. Assume, in addition, that the measured output $u_T(x)$ satisfies the regularity and consistency conditions (4.81). Then the inverse problem (4.1) and (4.2) has a unique solution. Furthermore, this solution possesses the convergent SVD given by (4.78), with the singular values defined in (4.75).*

Remark 4.5. As mentioned above, if the Fréchet gradient of Tikhonov functional is Lipschitz continuous, then CGA can be used to solve the inverse problem effectively. The same inverse problem can also be solved by truncated singular value decomposition (TSVD) algorithm if the regularized solution F_α has singular value decomposition (4.77) (see, [50]). This algorithm can be defined by taking the partial sum $\sum_{k,l=1}^{N} \frac{q(\alpha;\kappa_{kl})}{\kappa_{kl}} u_{T,kl}^{\sigma} v_{kl}(x), \ \ \alpha \geq 0$ of the series (4.77) and choosing the cut-off index N. Here $u_{T,kl}^{\sigma}$ is the kl-th Fourier coefficient of the noisy data $u_T^{\sigma}(x)$ and $\sigma > 0$ is the noise level.

4.4.1 Relationship between representation formulas

Since we have derived two representation formulas for the regularized solution, we analyze the relationship between these two representations. For this analysis, we write the equivalent form of gradient formula (4.48) in terms of the linear, compact operator Φ as follows (see, [50], Theorem 2.5.1): $\mathcal{J}'_\alpha(F) = \Phi^*(\Phi(F) - u_T) + \alpha F,\ \ F \in L^2(\Omega)$, where $\Phi^* : L^2(\Omega) \mapsto L^2(\Omega)$ is the adjoint of the operator Φ. Hence the representation (4.49) is an analogue of the representation $F_\alpha = (\Phi^*\Phi + \alpha I)^{-1}\Phi^* u_T$, for the solution of the regularized normal equation

$$(\Phi^*\Phi + \alpha I)F_\alpha = \Phi^* u_T. \tag{4.82}$$

Since the normal equation (4.82) has a unique solution for each $\alpha > 0$, the following coupled system also admits a unique solution (u, ψ) :

$$\begin{cases} u_{tt} = -\mu(x)u_t - (D(x)(u_{x_1x_1} + \nu u_{x_2x_2}))_{x_1x_1} \\ \qquad - (D(x)(u_{x_2x_2} + \nu u_{x_1x_1}))_{x_2x_2} - 2(1-\nu)\left(D(x)u_{x_1x_2}\right)_{x_1x_2} \\ \qquad -\frac{1}{\alpha}\left(\displaystyle\int_0^T \psi(x,t;F_\alpha)G(t)dt\right)G(t),\ (x,t) \in \Omega_T, \\ u(x,0) = 0,\ u_t(x,0) = 0,\ x \in \Omega, \\ u(x,t) = 0,\ u_{x_2x_2}(x,t) = 0,\ (x,t) \in (\Gamma_1 \cap \Gamma_3) \times [0,T], \\ u(x,t) = 0,\ u_{x_1x_1}(x,t) = 0,\ (x,t) \in (\Gamma_2 \cap \Gamma_4) \times [0,T], \end{cases} \tag{4.83}$$

$$\begin{cases} \psi_{tt} = \mu(x)\psi_t - (D(x)(\psi_{x_1x_1} + \nu \psi_{x_2x_2}))_{x_1x_1} \\ \qquad - (D(x)(\psi_{x_2x_2} + \nu \psi_{x_1x_1}))_{x_2x_2} \\ \qquad -2(1-\nu)\left(D(x)\psi_{x_1x_2}\right)_{x_1x_2},\ (x,t) \in \Omega_T, \\ \psi(x,T) = 0,\ \psi_t(x,T) = -[u(x,T;F) - u_T(x)],\ x \in \Omega, \\ \psi(x,t) = 0,\ \psi_{x_2x_2}(x,t) = 0,\ (x,t) \in (\Gamma_1 \cap \Gamma_3) \times [0,T], \\ \psi(x,t) = 0,\ \psi_{x_1x_1}(x,t) = 0,\ (x,t) \in (\Gamma_2 \cap \Gamma_4) \times [0,T]. \end{cases} \tag{4.84}$$

Theorem 4.7. *Suppose Assumption 4.1 hold true and let the pair (u, ψ) be the solution of coupled system (4.83)-(4.84) for a given $\alpha > 0$. Then $F_\alpha \in L^2(\Omega)$ defined by (4.49) is the* *solution of normal equation (4.82) if and only if*

$$F_\alpha(x) = \sum_{k,l=1}^{\infty} \frac{q(\alpha;\kappa_{kl})}{\kappa_{kl}} u_{T,kl} v_{kl}(x), \quad \alpha \geq 0, \tag{4.85}$$

where $q(\alpha;\kappa_{kl})$ *is the filter function and* κ_{kl} *is defined by (4.75).*

Proof. Assume that $F_\alpha \in L^2(\Omega)$ is the solution of the normal equation (4.82). Then the pair (u,ψ) is the solution of (4.83). Since eigenfunctions $\{v_{kl}\}_{k,l=1}^{\infty}$ form orthonormal basis for $L^2(\Omega)$, we can write Fourier series representation of $u(x,t;F_\alpha)$ as follows

$$u(x,t;F_\alpha) = \sum_{k,l=1}^{\infty} (u(t), v_{kl}) v_{kl}(x) = \sum_{k,l=1}^{\infty} u_{kl}(t;F_\alpha) v_{kl}(x),$$

where $u_{kl}(t;F_\alpha)$ is the solution of the ODE (4.76). By taking $t=T$ in the solution (4.77) of the ODE, we obtain the Fourier series representation of output data $u(x,T;F_\alpha)$ with Fourier coefficient

$$u_{kl}(T;F_\alpha) = \kappa_{kl} F_{\alpha,kl}, \tag{4.86}$$

where $F_{\alpha,kl}$ is the kl-th Fourier coefficient of F_α.

Applying the Fourier series method to adjoint problem (4.84), we obtain $\psi(x,t;F_\alpha) = \sum_{k,l=1}^{\infty} \psi_{kl}(t;F_\alpha) v_{kl}(x)$, where $\psi_{kl}(t;F_\alpha)$ is the solution of the ODE

$$\begin{cases} \psi''_{kl}(t) - \mu\psi'_{kl}(t) + \lambda_{kl}\psi_{kl}(t) = 0 \\ \psi_{kl}(T) = 0, \;\; \psi'_{kl}(T) = -\left(u_{kl}(T) - u_{T,kl}\right). \end{cases} \tag{4.87}$$

The roots of the characteristic equation corresponding to (4.87) is $m = \frac{\mu \pm \sqrt{\mu^2 - 4\lambda_{kl}}}{2}$. We obtain three possible solutions according to the sign of discriminant, which are given by

$$\psi_{kl}(t) = \begin{cases} \dfrac{[u_{kl}(T;F_\alpha) - u_{T,kl}]}{\omega_{kl}} e^{-\mu(T-t)/2} \sin(\omega_{kl}(T-t)), \\ \qquad \omega_{kl} = \frac{1}{2}\sqrt{4\lambda_{kl} - \mu^2}, \qquad \text{if } 0 < \mu < 2\sqrt{\lambda_{kl}} \\ [u_{kl}(T;F_\alpha) - u_{T,kl}]\,(T-t) e^{-\mu(T-t)/2}, \text{ if } \mu = 2\sqrt{\lambda_{kl}} \\ \dfrac{[u_{kl}(T;F_\alpha) - u_{T,kl}]}{2\widehat{\omega}_{kl}} e^{-\mu(T-t)/2} \left[e^{\widehat{\omega}_{kl}(T-t)} - e^{-\widehat{\omega}_{kl}(T-t)}\right] \\ \qquad \widehat{\omega}_{kl} = \frac{1}{2}\sqrt{\mu^2 - 4\lambda_{kl}}, \qquad \text{if } \mu > 2\sqrt{\lambda_{kl}}. \end{cases} \tag{4.88}$$

Multiplying both sides of equations (4.88) by $\frac{-G(t)}{\alpha}$, integrating on $(0,T)$ and using the expressions (4.75), we get

$$F_{\alpha,kl} = -\frac{1}{\alpha}\int_0^T \psi_{kl}(t;F_\alpha)G(t)dt = -\frac{\kappa_{kl}}{\alpha}\left[u_{kl}(T;F_\alpha) - u_{T,kl}\right].$$

Using the formula (4.86), we obtain the kl-th Fourier coefficient $F_{\alpha,kl}$ in terms of the parameter of regularization $\alpha > 0$ and Fourier coefficient $u_{T,kl}$ of measured output data:

$$F_{\alpha,kl} = \frac{\kappa_{kl}}{\kappa_{kl}^2 + \alpha}\, u_{T,kl} \;= \frac{q(\alpha;\kappa_{kl})}{\kappa_{kl}}\, u_{T,kl}, \quad k,l = 1,2,3,...,$$

where $q(\alpha;\kappa_{kl})$ is the filter function, and hence (4.77) follows.

Converse part directly follows from $F_{\alpha,kl}$. Indeed, assume that $F_\alpha \in L^2(\Omega)$ has the series representation (4.77). Hence kl-th Fourier coefficient of F_α is given by

$$F_{\alpha,kl} = \frac{q(\alpha;\kappa_{kl})}{\kappa_{kl}}\, u_{T,kl} = \frac{\kappa_{kl}}{\kappa_{kl}^2 + \alpha}\, u_{T,kl}.$$

It leads to the following

$$F_{\alpha,kl} = -\frac{\kappa_{kl}}{\alpha}\left(u_{kl}(T,F_\alpha) - u_{T,kl}\right), \tag{4.89}$$

since $u_{kl}(T,F_\alpha) = \kappa_{kl}F_{\alpha,kl}$. Substituting (4.75) and (6.2) in (4.89), we get the integral form of kl-th Fourier coefficient $F_{\alpha,kl}$ as follows

$$F_{\alpha,kl} = -\frac{1}{\alpha}\int_0^T \psi_{kl}(t)G(t)dt.$$

By the integral form of the coefficient $F_{\alpha,kl}$, we obtain the integral representation (4.49) of $F_\alpha(x)$, that is, $F_\alpha \in L^2(\Omega)$ is the solution of normal equation (4.82). This completes the proof. □

The above Theorem 4.7 shows that the formula (4.49) can be treated as an integral form of singular value decomposition. This Theorem shows that the solution obtained by these two methods, Tikhonov regularization and singular value decomposition are equivalent. These theoretical results lead to the comparison of numerical methods such as CGA and TSVD. This comparison was done for inverse source problem of heat and wave equation by illustrating several numerical examples in [49]. These examples show that the new version of CGA with TSVD initialization is more effective than TSVD and CGA alone.

4.5 Stability estimates: spectral approach

In this section, we establish the stability estimate for the solution of inverse problem (4.1) and (4.2) with $\rho_h(x) = \rho = const.$, $D(x) = D = const.$ and $\mu(x) = \mu = const,$ using the singular values κ_{kl} of the input-output operator Φ. The variable coefficients case is studied in the same way, using the property

$$M_{*,\lambda}\lambda_{kl}^2 \leq \widetilde{\lambda}_{kl}^2 \leq M_\lambda^* \lambda_{kl}^2,\ k,l = 0,1,2,3,\ldots$$

of the eigenvalues $\widetilde{\lambda}_{kl}^2$ corresponding to the variable coefficient Kirchhoff operator.

First consider the case with $\alpha = 0$. We use the following sufficient condition given by Theorem 11.4.2 of [50] for the positivity $\kappa_{kl} > 0$ for all $k,l = 1,2,3,...$ of the singular values of the operator Φ.

Lemma 4.3. *Let Assumption 4.1 hold. Suppose that the damping coefficient satisfies the condition $0 < \mu < 2\sqrt{\lambda_{11}}$ and the temporal load $G(t)$ belongs to $H^1(0,T)$. In addition, assume that the coefficient, final time and the temporal load satisfy the inequality:*

$$\begin{aligned} G(T) \ &> \ \left(G(0)e^{-\mu T/2} + \left((1-e^{-\mu T})/\mu\right)^{1/2} \|G'\|_{L^2(0,T)}\right) \\ &\quad \times \left(1 - \left(\mu/(2\sqrt{\lambda_{11}})\right)^2\right)^{-1/2} := \mathfrak{M}(G,\mu,T). \end{aligned} \tag{4.90}$$

Then all eigenvalues κ_{kl}, $k,l = 1,2,3,...$, of the input-output operator Φ are positive.

The proof of this lemma is based on the inequality (Theorem 11.4.2 of [50]):

$$\kappa_{kl} \ \geq \ \frac{1}{\lambda_{kl}} \left\{G(T) - \mathfrak{M}(G,\mu,T)\right\}. \tag{4.91}$$

Theorem 4.8. *Let Assumption 4.1 hold. Assume that the damping coefficient satisfies the condition $0 < \mu < 2\sqrt{\lambda_{11}}$ and the temporal load $G(t)$ belongs to $H^1(0,T)$. Suppose in addition the inequality (4.90) holds. Then*

$$\|F_* - \widetilde{F}_*\|_{L^2(\Omega)} \leq \bar{C}_{ST}\, \|u_T - \widetilde{u}_T\|_{H^4(\Omega)}, \tag{4.92}$$

with the stability constant

$$\bar{C}_{ST} = \frac{1}{G(T) - \mathfrak{M}(G,\mu,T)} > 0,$$

where $F_*, \widetilde{F}_* \in L^2(\Omega)$ are the unique solutions of the inverse problem (4.1)-(4.2) corresponding to the measured outputs $u_T, \widetilde{u}_T \in H^4(\Omega)$ satisfying the consistency conditions (4.81), and $\mathfrak{M}(G, \mu, T) > 0$ is the constant introduced in (4.90).

Proof. By the above assumptions and inequalities (4.90)-(4.91), we obtain

$$\frac{1}{\kappa_{kl}} \leq \frac{1}{G(T) - \mathfrak{M}(G, \mu, T)} \lambda_{kl}.$$

Using the relationship between the Fourier coefficients $F_{*,kl} = \frac{1}{\kappa_{kl}} u_{T,kl},\ k, l = 1, 2, 3, \ldots,$ we get

$$F_{*,kl} \leq \frac{1}{G(T) - \mathfrak{M}(G, \mu, T)} \lambda_{kl} u_{T,kl},\ k, l = 1, 2, 3, \ldots .$$

Squaring on both sides and then taking into account (4.73), we have

$$F_{*,kl}^2 \leq \frac{1}{[G(T) - \mathfrak{M}(G, \mu, T)]^2} \left[k^2 + l^2\right]^4 u_{T,kl}^2,\ k, l = 1, 2, 3, \ldots .$$

or

$$\sum_{k,l=1}^{\infty} F_{*,kl}^2 \leq \frac{1}{[G(T) - \mathfrak{M}(G, \mu, T)]^2} \sum_{k,l=1}^{\infty} \left[k^2 + l^2\right]^4 u_{T,kl}^2.$$

In view of Parseval's identity and the characterization of the Sobolev space by the Fourier transform, the above inequality leads to the desired result (4.92). □

Example 4.2. Consider a pure spatial loading case, that is, $G(t) = 1$. Then

$$\mathfrak{M}(G, \mu, T) = e^{-\mu T/2} \left(1 - \left(\mu/(2\sqrt{\lambda_{11}})\right)^2\right)^{-1/2}$$

and the stability constant is

$$\bar{C}_{\text{ST}} = \left(1 - e^{-\mu T/2} \left(1 - \left(\mu/(2\sqrt{\lambda_{11}})\right)^2\right)^{-1/2}\right)^{-1}.$$

Evidently, the sufficient condition (4.90) holds for all large enough values of the final time $T > 0$. Furthermore, for these values, $\bar{C}_{\text{ST}} > 1$.

Consider now the regularized inverse problem. We use the relationship

$$F_{*,\alpha,kl} = \frac{\kappa_{kl}}{\kappa_{kl}^2 + \alpha}\, u_{T,kl},\ k,l = 1,2,3,\ldots\ . \tag{4.93}$$

between the Fourier coefficients $F_{*,\alpha,kl}$ and $u_{T,kl}$ of the solution $F_{*,\alpha}(x)$ of the regularized inverse problem and the measured output $u_T(x)$, which follows from (4.85). Use the following lower bound in (4.93):

$$\kappa_{kl}^2 + \alpha = \left(\kappa_{kl} - \sqrt{\alpha}\right)^2 + 2\kappa_{kl}\sqrt{\alpha} \geq 2\kappa_{kl}\sqrt{\alpha},\ k,l = 1,2,3,\ldots\ .$$

for all $\alpha > 0$. Then (4.93) yields the inequality $F_{*,\alpha,kl} \leq \frac{1}{2\sqrt{\alpha}}\, u_{T,kl},\ k,l = 1,2,3,\ldots$. Consequently, we have:

Theorem 4.9. *Let the basic conditions (4.4) are satisfied. Then for the regularized solution of the inverse problem (4.1) and (4.2), the following Lipschitz stability estimate holds:*

$$\|F_{*,\alpha} - \widetilde{F}_{*,\alpha}\|_{L^2(\Omega)} \leq C_{ST}(\alpha)\, \|u_T - \widetilde{u}_T\|_{L^2(\Omega)}, \tag{4.94}$$

where the stability constant $C_{ST}(\alpha) = \frac{1}{2\sqrt{\alpha}}$ and $F_{*,\alpha}, \widetilde{F}_{*,\alpha} \in L^2(\Omega)$ are the unique regularized solutions of the inverse problem (4.1) and (4.2) corresponding to the measured outputs $u_T, \widetilde{u}_T \in L^2(\Omega)$, and $\alpha > 0$ is the parameter of regularization.

Remark 4.6. It is clear from the estimate (4.94) that a small value of the regularization parameter α magnifies the error between the measured outputs $u_T, \widetilde{u}_T \in L^2(\Omega)$. A similar consequence is observed in the stability estimates proved in Theorem 4.3 via the variational inequalit (4.52).

Chapter 5

Simultaneous identification of a spatial load and external heat source in a structurally damped thermoelastic plate

5.1 Introduction

In this final chapter, we study the thermoelastic plate model, a coupled system of the Kirchhoff model for describing the vibration of the plate, and the heat equation for modeling the temperature distribution in the plate. In engineering practice, there are circumstances where structural elements, in particular plates, are exposed to non-uniform thermal gradients when they are externally loaded. These loading conditions typically lead to deformation or geometry change due to thermal expansion, the reduction of the strength and stiffness of the material. This motivates us to extend the beam and plate equations and examine the inverse source problem of simultaneously identifying the mechanical load and heat source from single measured data in a thermoelastic plate equation with structural damping. We demonstrate the results obtained in the previous chapters using the regularization method to the advanced system of the Kirchhoff-Love plate and the heat equations.

Consider the homogeneous, elastically and thermally isotropic thermoelastic plate equations with structural damping $-\mathrm{w}\Delta u_t$ as follows:

$$
\begin{cases}
\rho h u_{tt} - r\Delta u_{tt} - \mathrm{w}\Delta u_t + D\Delta^2 u + \beta_1 \Delta\theta = F(x,t), \quad (x,t) \in \Omega_T, & \text{(5.1a)}\\
\beta_2 \theta_t - \Delta\theta - \beta_3 \Delta u_t + \beta_4 \theta = S(x,t), \qquad (x,t) \in \Omega_T, & \text{(5.1b)}\\
u = \frac{\partial u}{\partial \mathrm{n}} = 0,\ \theta = 0, \qquad (x,t) \in \Gamma_T & \\
u(x,0) = u_0, u_t(x,0) = v_0,\ \theta(x,0) = \theta_0 \qquad x \in \Omega, &
\end{cases}
$$

where $\Omega_T := \Omega \times (0,T]$, $\Gamma_T := \Gamma \times [0,T]$, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ, and n is the outward normal to Γ. For the derivation of this model without damping effect, one may look at the classical books [66], [65], and for various damping effects on this model, we refer to [18]. The mechanical variable $u(x,t)$ denotes the vertical displacement and the thermal variable $\theta(x,t)$ denotes the relative temperature at position x and time t. The initial state of the plate specified by (u_0, v_0, θ_0) are given functions. The functions $F(x,t)$ and $S(x,t)$ represent the mechanical load and external heat source applied on the plate. The structural damping $-\mathrm{w}\Delta u_t$ is a consequence of mechanical-energy dissipation due to rubbing friction resulting from a relative motion between components and intermittent contact at the joints in a mechanical structure. Further, structural damping describes a scenario where higher-order frequencies are more strongly damped than lower frequencies (see, [25]). The term $-r\Delta u_{tt} > 0$ accounts for the rotational inertia of the plate filaments, and r is proportional to the square of the plate thickness. In the model (5.1), hereafter, we assume that the thickness of the plate is very small, which corresponds to $r = 0$ ([72]). Finally, we assume that the plate is clamped at the edges, which results in the deflection and the slope of deflection normal to the edge are zero, and a constant temperature on the boundary. Both of these assumptions are incorporated by the boundary conditions: $u = \frac{\partial u}{\partial \mathrm{n}} = 0,\ \theta = 0,\ \forall (x,t) \in \Gamma_T$. The other coefficients in the model (5.1) takes the following form:

$$
\begin{aligned}
D &= \frac{Eh^3}{12(1-\nu^2)}, \quad \beta_1 = \frac{\eta\,(1+\nu)\,D}{2}, \\
\beta_2 &= \frac{\rho C}{\eta}, \quad \beta_3 = \frac{E\eta r_0}{\lambda_0^2(1-2\nu)}, \\
\beta_4 &= \frac{12\lambda_0}{h^2\eta}\left(1 + \frac{h\lambda_1}{2}\right).
\end{aligned}
$$

The parameter $\mathrm{w} > 0$ represents the structural damping coefficient that plays a major role in the uniqueness study of regularized inverse source problems. More precisely, the final time inverse problem with the damping parameter is motivated by two different factors from the standpoint of the physical and mathematical models. For the Euler-Bernoulli beam equation, the role of four separate damping mechanisms (air damping or viscous damping, strain rate damping, spatial hysteresis, and time hysteresis) was first studied experimentally and theoretically in [6]. Based on the analysis in this article, it was concluded that the nature of the damping mechanisms drastically changes the nature of the solution to the vibration problem and hence controls the response of the beam. The second mathematical reason is related to the non-uniqueness of the final data inverse source problem for undamped wave, beam, and plate equations, as observed in [50]; for these *undamped equations, the unknown spatial load can not be determined uniquely from the final time measured output.* The detailed analysis of the effect of viscous damping μu_t in the unique determination of the source term from final time measurement in the case of Euler-Bernoulli beam is studied in chapter 2 (see, also [51]) for various temporal loads, and Kirchoff-Love plate has been analyzed in chapter 4 when the viscous damping coefficient $\mu > 0$. The important reasons above lead to the assumption that the damping parameter w is positive.

Although there have been tremendous advancements in domains such as nuclear engineering, aircraft, machine construction, and several other related fields over the past few decades, the coupled effect between deformation and temperature has been a key factor for the solution of many thermal shock problems. The analysis of the impact of the temperature field and stresses produced during thermal shock, which might result in premature failure, and understanding of thermoelastic damping in devices is crucial in many engineering situations (see, [76], [14], [92]). Therefore, it is extremely important to understand the effect of the thermal and mechanical sources that result in the material deflection.

Given all the parameters, initial data, and source terms of (5.1), finding the solution (u, θ) is referred to as the *direct problem*. In this chapter, we study the problem of determining the unknown mechanical load $F(x,t)$ and external heat source $S(x,t)$ in thermoelastic plate equation (5.1) from the final time measured *vertical displacement*

$$u_T(x) := u(x,T), \quad x \in \Omega. \tag{5.2}$$

The later problem is referred to as the *inverse source problem.* Mathematically, the above inverse problem is formulated as follows. For the given constants M_1,$M_2 > 0$, we consider the admissible set of mechanical loads

$$\mathcal{F} = \{F \in L^2(0,T;L^2(\Omega)) : \|F\|_{L^2(0,T;L^2(\Omega))} \leq M_1\},$$

and the admissible set of external heat sources

$$\mathcal{G} = \{S \in L^2(0,T;L^2(\Omega)) : \|S\|_{L^2(0,T;L^2(\Omega))} \leq M_2\}.$$

We define the input-output operator

$$\begin{aligned} &\Phi : \mathcal{F} \times \mathcal{G} \subset L^2(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Omega)) \mapsto H_0^1(\Omega) \subset L^2(\Omega), \\ &\Phi[F,S](x) := u(x,t;F,S)|_{t=T}. \end{aligned} \tag{5.3}$$

The simultaneous identification of (F, S) from the measured final time data $u_T \in L^2(\Omega)$ can be defined in terms of the input-output operator as follows

$$\Phi[F, S] = u_T,\ u_T \in L^2(\Omega). \tag{5.4}$$

We see that, with noiseless measured output data u_T, the solution of the inverse problem is the solution of the functional equation (5.4). However, due to measurement error in the measured output $u_T(x)$, the exact equality in functional equation (5.4) is not possible in practice. We formulate this inverse problem as a minimization problem defined by a *Tikhonov functional* as follows:

$$\mathcal{J}(F, S) := \frac{1}{2}\|\Phi[F, S] - u_T\|^2_{L^2(\Omega)},\ (F, S) \in \mathcal{F} \times \mathcal{G}. \tag{5.5}$$

The regularized form of the Tikhonov functional is given by

$$\mathcal{J}_\alpha(F, S) := \frac{1}{2}\|\Phi[F, S] - u_T\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left(\|F\|^2_{L^2(0,T;L^2(\Omega))} + \|S\|^2_{L^2(0,T;L^2(\Omega))}\right), \tag{5.6}$$

where $\alpha > 0$ is the parameter of regularization.

In view of the Tikhonov functional, we reformulate the inverse problem of (5.1) with measurement (5.2) is refer to as the following minimization problem

$$\inf_{(F,S)\in\mathcal{F}\times\mathcal{G}} \mathcal{J}_\alpha(F, S), \tag{5.7}$$

in the set of admissible mechanical loads and external heat sources $(\mathcal{F}, \mathcal{G})$. A solution of this problem is called a quasi-solution or a least squares solution of the inverse problem.

Remark 5.1. For the numerical analysis of this specific inverse problem, one may consider the Tikhonov functional given by

$$\mathcal{J}_\alpha(F, S) := \frac{1}{2}\|\Phi[F, S] - u_T\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\|F\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha_2}{2}\|S\|^2_{L^2(0,T;L^2(\Omega))}$$

instead of the Tikhonov functional (5.6), where $\alpha_1 > 0$ and $\alpha_2 > 0$ are two different regularization parameters. In the context of obtaining numerical solutions, it is common to explore different choices of regularization parameters, and this choice depends on the noise level of measured data (see, [10], [12], [63]). Since we don't analyze the effect of two different regularaization parameters in the theoretical study, we have considered only one parameter for both F and S.

Our main contributions in this chapter are summarized as follows:

1. The existence of a unique mild solution of a related system (5.1) has been proved in [65] using semigroup theory. However, it doesn't provide any precise estimates for the solution of this system. By using the Galerkin method, we rigorously proved the well-posedness of the system (5.1) under Assumption 5.1 and derived optimal energy estimates, which are crucial for the Lipschitz continuity of the Fréchet derivative of the functional $\mathcal{J}(\cdot,\cdot)$ and stability estimates for the unknown coefficients (F,S) given in Theorem 5.5. It is worth noting that the strong coupling arising from $\Delta\theta$ of the thermal variable θ in (5.1a), Δu_t of the displacement u in (5.1b) and fourth-order spatial derivatives of u makes the model more complicated than other classical thermoelastic system mentioned above. However, proper scaling of coefficients in the model and structural damping help to tackle these challenges in the solvability of the direct problem and adjoint problem.

2. The inverse source problem of simultaneously identifying two sources $F(x,t)$ and $S(x,t)$ from a single measured data $u_T(x) := u(x,T)$ is solved using the Tikhonov regularization method coupled with the weak solution approach for the direct problem (5.1). The vector form of the Fréchet gradient of the Tikhonov functional is expressed in terms of the solutions to the adjoint problem. Unlike the single plate equation, the adjoint system, which involves with heat equation, can not be rewritten as a direct problem by the time variable transformation $\tau = T - t$ (see, (5.39a)-(5.39b)) due to the presence of the coupling term $\beta_3\Delta\phi_t$. Hence we proved the well-posedness of the adjoint system separately (see, Theorem 5.3). The Lipschitz continuity of the Fréchet derivative is proved, which is essential for the gradient-based numerical algorithm for the inverse problem.

3. A final significant result is the Lipschitz stability estimate (Theorem 5.5) for the unknowns (F,S) in terms of the given single measurement $u_T(x)$ using the first-order necessary optimality condition that is met by an optimal pair $(u(x,t;F_{*,\alpha},S_{*,\alpha}),$ $\theta(x,t;F_{*,\alpha},S_{*,\alpha}),F_{*,\alpha},S_{*,\alpha})$. The reconstruction procedure for the regularaized inverse source problem is stable and also unique whenever the final time T is sufficiently small. Altogether, different from the literature described above, determining two sources from a single measurement by the method of Tikhonov regularization is the highlight of this study, and the solvability of the inverse source problem and stability of the solution are established in this chapter.

The remaining sections of this chapter are structured as follows. A detailed analysis of the well-posedness of the direct problem (5.1) is given in section 5.2. In section 5.3, we investigate the compactness and Lipschitz continuity of the input-output operator. The existence of a minimizer for the functional is given in this section. The Fréchet derivative of the functional and its Lipschitz continuity are derived in section 5.4. Finally, stability of the solution to the inverse problem is given in section 5.5.

5.2 Well-posedness of the thermoelastic system

In this section, we study the existence and uniqueness of appropriately defined weak solutions. To verify the well-posedness of the direct problem as well as to analyse the inverse problem, we make the following assumptions on the coefficients, initial data, and source term.

Assumption 5.1.

$$\begin{cases} \rho,\ h, D, \beta_1, \beta_2, \beta_3, \text{w are positive constants,} \\ \beta_4 \geq 0, \text{non negative constant,} \\ F \in L^2(0,T;L^2(\Omega)),\ \text{and}\ S \in L^2(0,T;L^2(\Omega)) \\ u_0 \in H^2(\Omega), v_0 \in L^2(\Omega) \text{ and } \theta_0 \in L^2(\Omega). \end{cases}$$

Definition 5.1. We say a function $(u,\theta) \in L^2(0,T;\mathcal{V}_1^2(\Omega)) \times L^2(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;H_0^1(\Omega))$, $u_{tt} \in L^2(0,T;\mathcal{V}_1^2(\Omega)')$ and $\theta_t \in L^2(0,T;H^{-1}(\Omega))$, is called weak solution of (5.1) provided the following holds:
(i) for each $v \in \mathcal{V}_1^2(\Omega),\ w \in H_0^1(\Omega),$ a.e. $t \in [0,T]$,

$$\langle \rho h u_{tt}(t), v\rangle + (\text{w}\nabla u_t(t), \nabla v) + (D\Delta u(t), \Delta v) - \beta_1(\nabla\theta(t), \nabla v) = (F(t), v) \tag{5.8a}$$

$$\langle \beta_2\theta_t(t), w\rangle + (\nabla\theta(t), \nabla w) + \beta_3(\nabla u_t(t), \nabla w) + \beta_4(\theta(t), w) = (S(t), w), \tag{5.8b}$$

(ii) $u(0) = u_0,\ u_t(0) = v_0,$ and $\theta(0) = \theta_0,$

where $\mathcal{V}_1^2(\Omega)$ is defined by (1.11) and $\langle\cdot,\cdot\rangle$ denotes the duality pairing.

Remark 5.2. The regularity theory given in Theorem 1.6, Theorem 1.7, leads to the fact that $(u,\theta) \in C([0,T];H_0^1(\Omega)) \times C([0,T];L^2(\Omega))$ and $u_t \in C([0,T];\mathcal{V}_1^2(\Omega)')$. Thus the equality $u(0) = u_0, u_t(0) = v_0$ and $\theta(0) = \theta_0$ can be justified.

5.2.1 Existence and uniqueness of a weak solution

The objective is to generate a weak solution of the thermoelastic system (5.1) by first constructing solutions of finite-dimensional approximation to (5.1) and then passing to limits. This is known as Faedo-Galerkin's method. Specifically, let $\{z_m\}_{m=1}^{\infty}$ be an orthonormal basis in $L^2(\Omega)$ given by eigenfunctions of the operator $\mathcal{L} = -\Delta^2$ with eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$ in Ω with boundary condition $z_m = \frac{\partial z_m}{\partial \mathrm{n}} = 0$ on $\partial\Omega$. Then $\{z_m\}_{m=1}^{\infty}$ is smooth and can be taken as an orthogonal basis for $\mathcal{V}_1^2(\Omega)$.

Let $\{w_m\}_{m=1}^{\infty}$ be the complete set of normalized eigenfunctions for $\mathcal{L} = -\Delta$ in $H_0^1(\Omega)$, which form an orthogonal basis for $H_0^1(\Omega)$ and orthonormal basis for $L^2(\Omega)$. Then we define the finite dimensional subspaces as follows

$$Z_n = span\{z_1, z_2, ..., z_n\},\ W_n = span\{w_1, w_2, ..., w_n\}.$$

Now we look for a Faedo-Galerkin approximation $u_n(t) := u_n(x,t)$, $\theta_n(t) := \theta_n(x,t)$ of the form

$$u_n(t) = \sum_{m=1}^{n} r_{m,n}(t) z_m,\quad \theta_n(t) = \sum_{m=1}^{n} d_{m,n}(t) w_m$$

$$u_{0,n} = \sum_{m=1}^{n} p_{m,n} z_m,\quad v_{0,n} = \sum_{m=1}^{n} q_{m,n} z_m,\ \theta_{0,n} = \sum_{m=1}^{n} s_{m,n} w_m$$

where we intend to choose the coefficients $r_{m,n}, d_{m,n}, p_{m,n}, q_{m,n}$ and $s_{m,n}$ in such a way that $(u_n(t), \theta_n(t))$ for all $t \in [0,T]$ satisfies the following problem:

$$\begin{cases} \rho h(u_n''(t), z_m) + \mathrm{w}(\nabla u_n'(t), \nabla z_m) + D(\Delta u_n(t), \Delta z_m) \\ \qquad\qquad -\beta_1(\nabla\theta_n(t), \nabla z_m) = (F(t), z_m), & \text{(5.9a)} \\ \beta_2(\theta_n'(t), w_m) + (\nabla\theta_n(t), \nabla w_m) + \beta_3(\nabla u_n'(t), \nabla w_m) \\ \qquad\qquad +\beta_4(\theta_n, w_m) = (S(t), w_m), & \text{(5.9b)} \\ u_n(0) = u_{0,n},\ u_n'(0) = v_{0,n},\ \theta_n(0) = \theta_{0,n}. \end{cases}$$

Since $(u_n''(t), z_m) = r_{m,n}''(t)$, $(u_n'(t), z_m) = r_{m,n}'(t)$ and $(\theta_n'(t), w_m) = d_{m,n}'(t)$, (5.9) becomes the linear system of ODE

$$\begin{cases} \rho h M R_n''(t) + \mathrm{w} Y R_n'(t) + DPR_n(t) - \beta_1 Q S_n(t) = \bar{F}_{1,n}(t), & \text{(5.10a)} \\ \beta_2 \mathcal{M} S_n'(t) + [\beta_4\mathcal{M} + \mathcal{Q}] S_n(t) + \beta_3 Q R_n'(t) = \bar{S}_{1,n}(t), & \text{(5.10b)} \\ R_n(0) = \bar{U}_n,\ R_n'(0) = \bar{V}_n,\ \theta_n(0) = \bar{\Theta}_n, \end{cases}$$

where

$$R_n(t) = (r_{1,n}(t), r_{2,n}(t), ..., r_{n,n}(t)), S_n(t) = (d_{1,n}(t), d_{2,n}(t), ..., d_{n,n}(t)).$$

The entries of the matrix $M, P, Y, Q, \mathcal{Q}, \mathcal{M}$ are

$$\begin{aligned} M &= [(z_i, z_j)]^{\mathrm{T}}_{n\times n}, \ P = [(\Delta z_i, \Delta z_j)]^{\mathrm{T}}_{n\times n}, Q = [(\nabla w_i, \nabla z_j)]^{\mathrm{T}}_{n\times n}, \\ \mathcal{M} &= [(w_i, w_j)]^{\mathrm{T}}_{n\times n}, \text{ and } Y = [(\nabla z_i, \nabla z_j)]^{\mathrm{T}}_{n\times n}, \\ \mathcal{Q} &= [(\nabla w_i, \nabla w_j)]^{\mathrm{T}}_{n\times n} \end{aligned}$$

Further, we have

$$F_{1,j}(t) = (F(t), z_j), \bar{F}_{1,n}(t) = (F_{1,1}(t), F_{1,2}(t), ..., F_{1,n}(t)),$$

$$\begin{aligned} S_{1,j}(t) &= (S(t), z_j), \ \bar{S}_{1,n}(t) = (S_{1,1}(t), S_{1,2}(t), ..., S_{1,n}(t)) \\ U_j &= (u_0, z_j), \ V_j = (v_0, z_j), \Theta_j = (\theta_0, z_j), \\ \bar{U}_n &= (U_{1,}, U_2, ..., U_n)^{\mathrm{T}}, \bar{V}_n = (V_1, V_2, ..., V_n)^{\mathrm{T}}, \quad \bar{\Theta}_n = (\Theta_1, \quad \Theta_2, ..., \Theta_n)^{\mathrm{T}}. \end{aligned}$$

According to the standard theory of linear ODE, for every $n \geq 1$, there exists a unique function (R_n, S_n) satisfying (5.10a)-(5.10b) for a.e. $t \in [0, T]$. Hence, $(u_n, \theta_n) \in C^1([0, T]; Z_n) \times C([0, T]; W_n)$, solves (5.9) for a.e $t \in [0, T]$.

Theorem 5.1. *Suppose Assumption 5.1 holds true. Then the direct problem (5.1) has a unique weak solution as per Definition 5.1. Furthermore,*

$$\|u_t\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \frac{C_p(C_e+1)}{2\mathrm{w}} R_1(\theta_0, u_0, v_0, F, S), \tag{5.11}$$

$$\|u\|^2_{L^\infty(0,T;\mathcal{V}^2_1(\Omega))} \leq \frac{C'(C_e+1)}{D} R_1(\theta_0, u_0, v_0, F, S), \tag{5.12}$$

$$\|\theta\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \frac{C'\beta_3}{\beta_1} (C_e+1) R_1(\theta_0, u_0, v_0, F, S), \tag{5.13}$$

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \frac{\beta_3}{\beta_2\beta_1} (C_e+1) R_1(\theta_0, u_0, v_0, F, S), \tag{5.14}$$

and

$$\begin{aligned} \|u_{tt}\|^2_{L^2(0,T;\mathcal{V}^2_1(\Omega)')} &\leq C_5 R_1(\theta_0, u_0, v_0, F, S), \\ \|\theta_t\|^2_{L^2(0,T;H^{-1}(\Omega))} &\leq C_6 R_1(\theta_0, u_0, v_0, F, S), \end{aligned} \tag{5.15}$$

where $C_e = (\exp(T/\rho h) - 1)$, C', C_p *are from (1.16) and (1.2) respectively,*

$$R_1(\theta_0, u_0, v_0, F, S) = \|F\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\beta_1}{\beta_3\beta_4}\|S\|^2_{L^2(0,T;L^2(\Omega))} + \rho h\|v_0\|^2_{L^2(\Omega)} + D\|\Delta u_0\|^2_{L^2(\Omega)} + \frac{\beta_2\beta_1}{\beta_3}\|\theta_0\|^2_{L^2(\Omega)},$$

and

$$\begin{aligned} C_5 &= 4C_3^2\left[\left(\frac{\beta_3}{2\beta_1} + \frac{1}{2\mathrm{w}} + \frac{T}{D}\right)(C_e + 1) + 1\right], \\ C_6 &= 4C_4^2\left[\left(\frac{\beta_3}{2\beta_1} + \frac{\beta_3}{2\beta_4\beta_1} + \frac{1}{2\mathrm{w}}\right)(C_e + 1) + 1\right], \end{aligned} \tag{5.16}$$

while

$$C_3 = \frac{1}{\rho h}\max\{1, \mathrm{w}, D, \beta_1\},\ C_4 = \frac{1}{\beta_2}\max\{1, \beta_3, \beta_4\}. \tag{5.17}$$

Proof. We divide the proof into four steps, consisting of *a priori* estimates, the existence of weak solutions, the verification of initial data, and finally the uniqueness of solutions.

A Priori estimates: Multiplying both sides of (5.9a) and (5.9b) by $r'_{m,n}(t)$ and $\frac{\beta_1}{\beta_3}d_{m,n}(t)$, respectively, summing over $m = 1, 2, ..., n$, integrating over $(0, t)$, we establish the following energy identities by incorporating both non-homogeneous initial conditions and homogeneous boundary conditions in (5.1):

$$\begin{aligned} &\frac{\rho h}{2}\int_\Omega u_n'(t)^2dx + \mathrm{w}\int_0^t\int_\Omega |\nabla u_n'(\tau)|^2dxd\tau + \frac{D}{2}\int_\Omega |\Delta u_n(t)|^2dx \\ &-\beta_1\int_0^t\int_\Omega \nabla\theta_n(\tau).\nabla u_n'(\tau)dxd\tau = \int_0^t\int_\Omega F(x,\tau)u_n'(\tau)dxd\tau \\ &+\frac{\rho h}{2}\int_\Omega u_n'(0)^2dx + \frac{D}{2}\int_\Omega |\Delta u_n(0)|^2dx, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} &\frac{\beta_2\beta_1}{2\beta_3}\int_\Omega \theta_n(t)^2dx + \frac{\beta_1}{\beta_3}\int_0^t\int_\Omega |\nabla\theta_n(\tau)|^2dxd\tau \\ &+\beta_1\int_0^t\int_\Omega \nabla u_n'(\tau).\nabla\theta_n(\tau)dxd\tau + \frac{\beta_4\beta_1}{\beta_3}\int_0^t\int_\Omega \theta_n(\tau)^2dxd\tau \\ &= \frac{\beta_1}{\beta_3}\int_0^t\int_\Omega S(x,\tau)\theta_n(\tau)dxd\tau + \frac{\beta_2\beta_1}{2\beta_3}\int_\Omega \theta_n(0)^2dx. \end{aligned} \tag{5.19}$$

By adding (5.18), (5.19) and applying ϵ-inequality 1.14 we have

$$\begin{aligned}
&\rho h \int_\Omega u_n'(t)^2 dx + 2\text{w} \int_0^t \int_\Omega |\nabla u_n'(\tau)|^2 dx d\tau + D \int_\Omega |\Delta u_n(t)|^2 dx \\
&\quad + \frac{\beta_2\beta_1}{\beta_3} \int_\Omega \theta_n(t)^2 dx + \frac{2\beta_1}{\beta_3} \int_0^t \int_\Omega |\nabla \theta_n(\tau)|^2 dx d\tau + \frac{2\beta_4\beta_1}{\beta_3} \int_0^t \int_\Omega \theta_n(\tau)^2 dx d\tau \\
&\quad \leq \int_0^t \int_\Omega u_n'(\tau)^2 dx d\tau + \frac{\beta_1 \epsilon}{\beta_3} \int_0^t \int_\Omega \theta_n(\tau)^2 dx d\tau + \|F\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \frac{\beta_1}{\beta_3 \epsilon} \|S\|^2_{L^2(0,T;L^2(\Omega))} + \rho h \|v_{0,n}\|^2_{L^2(\Omega)} + D \|\Delta u_{0,n}\|^2_{L^2(\Omega)} \\
&\quad + \frac{\beta_2\beta_1}{\beta_3} \|\theta_{0,n}\|^2_{L^2(\Omega)}. \qquad (5.20)
\end{aligned}$$

Now choosing $\epsilon = \beta_4$, and employing Gronwall's inequality , we get

$$\|u_n'(t)\|^2_{L^2(\Omega)} \leq \frac{\exp(t/\rho h)}{\rho h} R_1(\theta_0, u_0, v_0, F, S).$$

This implies that

$$\begin{aligned}
&\max_{t\in[0,T]} \|u_n'(t)\|^2_{L^2(\Omega)} \leq \frac{(C_e+1)}{\rho h} R_1(\theta_0, u_0, v_0, F, S), \\
&\|u_n'\|^2_{L^2(0,T;L^2(\Omega))} \leq C_e\, R_1(\theta_0, u_0, v_0, F, S).
\end{aligned} \qquad (5.21)$$

Then energy estimate (5.20) leads to the following estimates

$$\begin{aligned}
\|\nabla \theta_n\|^2_{L^2(0,T;L^2(\Omega))} &\leq \frac{\beta_3}{2\beta_1} (C_e+1)\, R_1(\theta_0, u_0, v_0, F, S), && (5.22) \\
\max_{t\in[0,T]} \|\theta_n(t)\|^2_{L^2(\Omega)} &\leq \frac{\beta_3}{\beta_2\beta_1} (C_e+1)\, R_1(\theta_0, u_0, v_0, F, S), \\
\|\theta\|^2_{L^2(0,T;L^2(\Omega))} &\leq \frac{\beta_3}{2\beta_4\beta_1} (C_e+1)\, R_1(\theta_0, u_0, v_0, F, S), && (5.23) \\
\max_{t\in[0,T]} \|\Delta u_n(t)\|^2_{L^2(\Omega)} &\leq \frac{(C_e+1)}{D} R_1(\theta_0, u_0, v_0, F, S), && (5.24) \\
\|\nabla u_n'\|^2_{L^2(0,T;L^2(\Omega))} &\leq \frac{(C_e+1)}{2\text{w}} R_1(\theta_0, u_0, v_0, F, S). && (5.25)
\end{aligned}$$

To estimate $\|u_n''\|_{L^2(0,T;\mathcal{V}_1^2(\Omega)')}$, $\|\theta_n'\|_{L^2(0,T;H^{-1}(\Omega))}$, we proceed as follows. Fix any $v \in \mathcal{V}_1^2(\Omega)$, $w \in H_0^1(\Omega)$ with $\|v\|_{\mathcal{V}_1^2(\Omega)} \leq 1$, $\|w\|_{H_0^1(\Omega)} \leq 1$, and write $v = v_1 + v_2$, $w = w_1 + w_2$, where $v_1 \in Z_m$, $(v_2, z_m) = 0$, $w_1 \in W_m$, $(w_2, w_m) = 0$ $m = 1, 2, ..., n$, respectively. Since the functions $\{z_m\}_{m=1}^\infty$ and $\{w_m\}_{m=1}^\infty$ are orthogonal in $\mathcal{V}_1^2(\Omega)$, $H_0^1(\Omega)$ respectively, we have $\|v_1\|_{\mathcal{V}_1^2(\Omega)} \leq \|v\|_{\mathcal{V}_1^2(\Omega)} \leq 1$, $\|w_1\|_{H_0^1(\Omega)} \leq \|w\|_{H_0^1(\Omega)} \leq 1$. By utilizing (5.9a), (5.9b) and Cauchy's inequality, we get

$$\begin{aligned}
|(u_n''(t), v)| &\leq \frac{1}{\rho h}\bigg(\mathrm{w}\|\nabla u_n'(t)\|_{L^2(\Omega)}\|\nabla v_1\|_{L^2(\Omega)} + D\|\Delta u_n(t)\|_{L^2(\Omega)}\|\Delta v_1\|_{L^2(\Omega)} \\
&\quad +\beta_1\|\nabla\theta_n(t)\|_{L^2(\Omega)}\|\nabla v_1\|_{L^2(\Omega)} + \|v_1\|_{L^2(\Omega)}\|F(t)\|_{L^2(\Omega)}\bigg), \\
|(\theta_n'(t), w)| &\leq \frac{1}{\beta_2}\bigg(\|\nabla\theta_n(t)\|_{L^2(\Omega)}\|\nabla w_1\|_{L^2(\Omega)} + \beta_3\|\nabla u_n'(t)\|_{L^2(\Omega)}\|\nabla w_1\|_{L^2(\Omega)} \\
&\quad +\beta_4\|\theta_n(t)\|_{L^2(\Omega)}\|v_1\|_{L^2(\Omega)} + \|w_1\|_{L^2(\Omega)}\|S(t)\|_{L^2(\Omega)}\bigg).
\end{aligned}$$

Choosing $C_3, C_4 > 0$ as in (5.17), we obtain

$$\begin{aligned}
&\|u_n''(t)\|_{\mathcal{V}_1^2(\Omega)'} \\
&\quad \leq C_3\bigg[\|\nabla u_n'(t)\|_{L^2(\Omega)} + \|\Delta u_n(t)\|_{L^2(\Omega)} + \|\nabla\theta_n(t)\|_{L^2(\Omega)} + \|F(t)\|_{L^2(\Omega)}\bigg],
\end{aligned}$$

and

$$\begin{aligned}
&\|\theta_n'(t)\|_{H^{-1}(\Omega)} \\
&\quad \leq C_4\bigg[\|\nabla\theta_n(t)\|_{L^2(\Omega)} + \|\nabla u_n'(t)\|_{L^2(\Omega)} + \|\theta_n(t)\|_{L^2(\Omega)} + \|S(t)\|_{L^2(\Omega)}\bigg].
\end{aligned}$$

Squaring on both sides, integrating over $(0, T)$ and using the estimates (5.22), (5.23), (5.24) and (5.25), we have

$$\begin{aligned}
\|u_n''\|^2_{L^2(0,T;\mathcal{V}_1^2(\Omega)')} &\leq C_5 R_1(\theta_0, u_0, v_0, F, S), \\
\|\theta_n'\|^2_{L^2(0,T;H^{-1}(\Omega))} &\leq C_6 R_1(\theta_0, u_0, v_0, F, S),
\end{aligned}$$

where $C_5, C_6 > 0$ are the constants introduced in (5.16).

Existence of a weak solution: The estimates in the previous section show that the sequences $\{u_n\}, \{\Delta u_n\}, \{\nabla u_n'\}, \{u_n''\}, \{\theta_n\}, \{\nabla\theta_n\}$, and $\{\theta_n'\}$ are bounded in $L^2(0, T; \mathcal{V}_1^2(\Omega))$, $L^2(0, T; L^2(\Omega))$, $L^2(0, T; L^2(\Omega))$, $L^2(0, T; \mathcal{V}_1^2(\Omega)')$, $L^2(0, T; H_0^1(\Omega))$, $L^2(0, T; L^2(\Omega))$, $L^2(0, T; H^{-1}(\Omega))$ respectively.

By the Banach-Alagolu weak compactness theorem, there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{\theta_{n_k}\}$ of $\{\theta_n\}$ and functions $u \in L^2(0, T; \mathcal{V}_1^2(\Omega))$, $u' \in L^2(0, T; H_0^1(\Omega))$, $u'' \in$

$L^2(0,T;\mathcal{V}_1^2(\Omega)')$, $\theta \in L^2(0,T;H_0^1(\Omega))$, and $\theta' \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\begin{cases} u_{n_k} \rightharpoonup u & \text{weakly in } L^2(0,T;\mathcal{V}_1^2(\Omega)) \\ \Delta u_{n_k} \rightharpoonup \Delta u & \text{weakly in } L^2(0,T;L^2(\Omega)) \\ \nabla u'_{n_k} \rightharpoonup \nabla u' & \text{weakly in } L^2(0,T;L^2(\Omega)) \\ u''_{n_k} \rightharpoonup u'' & \text{weakly in } L^2(0,T;\mathcal{V}_1^2(\Omega)') \\ \theta_{n_k} \rightharpoonup \theta & \text{weakly in } L^2(0,T;H_0^1(\Omega)) \\ \nabla\theta_{n_k} \rightharpoonup \nabla\theta & \text{weakly in } L^2(0,T;L^2(\Omega)) \\ \theta'_{n_k} \rightharpoonup \theta' & \text{weakly in } L^2(0,T;H^{-1}(\Omega)), \end{cases} \tag{5.26}$$

as $k \to \infty$. Using standard arguments, by passing to limits $k \to \infty$ on (5.9), we build a weak solution to the direct problem (5.1), which satisfies the estimates (5.11)-(5.15).

Verification of initial data: Next we prove that the solution (u,θ) satisfies the initial conditions $u(0) = u_0, u_t(0) = v_0$ and $\theta(0) = \theta_0$. By Remark 5.2 we recall that $(u,\theta) \in C([0,T];H_0^1(\Omega)) \times C([0,T];L^2(\Omega))$. Choose a test functions $v \in C^2([0,T];\mathcal{V}_1^2(\Omega))$ and $w \in C^1([0,T];H_0^1(\Omega))$ with $v(T) = 0$, $v'(T) = 0$ and $w(T) = 0$ in the weak form of Definition 5.1, and integrate over $(0,T)$, integrating by parts with respect to time in the first two terms of (5.8a) and first, and third term of (5.8b), we get

$$\begin{aligned} I_1 &:= \int_0^T \Big[(\rho h u(t), v''(t)) - (\mathrm{w}\nabla u(t), \nabla v'(t)) + (D\Delta u(t), \Delta v(t)) \\ &\quad -(\beta_1 \nabla\theta(t), \nabla v(t))\Big] dt \qquad (5.27) \\ &= \int_0^T (F(t), v(t))dt - (\rho h u(0), v'(0)) + (\rho h u'(0), v(0)) + (\mathrm{w}\nabla u(0), \nabla v(0)), \\ I_2 &:= \int_0^T \Big[- (\beta_2\theta(t), w'(t)) + (\nabla\theta(t), \nabla w(t)) - (\beta_3\nabla u(t), \nabla w'(t) + (\beta_4\theta(t), w(t))\Big] dt \\ &= \int_0^T (S(t), w(t))dt + (\beta_2\theta(0), w(0)) + (\beta_3\nabla\theta(0), \nabla w(0)). \qquad (5.28) \end{aligned}$$

On the other hand, integrate over $(0,T)$, integrating by parts in the first two terms of (5.9a) and first, last terms of (5.9b), then passing the weak limits (5.26), we arrive at the following:

$$\begin{aligned} I_1 &= \int_0^T \Big[(F(t), v(t))dt - (\rho h\, u_0, v'(0))) + (\rho h\, v_0, v(0) \\ &\quad +(\mathrm{w}\nabla u_0, \nabla v(0))\Big] dt, \end{aligned} \tag{5.29}$$

and

$$I_2 = \int_0^T \left[(S(t), w(t))dt + (\beta_2\theta_0, w(0)) + (\beta_3 \nabla u_0, \nabla w(0)) \right] dt. \tag{5.30}$$

Comparing (5.27)-(5.28) and (5.29)-(5.30), we obtain the desired result.

Uniqueness: It suffices to check that the only weak solution of (5.1) with homogeneous initial data and $F = 0$, $S = 0$ is $(u, \theta) = 0$. Let $(u, \theta) = (u_1 - u_2,\ \theta_1 - \theta_2)$, where (u_1, θ_1) and (u_2, θ_2) are the two weak solutions of (5.1). By the existence of a weak solution, $(u, \theta) \in L^2(0, T; \mathcal{V}_1^2(\Omega)) \times L^2(0, T; H_0^1(\Omega))$ solves the direct problem (5.1) with homogeneous initial data and source terms $F = 0, S = 0$. Then the estimates (5.12) and (5.14) imply that $\|u\|_{L^\infty(0,T;\mathcal{V}_1^2(\Omega))} = 0$, $\|\theta\|_{L^\infty(0,T;L^2(\Omega))} = 0$, whence $(u, \theta) = 0,\ \ \forall (x, t) \in \Omega_T$. This completes the proof. □

5.3 Solvability of inverse problem

In this section, we mainly focus on the solvability of the inverse problem. We utilize a priori estimate for the weak solution of system (5.1) to demonstrate the compactness and Lipschitz continuity of the input-output operator Φ. By the compactness of the operator, we conclude that the inverse problem is ill-posed. The Lipschitz continuity leads to the lower semi-continuity of the functional $\mathcal{J}_\alpha$, which in turn leads to the existence of a unique minimizer for this functional.

Let $(u_1(x, t), \theta_1(x, t))$ and $(u_2(x, t), \theta_2(x, t))$ be the solutions of the direct problem (5.1) corresponding to the source terms (F_1, S_1) and (F_2, S_2) respectively for a common initial data u_0, v_0, and θ_0. Then $\delta u(x, t) = u_1(x, t) - u_2(x, t)$ and $\delta\theta(x, t) = \theta_1(x, t) - \theta_2(x, t)$ solves the following system

$$\begin{cases} \rho h \delta u_{tt} - \mathrm{w}\Delta\delta u_t + D\Delta^2\delta u + \beta_1\Delta\delta\theta = \delta F(x, t),\ (x, t) \in \Omega_T & \text{(5.31a)} \\ \beta_2\delta\theta_t - \Delta\delta\theta - \beta_3\Delta\delta u_t + \beta_4\delta\theta = \delta S(x, t),\ (x, t) \in \Omega_T, & \text{(5.31b)} \\ \delta u = \frac{\partial \delta u}{\partial \mathrm{n}} = 0,\ \delta\theta = 0,\ (x, t) \in \Gamma_T, \\ \delta u(x, 0) = 0, \delta u_t(x, 0) = 0,\ \delta\theta(x, 0) = 0,\ x \in \Omega, \end{cases}$$

where $\delta F(x, t) = F_1(x, t) - F_2(x, t)$, $\delta S(x, t) = S_1(x, t) - S_2(x, t)$.

Lemma 5.1. *Suppose Assumption 5.1 holds true. Then the input output operator Φ introduced in (5.3) is a compact operator. Furthermore, this operator is Lipschitz continuous,* *that is,*

$$\|\Phi[F_1, S_1] - \Phi[F_2, S_2]\|_{L^2(\Omega)} \le L_1 \left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))} \right)^{1/2}, \quad (5.32)$$

for all $(F, S) \in \mathcal{F} \times \mathcal{G}$*, where* $L_1 = (C_7 T C_e)^{1/2}$, $C_7 = \max\{1, \frac{\beta_1}{\beta_3\beta_4}\}$*, and* C_e *is defined in Theorem 5.1.*

Proof. Let $\{F_m\} \subset \mathcal{F},\ \{S_m\} \subset \mathcal{G},\ m = 1, 2, ...$, be the bounded sequence of sources in $L^2(0, T; L^2(\Omega))$, and $u(x, T; F_m, S_m)$ be the corresponding output represented by $\{u_{Tm}\}$. The sequence of output $\{u_{Tm}\}$ is bounded in $H^1_0(\Omega)$, as shown by the estimate (5.12) and Remark 5.2. Since $H^1_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, the output sequence $\{u_{Tm}\}$ is precompact in $L^2(\Omega)$, implying that the input output operator Φ is compact.

Next, we prove that Φ is Lipschitz continuous. From the definition of operator Φ, we have

$$\begin{aligned} \|\Phi[F_1, S_1] - \Phi[F_2, S_2]\|_{L^2(\Omega)} &= \|u(., T; F_1, S_1) - u(., T; F_2, S_2)\|_{L^2(\Omega)} \\ &= \|\delta u(., T)\|_{L^2(\Omega)}. \end{aligned} \quad (5.33)$$

Since

$$\|\delta u(., T)\|^2_{L^2(\Omega)} \le T \|\delta u_t\|^2_{L^2(0,T;L^2(\Omega))}, \quad (5.34)$$

and the second estimate in (5.21) holds for $\delta u(x, t; \delta F, \delta S)$, we get

$$\begin{aligned} \|\delta u(., T)\|^2_{L^2(\Omega)} &\le TC_e \left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\beta_1}{\beta_4\beta_3} \|\delta S\|^2_{L^2(0,T;L^2(\Omega))} \right), \\ &\le C_7 T C_e \left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))} \right), \end{aligned} \quad (5.35)$$

where $C_7 = \max\{1, \frac{\beta_1}{\beta_3\beta_4}\}$. Substituting (5.35) in (5.33) we arrive at the required result (5.32). □

Remark 5.3. The compactness of the input output operator Φ leads to the ill-posedness of the inverse problem (5.1) and (5.2) (see, [40] and also [50], Lemma 1.3.1).

Theorem 5.2. *Let Assumption (5.1) holds true. Suppose that* $u_T \in L^2(\Omega)$*. Then for any* $\alpha > 0$*, there exists a unique admissible mechanical load* $F_\alpha \in \mathcal{F}$ *and admissible heat source* $S_\alpha \in \mathcal{G}$ *minimizing the functional* $\mathcal{J}_\alpha(F, S)$*.*

Proof. We first show that the Tikhonov functional (5.5) is Lipschitz continuous. To this end we employ the identity

$$
\begin{aligned}
&|\mathcal{J}(F_1,S_1)-\mathcal{J}(F_2,S_2)|^2 \\
&\quad = \left|\sqrt{\mathcal{J}(F_1,S_1)}+\sqrt{\mathcal{J}(F_2,S_2)}\right|^2 \left|\sqrt{\mathcal{J}(F_1,S_1)}-\sqrt{\mathcal{J}(F_2,S_2)}\right|^2,
\end{aligned}
$$

and also the identity

$$
\begin{aligned}
&\left|\sqrt{\mathcal{J}(F_1,S_1)}-\sqrt{\mathcal{J}(F_2,S_2)}\right|^2 \\
&\quad = \frac{1}{2}\left|\|\Phi[F_1,S_1]-u_T\|_{L^2(\Omega)}-\|\Phi[F_2,S_2]-u_T\|_{L^2(\Omega)}\right|^2,
\end{aligned}
$$

which follows from the definition of the functional $\mathcal{J}(F,S)$.

In view of the above identities, the inequality $|\|a\|-\|b\||\leq\|a-b\|$ and Lemma 5.1, we deduce that

$$
\begin{aligned}
&|\mathcal{J}(F_1,S_1)-\mathcal{J}(F_2,S_2)|^2 \\
&\quad \leq \frac{1}{2}\left|\sqrt{\mathcal{J}(F_1,S_1)}+\sqrt{\mathcal{J}(F_2,S_2)}\right|^2 \|\Phi[F_1,S_1]-\Phi[F_2,S_2]\|^2_{L^2(\Omega)} \\
&\quad \leq L_1^2\left(\|\Phi(F_1,S_1)\|^2_{L^2(\Omega)}+\|\Phi(F_2,S_2)\|^2_{L^2(\Omega)}+2\|u_T\|^2_{L^2(\Omega)}\right) \\
&\quad \left(\|F_1-F_2\|^2_{L^2(0,T;L^2(\Omega))}+\|S_1-S_2\|^2_{L^2(0,T;L^2(\Omega))}\right),
\end{aligned}
$$

for all $(F,S)\in\mathcal{F}\times\mathcal{G}$.

Using the definition of Φ and the second estimate in (5.21) applied to the weak solution of the direct problem, we get

$$
\begin{aligned}
&\|\Phi[F,S]\|^2_{L^2(\Omega)} \leq T\|u_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq \; TC_e\left(M_1+\frac{\beta_1}{\beta_4\beta_3}M_2+\rho h\|v_0\|^2_{L^2(\Omega)}+D\|\Delta u_0\|^2_{L^2(\Omega)}+\frac{\beta_2\beta_1}{\beta_3}\|\theta_0\|^2_{L^2(\Omega)}\right),
\end{aligned}
$$

where $M_1>0$ and $M_2>0$ are the constants introduced in the definitions of the set of admissible mechanical loads $\mathcal{F}$ and external heat sources $\mathcal{G}$, respectively.

Hence, we have

$$
\begin{aligned}
&\left|\mathcal{J}(F_1,S_1)-\mathcal{J}(F_2,S_2)\right| \\
&\quad \leq \; L_2\left(\|F_1-F_2\|^2_{L^2(0,T;L^2(\Omega))}+\|S_1-S_2\|^2_{L^2(0,T;L^2(\Omega))}\right)^{1/2},
\end{aligned}
$$

where

$$L_2 = L_1\Bigg[2TC_e\Bigg(M_1 + \frac{\beta_1}{\beta_4\beta_3}M_2 + \rho h\|v_0\|^2_{L^2(\Omega)} + D\|\Delta u_0\|^2_{L^2(\Omega)} + \frac{\beta_2\beta_1}{\beta_3}\|\theta_0\|^2_{L^2(\Omega)}\Bigg) + 2\|u_T\|^2_{L^2(\Omega)}\Bigg]^{1/2},$$

which means the Lipschitz continuity of the Tikhonov functional (5.5).

Consequently, the functional $\mathcal{J}(F,S)$ is weakly lower semi continuous on a nonempty closed convex set $\mathcal{F}\times\mathcal{G}$. Hence by the generalized Weierstrass theorem, we conclude that the functional $\mathcal{J}(F,S)$ has a minimizer $(F,S)\in\mathcal{F}\times\mathcal{G}$.

Let $\{(F_n,S_n)\}\in\mathcal{F}\times\mathcal{G}$ be the minimizing sequence for the functional $\mathcal{J}(F,S)$, and $\{(u(x,t;F_n,S_n),\theta(x,t;F_n,S_n))\}$ is the corresponding sequence of weak solution to the direct problem (5.1). Now assume that $F_n \rightharpoonup F$ in $\mathcal{F}$ and $S_n \rightharpoonup S$ in $\mathcal{G}$. The estimates (5.12) and (5.13) show that the sequences $\{u(x,t;F_n,S_n)\}$, $\{\theta(x,t;F_n,S_n)\}$ are bounded in $L^2(0,T;\mathcal{V}_1^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$ respectively. Then there exist subsequences $\{u_n\}$ of $\{u(F_n,S_n)\}$ and $\{\theta_n\}$ of $\{\theta(F_n,S_n)\}$, such that $u_n \rightharpoonup u^*(x,t)$ in $L^2(0,T;\mathcal{V}_1^2(\Omega))$, $\theta_n \rightharpoonup \theta^*(x,t)$ in $L^2(0,T;H_0^1(\Omega))$. To prove that $u^*(x,t)=u(x,t;F,S)$ and $\theta^*(x,t)=\theta(x,t;F,S)$, we choose test functions $v\in C^2([0,T];\mathcal{V}_1^2(\Omega))$ and $w\in C^1([0,T];H_0^1(\Omega))$ with $v(T)=0$, $v'(T)=0$ and $w(T)=0$ for the weak form (5.8a)-(5.8b) , and integrate over $(0,T)$, we get

$$\begin{aligned}
&\int_0^T\int_\Omega \rho h u_n(t)v''(t)dxdt - \int_0^T\int_\Omega \mathrm{w}\nabla u_n(t)\cdot\nabla v'(t)dxdt \\
&\quad + \int_0^T\int_\Omega D\Delta u_n(t)\Delta v(t)dxdt - \int_0^T\int_\Omega \beta_1\nabla\theta_n(t)\cdot\nabla v(t)dxdt \\
&\quad = \int_0^T\int_\Omega F_n(t)v(t)dxdt - \int_\Omega \rho h u_n(0)v'(0)dx + \int_\Omega \rho h u_n'(0)v(0)dx \\
&\quad + \int_\Omega \mathrm{w}\nabla u_n(0)\cdot\nabla v(0)dx, \qquad (5.36)
\end{aligned}$$

and

$$\begin{aligned}
&-\int_0^T\int_\Omega \beta_2\theta_n(t)w'(t)dxdt + \int_0^T\int_\Omega \nabla\theta_n(t)\cdot\nabla w(t)dxdt \\
&\quad -\int_0^T\int_\Omega \beta_3\nabla u_n(t)\cdot\nabla w'(t)dxdt + \int_0^T\int_\Omega \beta_4\theta_n(t)w(t)dxdt \\
&\quad = \int_0^T\int_\Omega S_n(t)w(t)dxdt + \int_\Omega \beta_2\theta_n(0)w(0)dx + \int_\Omega \beta_3\nabla u_n(0)\cdot\nabla w(0)dx. \qquad (5.37)
\end{aligned}$$

Since $F_n \rightharpoonup F$, $S_n \rightharpoonup S$ in $L^2(0,T;L^2(\Omega))$, passing the limit $n \to \infty$ in (5.36)-(5.37), one can verify that $u^*(x,t) = u(x,t;F,S)$ and $\theta^*(x,t) = \theta(x,t;F,S)$ is a weak solution of the system (5.1).

The lower semi-continuity of the functional $\mathcal{J}(F,S)$ in $\mathcal{F} \times \mathcal{G}$ implies that the regularized Tikhonov functional $\mathcal{J}_\alpha(F_n,S_n)$ corresponding to F_n, S_n defined by (5.6) satisfies $\mathcal{J}_\alpha(F,S) \le \liminf_{n\to\infty} \mathcal{J}_\alpha(F_n,S_n)$ as $F_n \rightharpoonup F$ in $\mathcal{F}$ and $S_n \rightharpoonup S$ in $\mathcal{G}$. Therefore $\mathcal{J}_\alpha(F,S)$ is lower semi-continuous. Moreover, due to the linearity of the inverse source problem (5.1)-(5.2), we have

$$\Phi[\lambda(F_1,S_1) + (1-\lambda)(F_2,S_2)] = \lambda\Phi[F_1,S_1] + (1-\lambda)\Phi[F_2,S_2], \lambda \in (0,1),$$

whence the functional $\mathcal{J}_\alpha(F,S)$ is strictly convex. By incorporating all these argument and employing the generalized Weierstrass theorem, we conclude that the regularized Tikhonov functional $\mathcal{J}_\alpha(F,S)$ has a unique minimizer $(F_\alpha, S_\alpha) \in \mathcal{F} \times \mathcal{G}$. □

5.4 Fréchet differentiability of the Tikhonov functional

This section demonstrates the Fréchet differentiability of the functional, and it also shows how to express the Fréchet gradient in terms of both direct and adjoint problem solutions. We established the well-posedness of the adjoint problem, and proved the Lipschitz continuity of the Fréchet gradient by using the estimates for direct and adjoint problems. Now we establish an integral relationship which connects the change in the sources $\delta F(x,t) = F_1(x,t) - F_2(x,t)$ and $\delta S(x,t) = S_1(x,t) - S_2(x,t)$ to the change in output $\delta u(x,T) = u(x,T;F_1,S_1) - u(x,T;F_2,S_2)$.

Lemma 5.2. *Let the conditions of Theorem 5.2 hold true. Then, we have*

$$-\rho h \int_\Omega \delta u(x,T) q(x) dx = \int_{\Omega_T} \delta F(x,t)\psi(x,t) dx dt + \int_{\Omega_T} \delta S(x,t)\varphi(x,t) dx dt, \quad (5.38)$$

where $(\psi(x,t), \varphi(x,t))$ is the solution of the following adjoint problem

$$\begin{cases} \rho h\psi_{tt} + \mathrm{w}\Delta\psi_t + D\Delta^2\psi + \beta_3\Delta\varphi_t = 0,\ (x,t) \in \Omega_T & (5.39a) \\ -\beta_2\varphi_t - \Delta\varphi + \beta_1\Delta\psi + \beta_4\varphi = 0,\ (x,t) \in \Omega_T & (5.39b) \\ \psi = \frac{\partial \psi}{\partial n} = 0,\ \varphi = 0,\ (x,t) \in \Gamma_T \\ \psi(x,T) = 0, \psi_t(x,T) = q(x),\ \varphi(x,T) = 0,\ x \in \Omega. \end{cases}$$

Proof. Multiplying the both sides of equations (5.31a) and (5.31b) by arbitrary functions $\psi(x,t)$ and $\varphi(x,t)$ respectively, integrating over Ω_T, and then adding the obtained equations, we have

$$\begin{aligned}\int_{\Omega_T} &\left(\rho h\delta u_{tt} - \mathrm{w}\Delta\delta u_t + D\Delta^2\delta u + \beta_1\Delta\delta\theta\right)\psi(x,t)dxdt \\ &+ \int_{\Omega_T}\left(\beta_2\delta\theta_t - \Delta\delta\theta - \beta_3\Delta\delta u_t + \beta_4\delta\theta\right)\varphi(x,t)dxdt \\ &= \int_{\Omega_T}\delta F(x,t)\psi(x,t)dxdt + \int_{\Omega_T}\delta S(x,t)\varphi(x,t)dxdt.\end{aligned}$$

After performing integration by parts formula multiple times and using the initial and boundary conditions, we get

$$\begin{aligned}\int_{\Omega_T} &\left(\rho h\psi_{tt} + \mathrm{w}\Delta\psi_t + D\Delta^2\psi + \beta_3\Delta\varphi_t\right)\delta u(x,t)\,dxdt \\ &+ \int_{\Omega_T}\left(-\beta_2\varphi_t - \Delta\varphi + \beta_1\Delta\psi + \beta_4\varphi\right)\delta\theta(x,t)\,dxdt \\ &+ \int_{\Omega}\left[\varphi(x,t)\left(\beta_2\delta\theta(x,t) - \beta_3\Delta\delta u(x,t)\right)\right]_{t=0}^{T}dx \\ &+ \int_{\Omega}\left[\psi(x,t)\left(\rho h\delta u_t(x,t) - \mathrm{w}\Delta\delta u(x,t)\right)\right]_{t=0}^{T}dx - \int_{\Omega}\left[\psi_t(x,t)\rho h\delta u(x,t)\right]_{t=0}^{T}dx \\ &= \int_{\Omega_T}\delta F(x,t)\psi(x,t)dxdt + \int_0^T\int_{\Omega}\delta S(x,t)\varphi(x,t)dxdt.\end{aligned}$$

By considering the fact that the arbitrary function $(\psi(x,t),\varphi(x,t))$ solves the system (5.39), the first four left-hand side integrals become zero, and hence we obtain the desired integral relationship. □

Next, we prove the well-posedness of the adjoint problem (5.39a)-(5.39b). Unlike the single plate equation, the time reversal method ($\tau = T - t$) doesn't work to write straightaway the existence of the solution to the adjoint problem. This issue arises due to the coupling effect of the plate and heat equation and the impact of the *irreversible system.* Indeed, in the direct problem (5.1), the higher-order space-time derivative of the coupling term $\beta_3\Delta u_t$ has appeared in the second equation (5.1b). We multiplied the first equation (5.1a) by u_t and the second equation (5.1b) by $\frac{\beta_1}{\beta_3}\theta$, the integrals corresponding to the coupling terms $\beta_3\Delta u_t$ and $\beta_1\Delta\theta$ get canceled each other when we add them together after an integration by parts. This idea doesn't seem to work in the case of the adjoint problem since the coupling term $\beta_3\Delta\varphi_t$ has appeared in the first equation (5.39a), which makes this system complicated compared to the direct problem. To overcome this difficulty, we take the formal time derivative of the heat equation and directly apply the Galerkin method to demonstrate that the adjoint problem is well-posed.

Theorem 5.3. *Suppose Assumption 5.1 holds true and $q \in L^2(\Omega)$. Then the adjoint problem* (5.39) *has a unique weak solution $(\psi, \varphi) \in L^\infty(0,T;\mathcal{V}_1^2(\Omega)) \times L^\infty(0,T;H_0^1(\Omega))$. Furthermore,*

$$\|\psi\|^2_{L^\infty(0,T;\mathcal{V}_1^2(\Omega))} \leq \frac{C'}{D}\rho h\|q\|^2_{L^2(\Omega)}, \tag{5.40}$$

$$\|\psi_t\|^2_{L^2(0,T;L^2(\Omega))} \leq T\|q\|^2_{L^2(\Omega)}, \tag{5.41}$$

$$\|\varphi\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq \frac{T\beta_1}{\beta_3}\left(1+\frac{T}{\beta_2}\right)\rho h\|q\|^2_{L^2(\Omega)},$$

$$\|\varphi_t\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \frac{\beta_1}{\beta_2\beta_3}\rho h\|q\|^2_{L^2(\Omega)}, \tag{5.42}$$

$$\|\nabla\varphi_t\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{\beta_1}{\beta_3}\rho h\|q\|^2_{L^2(\Omega)}. \tag{5.43}$$

Proof. We employ the Galerkin approximation method, as we have done in subsection 5.2.1 for the direct problem, with Faedo-Galerkin approximation $\psi_n(t) := \psi_n(x,t), \varphi_n(t) := \varphi_n(x,t)$ of the form

$$\psi_n(t) = \sum_{m=1}^{n} a_{m,n}(t)z_m, \;\; \varphi_n(t) = \sum_{m=1}^{n} b_{m,n}(t)w_m, \; q_n = \sum_{m=1}^{n} c_{m,n}z_m,$$

where $\{z_m\}_{m=1}^\infty$ form an orthogonal basis for $\mathcal{V}_1^2(\Omega)$ and $\{w_m\}_{m=1}^\infty$ form an orthogonal basis for $H_0^1(\Omega)$. Now we have the following weak form corresponding to adjoint problem (5.39) for all $t \in [0,T]$:

$$\begin{cases} \rho h(\psi_n''(t), z_m) - \mathrm{w}(\nabla\psi_n'(t), \nabla z_m) + D(\Delta\psi_n(t), \Delta z_m) \\ \qquad\qquad -\beta_3(\nabla\varphi_n'(t), \nabla z_m) = 0, & (5.44a) \\ -\beta_2(\varphi_n'(t), w_m) + (\nabla\varphi_n(t), \nabla w_m) - \beta_1(\nabla\psi_n(t), \nabla w_m) + \beta_4(\varphi_n(t), w_m) = 0, & (5.44b) \\ \psi_n(T) = 0, \; \psi_n'(T) = q_n, \; \varphi_n(T) = 0. \end{cases}$$

By following the similar steps done in section (5.2.1), we obtain that (5.44) has a unique solution $(\psi_n, \varphi_n) \in C^1([0,T];Z_n) \times C([0,T];W_n)$. From equation (5.44b), $\forall w_m \in W_n, m =$

$1, 2 \cdots n$, we have

$$(\varphi_n'(T), w_m) = \frac{1}{\beta_2}\bigg((\nabla \varphi_n(T), \nabla w_m) - \beta_1(\nabla \psi_n(T), \nabla w_m) + \beta_4(\psi_n(T), w_m)\bigg) = 0,$$

whence $(\varphi_n'(T), w_m) = b_{m,n}'(T) = 0$.

In order to handle the term $(\nabla \varphi_n'(t), w_m)$, we differentiate (5.44b) formally with respect to the time variable to obtain that for all $t \in [0, T]$:

$$\begin{cases} \rho h(\psi_n''(t), z_m) - \mathrm{w}(\nabla \psi_n'(t), \nabla z_m) + D(\Delta \psi_n(t), \Delta z_m) \\ \qquad\qquad -\beta_3(\nabla \varphi_n'(t), \nabla z_m) = 0, & (5.45a) \\ -\beta_2(\varphi_n''(t), w_m) + (\nabla \varphi_n'(t), \nabla w_m) - \beta_1(\nabla \psi_n'(t), \nabla w_m) + \beta_4(\varphi_n'(t), w_m) = 0, & (5.45b) \\ \psi_n(T) = 0,\ \psi_n'(T) = q_n,\ \varphi_n(T) = 0,\ \varphi_n'(T) = 0. \end{cases}$$

Now multiplying the equation (5.45a) by $a_{m,n}'(t)$ and (5.45b) by $\frac{\beta_3}{\beta_1} b_{m,n}'(t)$, summing over $m = 1, 2, ..., n$, integrating over (t, T), we deduce the following inequality by incorporating both initial and boundary data in (5.45):

$$\begin{aligned} \frac{\rho h}{2} \int_\Omega \psi_n'(t)^2 dx + \mathrm{w} \int_t^T \int_\Omega |\nabla \psi_n'(\tau)|^2 dx d\tau + \frac{D}{2} \int_\Omega |\Delta \psi_n(t)|^2 dx \\ + \beta_3 \int_t^T \int_\Omega \nabla \varphi_n'(\tau). \nabla \psi_n'(\tau) dx d\tau = \frac{\rho h}{2} \int_\Omega q_n^2 dx, \end{aligned} \quad (5.46)$$

and

$$\begin{aligned} \frac{\beta_2 \beta_3}{2\beta_1} \int_\Omega \varphi_n'(t)^2 dx + \frac{\beta_3}{\beta_1} \int_t^T \int_\Omega |\nabla \varphi_n'(\tau)|^2 dx d\tau \\ -\beta_3 \int_t^T \int_\Omega \nabla \varphi_n'(\tau). \nabla \psi_n'(\tau) dx d\tau + \frac{\beta_4 \beta_3}{\beta_1} \int_t^T \int_\Omega \varphi_n'(\tau)^2 dx d\tau = 0. \end{aligned} \quad (5.47)$$

By adding (5.46) and (5.47), we get

$$\begin{aligned} &\frac{\rho h}{2} \int_\Omega \psi_n'(t)^2 dx + \mathrm{w} \int_t^T \int_\Omega |\nabla \psi_n'(\tau)|^2 dx d\tau + \frac{D}{2} \int_\Omega |\Delta \psi_n(t)|^2 dx \\ &\quad + \frac{\beta_2 \beta_3}{2\beta_1} \int_\Omega \varphi_n'(t)^2 dx + \frac{\beta_3}{\beta_1} \int_t^T \int_\Omega |\nabla \varphi_n'(\tau)|^2 dx d\tau + \frac{\beta_4 \beta_3}{\beta_1} \int_t^T \int_\Omega \varphi_n'(\tau)^2 dx d\tau \\ &\quad \leq \frac{\rho h}{2} \int_\Omega q_n^2 dx. \end{aligned}$$

This implies that the following estimates hold:

$$\begin{aligned}
\|\psi_n\|^2_{L^\infty(0,T;\mathcal{V}^2_1(\Omega))} &\le \frac{C'}{D}\rho h\|q_n\|^2_{L^2(\Omega)}, \\
\|\psi'_n\|^2_{L^2(0,T;L^2(\Omega))} &\le T\|q_n\|^2_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
\|\varphi'_n\|^2_{L^\infty(0,T;L^2(\Omega))} &\le \frac{\beta_1}{\beta_2\beta_3}\rho h\|q_n\|^2_{L^2(\Omega)}, \\
\|\nabla\varphi'_n\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{\beta_1}{\beta_3}\rho h\|q_n\|^2_{L^2(\Omega)}, \\
\|\nabla\psi'_n\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{\rho h}{2\mathrm{w}}\|q_n\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using the condition $\varphi_n(\cdot,T)=0, \forall x\in\Omega$, we have

$$\begin{aligned}
\|\varphi_n\|^2_{L^\infty(0,T;L^2(\Omega))} &\le T\|\varphi'_n\|^2_{L^2(0,T;L^2(\Omega))} \le T^2\rho h\frac{\beta_1}{\beta_2\beta_3}\|q_n\|^2_{L^2(\Omega)}, \\
\|\nabla\varphi_n\|^2_{L^\infty(0,T;L^2(\Omega))} &\le T\|\nabla\varphi'_n\|^2_{L^2(0,T;L^2(\Omega))} \le T\rho h\frac{\beta_1}{\beta_3}\|q_n\|^2_{L^2(\Omega)}.
\end{aligned}$$

By employing the similar steps in Theorem 5.1 , which is used to find the estimate for $\|u''_n\|_{L^2(0,T;\mathcal{V}^2_1(\Omega)')}$, we get the estimate

$$\|\psi''_n\|_{L^2(0,T;\mathcal{V}^2_1(\Omega)')} \le 3\rho h C_8^2\left(\frac{TC_1}{D}+\frac{1}{2\omega}+\frac{\beta_1}{\beta_3}\right)\|q_n\|^2_{L^2(\Omega)},$$

where $C_8 = \frac{1}{\rho h}\max\{\omega, D, \beta_3\}$.

The existence of a weak solution to the adjoint problem (5.39), which satisfies the estimates (5.40)-(5.43) can be proved using the same arguments as those used to show the existence of a weak solution to the direct problem in Theorem 5.1. Using the same steps as for the direct problem, one can verify the initial data and establish the uniqueness of the solution. □

Now consider the increment of the functional $\delta\mathcal{J}(F,S) = \mathcal{J}(F+\delta F, S+\delta S) - \mathcal{J}(F,S)$, which satisfy the identity

$$\begin{aligned}
\delta\mathcal{J}(F,S) &= \int_\Omega [u(x,T;F,S)-u_T(x)]\delta u(x,T;F,S)dx \\
&\quad +\frac{1}{2}\int_\Omega [\delta u(x,T;F,S)]^2 dx, \qquad (5.48)
\end{aligned}$$

for all $S, S + \delta S \in \mathcal{G}$, $F, F + \delta F \in \mathcal{F}$, and

$$\delta u(x, T; F, S) = u(x, T; F + \delta F, S + \delta S) - u(x, T; F, S).$$

Making use of Lemma 5.2 and choosing arbitrary input $q \in L^2(\Omega)$ in (5.38) as

$$q(x) = -\frac{1}{\rho h} \left[u(x, T; F, S) - u_T(x)\right] \in L^2(\Omega), \tag{5.49}$$

we express the first integral of (5.48) as follows

$$\begin{aligned}
&\int_{\Omega_T} [u(x, T; F, S) - u_T(x)]\delta u(x, T; F, S)dx \\
&\qquad = \int_{\Omega_T} \delta F(x, T)\psi(x, t)dxdt + \int_{\Omega_T} \delta S(x, t)\varphi(x, t)dxdt.
\end{aligned} \tag{5.50}$$

Remark 5.4. From Theorem 5.1 and Remark 5.2 , it is clear that $u(\cdot, T; F, S) \in L^2(\Omega)$ and the measured data $u_T \in L^2(\Omega)$. Consequently, the right-hand side of (5.49) belongs to $L^2(\Omega)$. Hence, the existence and uniqueness of the weak solution to the adjoint problem (5.39) is proved when $q \in L^2(\Omega)$ is justified.

Proposition 5.1. *Suppose Assumption 5.1 holds true. Then for the Fréchet gradient $\nabla \mathcal{J}(F, S)$ of the Tikhonov functional $\mathcal{J}(F, S)$ defined by (5.5), the following gradient formula holds:*

$$\nabla \mathcal{J}(F, S) = \left(\psi(x, t; F, S), \varphi(x, t; F, S)\right)^{\mathrm{T}}, \ (F, S) \in \mathcal{F} \times \mathcal{G}, \tag{5.51}$$

where

$$(u(x, t; F, S), \theta(x, t; F, S)) \textit{ and } (\psi(x, t; F, S), \varphi(x, t; F, S))$$

are the weak solutions of direct problem (5.1) and adjoint problem (5.39), respectively.

Proof. Using the integral identity (5.48) and (5.50), we get

$$\begin{aligned}
\Big| \delta \mathcal{J}(F, S) \ &- \ \int_{\Omega_T} \left(\psi(x, t; F, S), \varphi(x, t; F, S)\right)^{\mathrm{T}} (\delta F(x, t), \delta S(x, t))dxdt \Big| \\
&= \ \frac{1}{2} \int_{\Omega} \delta u(x, T)^2 dx.
\end{aligned}$$

Applying Holder's inequality and the trace estimate

$$\|\delta u(., T)\|^2_{L^2(\Omega)} \leq T \|\delta u_t\|^2_{L^2(0,T;L^2(\Omega))},$$

we get

$$\left|\delta\mathcal{J}(F,S) - \int_{\Omega_T}(\psi(x,t;F,S),\varphi(x,t;F,S))^{\mathrm{T}}(\delta F(x,t),\delta S(x,t))dxdt\right| \leq \frac{T}{2}\|\delta u_t\|^2_{L^2(0,T;L^2(\Omega))}. \tag{5.52}$$

Thus, invoking (5.34) and (5.35) into (5.52), we infer that the following notion of the Fréchet derivative holds:

$$\begin{aligned}\delta\mathcal{J}(F,S) &= \left((\psi(x,t;F,S),\varphi(x,t;F,S))^{\mathrm{T}},(\delta F(x,t),\delta S(x,t))\right)_{L^2(\Omega_T)} \\ &\quad + O\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))}\right) + O\left(\|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned}$$

This leads to the gradient formula (5.51). □

The following corollary illustrates the Fréchet gradient of the regularized Tikhonov functional $\mathcal{J}_\alpha(F,S)$.

Corollary 5.1. *Suppose the conditions of Proposition 5.1 hold true. Then for the regularized Tikhonov functional $\mathcal{J}_\alpha(F,S)$ defined by (5.6), the following gradient formula holds:*

$$\nabla\mathcal{J}_\alpha(F,S) = (\psi(x,t;F,S),\varphi(x,t;F,S))^{\mathrm{T}} + \alpha\,(F(x,t),S(x,t))^{\mathrm{T}}. \tag{5.53}$$

Proof. Using the similar arguments of the proof of Proposition 5.1, one can obtain the Fréchet derivative (5.53). □

Remark 5.5. Consider the typical source functions $(F(x,t),S(x,t))$ in (5.1) such that $F(x,t) = F_\ell(x)S_\ell(t), S(x,t) = F_s(x)S_s(t)$, where the space-wise dependent sources $F_\ell(x)$, $F_s(x)$ are unknowns that need to be determined from the final time output $u_T(x)$ defined in (5.2), and the temporal sources $S_\ell(t)$, $S_s(t)$ are known functions.

In this case, for the Fréchet gradient of the Tikhonov functional $\mathcal{J}(F)$, $F(x) := (F_\ell(x),F_s(x))$ defined in (5.5) the following gradient formula holds:

$$\nabla\mathcal{J}[F](x) = \left(\int_0^T \psi(x,t;F)S_\ell(t)dt, \int_0^T \varphi(x,t;F)S_s(t)dt\right)^{\mathrm{T}},\ F \in \mathcal{F}_\ell \times \mathcal{F}_s,$$

where the class of admissible sources:

$$\mathcal{F}_\ell = \{F_\ell \in L^2(\Omega) : \|F_\ell\|_{L^2(\Omega)} \leq M_1'\},\ \mathcal{F}_s = \{F_s \in L^2(\Omega) : \|F_s\|_{L^2(\Omega)} \leq M_2'\}.$$

The following theorem demonstrates the Lipschitz continuity of the Fréchet gradient $\nabla\mathcal{J}(F,S)$. It is highly useful when we apply gradient-based methods to solve the inverse problem. Indeed, in the case of gradient type algorithms such as conjugate gradient algorithm or Landweber iteration algorithm, the relaxation parameter can be estimated using the Lipschitz constant associated with the Lipschitz continuity of $\nabla\mathcal{J}(F,S)$, and that can be used to discuss the convergence of the iterative scheme as well (see, [50]).

Theorem 5.4. *Let the conditions of Proposition 5.1 hold true and the input data $F \in \mathcal{F}$, $S \in \mathcal{G}$ and measured data $u_T \in L^2(\Omega)$. Then the Fréchet gradient $\nabla\mathcal{J}(F,S)$ defined by (5.51) is Lipschitz continuous. Moreover*

$$\begin{aligned}&\|\nabla\mathcal{J}(F+\delta F,S+\delta S)-\nabla\mathcal{J}(F,S)\|_{L^2(\Omega)}\\&\quad\le L_2\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))}+\|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right)^{1/2},\end{aligned}$$

where the Lipschitz constant

$$L_2 = T^2\left(\frac{C_7}{2}\left(1+\frac{\rho h\beta_1}{\beta_2\beta_3}\right)C_e\right)^{1/2},$$

and the constants C_e, and C_7 are defined in Theorem 5.1 and Lemma 5.1, respectively.

Proof. By taking the differences $\delta\psi(x,t;F,S)=\psi(x,t;F+\delta F,S+\delta S)-\psi(x,t;F,S)$, $\delta\varphi(x,t;F,S)=\varphi(x,t;F+\delta F,S+\delta S)-\varphi(x,t;F,S)$, and using the Fréchet gradient (5.51), we get

$$\begin{aligned}&\|\nabla\mathcal{J}(F+\delta F,S+\delta S)-\nabla\mathcal{J}(F,S)\|^2_{L^2(0,T;L^2(\Omega))}\\&\quad=\int_{\Omega_T}\delta\psi(x,t;F,S)^2dxdt+\int_{\Omega_T}\delta\varphi(x,t;F,S)^2dxdt\\&\quad\le\frac{T^2}{2}\left(\|\delta\psi_t\|^2_{L^2(0,T;L^2(\Omega))}+\|\delta\varphi_t\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned}$$

Now using the estimates (5.41) and (5.42), which also hold for $(\delta\psi(x,t),\delta\varphi(x,t))$ with $\delta\psi_t(x,T)=\delta u(x,T)$, we get

$$\begin{aligned}&\|\nabla\mathcal{J}(F+\delta F,S+\delta S)-\nabla\mathcal{J}(F,S)\|^2_{L^2(0,T;L^2(\Omega))}\\&\qquad\le\frac{T^3}{2}\left(1+\frac{\rho h\beta_1}{\beta_2\beta_3}\right)\|\delta u(.,T)\|^2_{L^2(\Omega)}.\end{aligned}\tag{5.54}$$

By employing the estimate (5.35), one can deduce from (5.54) that

$$\begin{aligned}&\|\nabla \mathcal{J}(F+\delta F, S+\delta S)-\nabla \mathcal{J}(F,S)\|^2_{L^2(0,T;L^2(\Omega))}\\&\quad\le \frac{C_7T^4}{2}\left(1+\frac{\rho h\beta_1}{\beta_2\beta_3}\right)C_e\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))}+\|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned}$$

This completes the proof. □

5.5 Stability

In this section, we first develop a first-order necessary optimality condition that must be met by an optimal solution to the minimization problem (5.7). This optimality condition is crucial in deriving the stability estimate for the inverse problem (5.1)-(5.2).

We obtain the necessary optimality condition by using the classical calculus of variation result given by Theorem 1.9. The following proposition shows the variational inequality for the optimal solution to the minimization problem (5.7).

Proposition 5.2. *Suppose the conditions of Theorem 5.4 hold. Then, for the considered inverse source problem with unique minimizer* $(F_{*,\alpha}, S_{*,\alpha}) \in \mathcal{F}\times\mathcal{G}$, *the following variational inequality holds*

$$\begin{aligned}&\int_{\Omega_T}[F(x,t)-F_{*\alpha}(x,t)]\psi(x,t;F_{*,\alpha},S_{*,\alpha})dxdt\\&\quad+\int_{\Omega_T}[S(x,t)-S_{*\alpha}(x,t)]\varphi(x,t;F_{*,\alpha},S_{*,\alpha})dxdt\\&\quad+\alpha\int_{\Omega_T}F_{*\alpha}(x,t)[F(x,t)-F_{*\alpha}(x,t)]dxdt\\&\quad+\alpha\int_{\Omega_T}S_{*\alpha}(x,t)[S(x,t)-S_{*\alpha}(x,t)]dxdt\ge 0,\ \ \forall(F,S)\in\mathcal{F}\times\mathcal{G}. \qquad (5.55)\end{aligned}$$

Proof. It virtue of Theorem 5.2 and Proposition 5.1, it is clear that the Tikhonov functional $\mathcal{J}_\alpha(F,S)$ defined by (5.6) is a Fréchet differentiable functional with gradient (5.51) on a nonempty closed convex subset $\mathcal{F}\times\mathcal{G}\subset L^2(0,T;L^2(\Omega))\times L^2(0,T;L^2(\Omega))$.

By invoking (1.19), the following variational inequality holds:

$$(\nabla\mathcal{J}_\alpha(F_{*,\alpha},S_{*,\alpha}),(F-F_{*,\alpha},S-S_{*,\alpha}))_{L^2(\Omega_T)}\ge 0. \qquad (5.56)$$

Substituting (5.53) in (5.56), we get

$$\begin{aligned}\left((\psi(x,t;F_{*,\alpha},S_{*,\alpha}),\varphi(x,t;F_{*,\alpha},S_{*,\alpha}))^{\mathrm{T}},(F-F_{*,\alpha},S-S_{*,\alpha})\right)_{L^2(\Omega_T)}\\ +\alpha\left((F_{*,\alpha},S_{*,\alpha})^{\mathrm{T}},(F-F_{*,\alpha},S-S_{*,\alpha})\right)_{L^2(\Omega_T)}\geq 0.\end{aligned}\tag{5.57}$$

From the inequality (5.57), we obtain the desired result (5.55). □

Next, we obtain the conditions on the final time to obtain stability estimate for the mechanical load $F \in \mathcal{F}$ and heat source $S \in \mathcal{G}$.

Theorem 5.5. *Let Assumption 5.1 holds true and suppose* $(F_{*,\alpha}, S_{*,\alpha}), (\hat{F}_{*\alpha}, \hat{S}_{*\alpha}) \in \mathcal{F} \times \mathcal{G}$ *are unique minimizers of the regularized Tikhonov functional* $\mathcal{J}_\alpha$ *defined by (5.6) corresponding to the measured outputs* $u_T, \hat{u}_T \in L^2(\Omega)$*, respectively. Suppose the final time* T *satisfies the condition*

$$0 < T \leq \frac{\alpha^{2/5}(\rho h)^{2/5}}{\left[2C_7\left(\frac{1}{\rho h}+\frac{\beta_1}{\beta_2\beta_3}\right)\right]^{1/5}},\tag{5.58}$$

where $C_7 > 0$ *is the constant introduced in Lemma 5.1. Then the following stability estimate holds:*

$$\|\hat{F}_{*,\alpha}-F_{*,\alpha}\|^2_{L^2(0,T;L^2(\Omega))}+\|\hat{S}_{*,\alpha}-S_{*,\alpha}\|^2_{L^2(0,T;L^2(\Omega))}\leq C_{ST}\|\hat{u}_T-u_T\|^2_{L^2(\Omega)},\tag{5.59}$$

where $C_{ST} = (TC_7C_e)^{-1}$ *is the stability constant.*

Proof. By writing the variational inequality (5.55) for $(\hat{F}_{*,\alpha}, \hat{S}_{*,\alpha}) \in \mathcal{F} \times \mathcal{G}$ instead of $(F, S) \in \mathcal{F} \times \mathcal{G}$, we have

$$\begin{aligned}&\int_{\Omega_T}(\hat{F}_{*,\alpha}(x,t)-F_{*,\alpha}(x,t))\psi(x,t;F_{*,\alpha},S_{*,\alpha})dxdt\\ &\quad+\int_{\Omega_T}(\hat{S}_{*,\alpha}(x,t)-S_{*,\alpha}(x,t))\varphi(x,t;F_{*,\alpha},S_{*,\alpha})dxdt\\ &\quad+\alpha\int_{\Omega_T}F_{*,\alpha}(x,t)\left(\hat{F}_{*,\alpha}(x,t)-F_{*,\alpha}(x,t)\right)dxdt\\ &\quad+\alpha\int_{\Omega_T}S_{*,\alpha}(x,t)\left(\hat{S}_{*,\alpha}(x,t)-S_{*,\alpha}(x,t)\right)dxdt\geq 0.\end{aligned}\tag{5.60}$$

Further, by changing $F_{*,\alpha}(x, t)$ and $F(x, t)$ by $\hat{F}_{*,\alpha}(x, t)$ and $F_{*,\alpha}(x, t)$, respectively, and

$S_{*,\alpha}(x,t)$ and $S(x,t)$ by $\hat{S}_{*,\alpha}(x,t)$ and $S_{*,\alpha}(x,t)$, respectively in (5.55), we obtain

$$\begin{aligned}
&\int_{\Omega_T} (F_{*,\alpha}(x,t) - \hat{F}_{*,\alpha}(x,t))\psi(x,t;\hat{F}_{*,\alpha},\hat{S}_{*,\alpha})dxdt \\
&\quad + \int_{\Omega_T} (S_{*,\alpha}(x,t) - \hat{S}_{*,\alpha}(x,t))\varphi(x,t;\hat{F}_{*,\alpha},\hat{S}_{*,\alpha})dxdt \\
&\quad +\alpha \int_{\Omega_T} \hat{F}_{*,\alpha}(x,t)\left(F_{*,\alpha}(x,t) - \hat{F}_{*,\alpha}(x,t)\right)dxdt \\
&\quad +\alpha \int_{\Omega_T} \hat{S}_{*,\alpha}(x,t)\left(S_{*,\alpha}(x,t) - \hat{S}_{*,\alpha}(x,t)\right)dxdt \geq 0. \qquad (5.61)
\end{aligned}$$

From the inequalities (5.60) and (5.61), we deduce that

$$\begin{aligned}
&\alpha \int_{\Omega_T} \left(\hat{F}_{*,\alpha}(x,t) - F_{*,\alpha}(x,t)\right)^2 dxdt + \alpha \int_{\Omega_T} \left(\hat{S}_{*,\alpha}(x,t) - S_{*,\alpha}(x,t)\right)^2 dxdt \\
&\qquad \leq \int_{\Omega_T} \left(\hat{F}_{*,\alpha}(x,t) - F_{*,\alpha}(x,t)\right)\delta\psi(x,t)dxdt \\
&\qquad\quad + \int_{\Omega_T} \left(\hat{S}_{*,\alpha}(x,t) - S_{*,\alpha}(x,t)\right)\delta\varphi(x,t)dxdt, \qquad (5.62)
\end{aligned}$$

where $\delta\psi(x,t) = \psi(x,t;F_{*,\alpha},S_{*,\alpha}) - \psi(x,t;\hat{F}_{*,\alpha},\hat{S}_{*,\alpha})$, $\delta\varphi(x,t)) = \varphi(x,t;F_{*,\alpha},S_{*,\alpha}) - \varphi(x,t;\hat{F}_{*,\alpha},\hat{S}_{*,\alpha})$, and $(\delta\psi(x,t),\delta\varphi(x,t))$ is the solution of the adjoint problem (5.39) with $\delta\psi_t(x,T) = \frac{-1}{\rho h}\left(\delta u(x,T) - (u_T - \hat{u}_T)\right)$. By setting $\delta F(x,t) = \hat{F}_{*,\alpha}(x,t) - F_{*,\alpha}(x,t)$ and $\delta S(x,t) = \hat{S}_{*,\alpha}(x,t) - S_{*,\alpha}(x,t)$, applying Cauchy's ϵ-inequality with $\epsilon = \alpha$ to the right-hand side integral of (5.62), we get

$$\begin{aligned}
&\int_{\Omega_T} \delta F(x,t)\delta\psi(x,t)dxdt + \int_{\Omega_T} \delta S(x,t)\delta\varphi(x,t)dxdt \\
&\quad \leq \frac{\alpha}{2}\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right) \\
&\qquad + \frac{1}{2\alpha}\left(\|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta\varphi\|^2_{L^2(0,T;L^2(\Omega))}\right).
\end{aligned}$$

The inequality (5.62) further reduces to the following:

$$\begin{aligned}
&\alpha^2\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right) \\
&\quad \leq \|\delta\psi\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta\varphi\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad \leq \frac{T^2}{2}\left(\|\delta\psi_t\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta\varphi_t\|^2_{L^2(0,T;L^2(\Omega))}\right). \qquad (5.63)
\end{aligned}$$

Now we need to find estimates for $\|\delta\psi_t\|^2_{L^2(0,T;L^2(\Omega))}$ and $\|\delta\varphi_t\|^2_{L^2(0,T;L^2(\Omega))}$. Using the estimate (5.41), which also holds for $\delta\psi_t(x,t)$, and (5.35), we get

$$\begin{aligned}
&\|\delta\psi_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq \frac{2T}{(\rho h)^2}\left[\|\delta u(.,T)\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta u_T\|^2_{L^2(\Omega)}\right] \\
&\leq \frac{2T^2 C_e C_7}{(\rho h)^2}\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right) + \frac{2T}{(\rho h)^2}\|\delta u_T\|^2_{L^2(\Omega)},
\end{aligned} \tag{5.64}$$

where recall that $C_7 > 0$ is the constant introduced in Lemma 5.1. Similarly making use of (5.42) and (5.35), one can get

$$\begin{aligned}
&\|\delta\varphi_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq \frac{2T^2\beta_1 C_7}{\beta_2\beta_3\rho h} C_e\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right) + \frac{2T\beta_1}{\beta_2\beta_3\rho h}\|\delta u_T\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.65}$$

Substituting (5.65) and (5.64) in (5.63), we get

$$\begin{aligned}
&\alpha^2\left(\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right) \\
&\quad \leq \frac{T^4 C_7 C_e}{\rho h}\left(\frac{1}{\rho h} + \frac{\beta_1}{\beta_2\beta_3}\right)\left[\|\delta F\|^2_{L^2(0,T;L^2(\Omega))} + \|\delta S\|^2_{L^2(0,T;L^2(\Omega))}\right] \\
&\quad + \frac{T^3}{\rho h}\left(\frac{1}{\rho h} + \frac{\beta_1}{\beta_2\beta_3}\right)\|\delta u_T\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.66}$$

Let the value of the final time $T > 0$ be chosen so that the following constraint holds:

$$\frac{T^4 C_7 C_e}{\rho h}\left(\frac{1}{\rho h} + \frac{\beta_1}{\beta_2\beta_3}\right) = \frac{\alpha^2}{2}. \tag{5.67}$$

In view of the equality (5.67) and the fact that $C_e = \exp(T/(\rho h)) - 1 \geq T/(\rho h)$, we obtain

$$T \leq \frac{\alpha^{2/5}(\rho h)^{2/5}}{\left[2C_7\left(\frac{1}{\rho h} + \frac{\beta_1}{\beta_2\beta_3}\right)\right]^{1/5}} =: T^*.$$

From (5.67) and (5.66), we arrive at the desired stability estimate (5.59) with the stability constant C_{ST}. □

The following example illustrates that how the regularization parameter α and final time T directly have impact on the stability constant C_{ST}.

Table 5.1: Stability constant C_{ST} corresponding to T and α.

α	$T^* = \frac{\alpha^{2/5}(\rho h)^{2/5}}{\left[2C_7\left(\frac{1}{\rho h}+\frac{\beta_1}{\beta_2\beta_3}\right)\right]^{1/5}}$	$C_{\text{ST}} = (TC_7C_e)^{-1}$
$10^{-5/2}$	0.14	6.9
10^{-3}	0.09	10.87
10^{-4}	$.04$	24.75

Example 5.1. In order to make explanations and comments more understandable, it is assumed that $\rho = 2, h = 2$, $\beta_1 = .6, D = 1, r = .08, \beta_2 = .5, \beta_3 = 1, \beta_4 = 1$. Analyzing the Table 1 shows that the final time $T > 0$ is directly proportional to the parameter of regularization $\alpha > 0$. The decrease in the values of α leads to a decrease in the upper limit T^* of final time. The decrease in the value of T increases the value of the stability constant C_{ST}. Further, it is worth noting that the coefficients of the system (5.1) don't have any direct relation with the final time T and regularization parameter α. Therefore, the above conclusion of the stability estimate, in terms α and T holds, irrespective of the different values of the coefficients in system (5.1).

The Lipschitz type stability result given by Theorem 5.5 also gives the following uniqueness of solution to the regularaized inverse problem.

Corollary 5.2. *Suppose the conditions of Theorem 5.5 are satisfied. If the measured data are equal, that is* $u_T(x) = \hat{u}_T(x)$, *for all* $x \in \Omega$, *and the final time* T *satisfies the condition* (5.58)*, then the minimizers* $(F_{*,\alpha}, S_{*,\alpha}), (\hat{F}_{*,\alpha}, \hat{S}_{*,\alpha}) \in \mathcal{F} \times \mathcal{G}$ *corresponding to measured data* $u_T, \hat{u}_T$, *respectively, are equal:*

$$(F_{*,\alpha}, S_{*,\alpha}) = (\hat{F}_{*,\alpha}, \hat{S}_{*,\alpha}), \ a.e. \ (x,t) \in \Omega_T.$$

Proof. If $u_T = \hat{u}_T$, for all $x \in \Omega$, then by the stability estimate (5.59), it is clear that

$$\|\hat{F}_{*,\alpha} - F_{*,\alpha}\|^2_{L^2(0,T;L^2(\Omega))} + \|\hat{S}_{*,\alpha} - S_{*,\alpha}\|^2_{L^2(0,T;L^2(\Omega))} \leq 0.$$

Hence, $\|\hat{F}_{*,\alpha} - F_{*,\alpha}\|_{L^2(0,T;L^2(\Omega))} = 0$ and $\|\hat{S}_{*,\alpha} - S_{*,\alpha}\|_{L^2(0,T;L^2(\Omega))} = 0$. This completes the proof. □

Remark 5.6. The arguments and methodology presented in this chapter, in particular Theorem 5.5, can be extended to the case when the measured data $u_T(x)$ is known over only in a suitable open subset $\Omega_1 \subset \Omega$. We can address this extension of the result by modifying the final time data in the adjoint problem (5.39) as follows

$$\psi_t(x,T) = q(x) = \begin{cases} \tilde{q}(x), & x \in \Omega_1, \\ 0, & x \in \Omega_1 \backslash \Omega, \end{cases}$$

where $\tilde{q}(x) = -\frac{1}{\rho h}\left[u(x,T;F,G) - u_T(x)\right] \in L^2(\Omega_1)$.

Chapter 6

Conclusion and future work

This thesis has delved into the analysis and applicability of inverse source problems in the damped Euler-Bernoulli beam, the damped Kirchhoff-Love plate, and the damped thermoelastic plate equations. Our research has provided valuable insights and contributions to understanding these inverse source problems, shedding light on their solvability and the impact of various damping factors, such as viscous damping, structural damping, and Kelvin-Voigt damping.

In Chapter 1, we have examined the feasibility of inverse source problems in the Euler-Bernoulli beam with viscous damping, focusing on determining the unknown spatial load from the final time output. This research contrasts the scope of papers [43]-[48], which primarily discussed identifying unknown source terms in undamped Euler-Bernoulli beams through regularized solutions. Unlike these papers, our work demonstrates the uniqueness of non-regularized solutions and establishes a series representation for such solutions for constant and exponentially decaying temporal loads by applying SVD. We have also identified admissible and optimal final time intervals essential for precise final time output measurement. We have partially answered the non-uniqueness question of inverse source problems for the beam and wave equations (from the final time displacement measurement) posed in [50] for certain specific temporal loads under conditions on the final time T and damping coefficient $\mu > 0$. For any given arbitrary temporal load and final T, a broader class of sufficient conditions for the unique reconstruction of the spatial load in the beam equation still needs to be obtained.

Chapter 2 expands the analysis to take a closer look at the role of different damping terms in beam models and generalized the results given in [47] and [46] for the identification of shear force (boundary data) in Euler-Bernoulli beam by incorporating all physical coefficients and including Kelvin-Voigt damping. We have shown that in contrast to the above works, which exclusively considered external damping, the weak and regular weak solutions of the Euler-Bernoulli beam equation with Kelvin-Voigt damping term $(\kappa(x)u_{xxt})_{xx}$ has more enhanced regularity property than corresponding weak solutions of this equation without this term. This property also helped in solving the adjoint problem with less regular data in the quasi-solution approach and a less regular class of admissible shear force in the context of the inverse boundary value problem compared to [47]. As we look ahead, our future research will explore intriguing avenues, notably in non-linear coefficient identification within the physically relevant model (3.1). We aim to tackle the captivating inverse problem of identifying one or multiple parameters, such as $\rho(x), r(x), \mu(x), \kappa(x)$ from a suitable measured data.

In Chapter 3, we extended the analysis of the inverse source problem of one-dimensional beams to a two-dimensional problem, focusing on the Kirchhoff-Love plate equation with viscous damping. This pioneering work represents the first exploration of the inverse problem within the context of the rectangular Kirchhoff-Love plate equation. A key highlight of our study lies in the unique comparison between two essential methods: Tikhonov regularization and Singular Value Decomposition, a comparative analysis hitherto unexplored in prior research of the related model [39], [52], [95]. Our research also yields valuable insights into these notable methods through the solvability of inverse source problems and stability analysis of this problem. The paper [95] has obtained sufficient conditions for recovering spatial loads from final time measurements but lacked a series representation of the unknown spatial load by the SVD in terms of the given measurement $u_T(x)$ under feasible conditions concerning final time and damping terms. As our research marks a primary effort in this specific model, it opens up numerous avenues for future exploration, including boundary data identification problems and coefficient identification problems.

Finally, Chapter 5 of the thesis addresses the inverse source problem of identifying the spatial and temporal loads in the thermoelastic plate. To our knowledge, this is the first work to simultaneously identify both types of loads in the structurally damped thermoelastic plate from a single set of final time displacement data. This work generalizes the previous studies [13], [96], [99], [100] that explored spatially varying load/coefficient identification problems in the classical thermoelastic system consists of a hyperbolic equation for displacement and the heat equation for temperature. By appropriately scaling the coefficients of the model, we solved the strongly coupled system and the corresponding adjoint system by applying the Faedo-Galerkin method and the regularizing effect of the structural damping. The quasi-solution approach ensures the existence of solutions to the inverse source problem, and utilizing the adjoint problem approach, derived the Fréchet gradient of the Tykhonov functional. One of the added advantages of the Lipschitz continuity of the Fréchet gradient is that the Lipschitz constant can be effectively used in the gradient-based numerical reconstruction procedure ([50]). It is evident from the Lipschitz-type stability estimate, which is derived through the method of Tikhonov regularization and a variational approach, that the reconstruction procedure is stable, provided the final time T is feasibly small. This estimate also ensures the uniqueness of solutions to the regularized inverse source problem of the thermoelastic system. The inverse coefficient problems of determining the flexural rigidity $D(x)$, density $\rho(x)$ either simultaneously or separately would certainly be an interesting research problem to address in the future. Another mathematically challenging and more physically relevant future research problem could be the inverse boundary value problem of the thermoelastic plate. More precisely, the following mathematical model can be considered (see, [65], chapter 6)

$$\begin{cases} \rho h u_{tt} - r\Delta u_{tt} - \mathrm{w}\Delta u_t + D\Delta^2 u + \beta_1 \Delta\theta = 0, \quad (x,t) \in \Omega_T, & (6.1a) \\ \beta_2 \theta_t - \Delta\theta - \beta_3 \Delta u_t + \beta_4 \theta = 0, \qquad (x,t) \in \Omega_T, & (6.1b) \\ u(x,0) = 0, u_t(x,0) = 0, \ \theta(x,0) = 0 \qquad x \in \Omega, \end{cases}$$

with the boundary conditions

$$\begin{cases} u = 0 \text{ on } \Gamma \times [0,T] \\ \frac{\partial u}{\partial \mathrm{n}} = \begin{cases} g(x,t) \text{ on } \Gamma_0 \times [0,T] \\ 0 \qquad \text{on } \Gamma_1 \times [0,T], \ \Gamma_1 := \Gamma \backslash \Gamma_0 \end{cases} \\ \theta = 0 \text{ on } \Gamma \times [0,T], \end{cases} \tag{6.2}$$

where Γ_0 is a portion of the boundary Γ of the domain Ω.
A possible future work could be the determination of the unknown boundary source g from the measured data such as displacement and temperature given at final time T for all $x \in \Omega$.

Overall, our research has significantly contributed to the analysis of the damping term in the inverse source problems of the damped Euler-Bernoulli beam and Kirchhoff-Love plate equations. Establishing the well-posedness of the direct problem, the successful formulation and analysis of the inverse problem using regularization techniques and singular value decomposition, and the derivation of stability estimates have advanced our knowledge in this area of study. These findings provide a solid foundation for further investigations in related fields. While this thesis has been entirely theoretical in nature, it opens the door for future advancements, such as extending these studies into the realm of numerical analysis, allowing for a more comprehensive exploration of these complex problems.

List of Publications

1. D. Anjuna, A. Hasanov, K. Sakthivel and C. Sebu, On unique determination of an unknown spatial load in damped Euler-Bernoulli beam equation from final time output, *Journal of Inverse and Ill-posed Problems,* **30** (2022), 581-593, http://doi.org/10.1515/jiip-2021-0009.

2. D. Anjuna, K. Sakthivel and A. Hasanov, Determination of a spatial load in a damped Kirchhoff-Love plate equation from final time measured data, *Inverse Problems,* **38**(2022), 15009 (35pp), https://doi.org/10.1088/1361-6420/ac346c.

3. K. Sakthivel, A. Hasanov and D. Anjuna, Inverse problems of identifying the unknown transverse shear force in the Euler-Bernoulli beam with Kelvin-Voigt damping, *Journal of Inverse and Ill-posed Problems,* **32**(2024), 75-106, https://doi.org/10.1515/jiip-2022-0053.

4. D. Anjuna, A. Hasanov and K. Sakthivel, Simultaneous identification of spatial load and external heat source in a thermoelastic plate from final time measured displacement, *Inverse Problems and Imaging,* 10.3934/ipi.2023053.

Presentations in Conferences

1. On unique determination of an unknown spatial load in damped Euler-Bernoulli beam equation, *Symposium on Differential Equations: Analysis, Computation and Applications,* IIT Roorkee, 2021.

2. On unique determination of an unknown spatial load in damped Euler-Bernoulli beam equation, *Third National Conference on Control and Inverse Problems,* Jointly organized by Central University of Kerala, Central University of Tamilnadu and Periyar University, 2022.

3. Determination of spatial load in damped Kirchhoff-Love plate equation, *International Conference on Analysis, Inverse Problems and Applications,* IIT Madras, 2022.

4. Inverse problems of Euler-Bernoulli beam equation with Kelvin-Voigt damping, *International Conference on Differential Equations and Control Problems,* IIT Mandi, 2023.

5. Determination of a spatial load in a damped Kirchhoff-Love plate equation, *Quasi linear Equations, Inverse Problems and Their Applications,* Moscow Institute of Physics and Technology, Russia, December 4-6, 2023.

Bibliography

[1] K. Ammari, F. Hassine and L. Robbiano, Stabilization for the wave equation with singular Kelvin-Voigt damping, *Archive for Rational Mechanics and Analysis*, **236**(2020), 577-601.

[2] C. Alves, A. L. Silvestre, T. Takahashi and M. Tucsnak, Solving inverse source problems using observability. applications to the Euler-Bernoulli plate equation, *SIAM Journal on Control and Optimization* , **48** (2009), 1632-1659.

[3] M. Antognozzi, D. R. Binger, A.D.L Humphris, P.J. James and M.J Miles, Modeling of cylindrically tapered cantilevers for transverse dynamic force microscopy (TDFM), *Ultramicroscopy,* **86**(2001), 223–232.

[4] S.R Arridge, Optical tomography in medical imaging, *Inverse Problems,* **15**(1999), R41.

[5] H.T Banks, Inverse problems for distributed parameter systems, *29th IEEE Conference on Decision and Control*, 1990, 13-16.

[6] H. T. Banks and D. J. Inman, On damping mechanisms in beams, *Journal of Applied Mechanics,* **58**(1991), 716–723.

[7] V. Barcilon, Inverse problem for the vibrating beam in the free-clamped configuration, *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences,* **304**(1982), 211-251.

[8] V. Barcilon, Inverse eigenvalue problems, *Lecture Notes in Mathematics, Springer,* **1225**(1986), 1–51.

[9] O. Baysal and A. Hasanov, Solvability of the clamped Euler–Bernoulli beam equation, *Applied Mathematics Letters,* **93**(2019) ,85–90.

[10] L. Beilina and S. Hosseinzadegan, An adaptive finite element method in reconstruction of coefficients in Maxwell's equations from limited observations, *Applications of Mathematics,* **61**(2016), 253-286.

[11] L. Beilina and M.V.Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems,* Springer, 2012.

[12] L. Beilina, Numerical analysis of least squares and perceptron learning for classification problems, *Open Journal of Discrete Applied Mathematics*, **3**(2020), 30-49.

[13] M.Bellassoued, and M. Yamamoto, Carleman estimates and an inverse heat source problem for the thermoelasticity system, *Inverse Problems*, **27**(2010), 015006.

[14] S. K. Bhullar and J. L. Wegner, Some transient thermoelastic plate problems, *Journal of Thermal Stresses,* **32** (2009), 768-790.

[15] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations,* Springer, 2010.

[16] J. D. Chang and B. Z. Guo, Identification of variable spacial coefficients for a beam equation from boundary measurements, *Automatica,* **43**(2007), 732–737.

[17] W.J, Chang, T.H. Fang and C.I. Weng, Inverse determination of the cutting force on nanoscale processing using atomic force microscopy, *Nanotechnology,* **15**(2004), 427–430.

[18] W. Chen, Cauchy problem for thermoelastic plate equations with different damping mechanisms, *Communications in Mathematical Sciences,* **18**(2020), 429–457.

[19] J. Cheng and H. Xu, Inner mass impact damper for attenuating structure vibration, *International Journal of Solids and Structures,* **43**(2006), 5355–5369.

[20] R. W. Clough and J. Penzien, *Dynamics of Structure,* 2nd edn. McGraw-Hill, 1993.

[21] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, 1955.

[22] A. Coronel, L. Friz, I. Hess and M. Zegarra, On the existence and uniqueness of an inverse problem in epidemiology, *Applicable Analysis*, **100**(2021), 513-516.

[23] I. Craige and J. Brown, *Inverse Problems in Astronomy,* Adam Hilger Ltd., Accord, MA, 1986.

[24] A. Demir and A. Hasanov, Identification of the unknown diffusion coefficient in a linear parabolic equation by the semigroup approach, *Journal of Mathematical Analysis and Applications,* **340**(2008), 5–15.

[25] R. Denk and R. Schnaubelt, A structurally damped plate equation with Dirichlet-Neumann boundary conditions, *Journal of Differential Equations,* **259** (2015), 1323-1353.

[26] A. M. Djfari, *Inverse Problems in Vision and 3D Tomography,* John Wiley and Sons, 2010.

[27] L. C. Evans, *Partial Differential Equations,* American Mathematical Soc, 2010.

[28] E. De Vito, L. Rosasco, A. Caponnetto, U. De Giovannini and F. Odone, Learning from examples as an inverse problem, *Journal of Machine Learning Research,* **6**(2005).

[29] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems,* Springer Science & Business Media, 1996.

[30] A. Friedmann and F. Reitich, Parameter identification in reaction-diffusion model, *Inverse Problems*, **8**(1992), 187-192.

[31] L. Frýba, *Dynamics of Railway Bridges,* Thomas Telford, 1996.

[32] L. Frýba, *Vibrations of the Solids and Structures Under Moving Loads,* Thomas Telford, London, 1999.

[33] G. M. L. Gladwell, The inverse problem for the Euler-Bernoulli beam, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences,* **407**(1986), 199-218.

[34] G.M.L. Gladwell, *Inverse Problems in Vibration,* Kluwer Academic Publishers, 2nd ed. , Dordrecht, 2004.

[35] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer, 1998.

[36] E. Graif and K. Kunisch, Parameter estimation for the Euler-Bernoulli beam, *Matemática Aplicadae Computacional,* **4**(1985), 95-124.

[37] M. Grimmonprez, L. Marin, and K. Van Bockstal, The reconstruction of a solely time-dependent load in a simply supported non-homogeneous Euler–Bernoulli beam, *Applied Mathematical Modelling,* **79**(2020), 914–933.

[38] M. Grimmonprez and M. Slodičcka, Reconstruction of an unknown source parameter in a semilinear parabolic problem, *Journal of Computational and Applied Mathematics,* **289** (2015), 331–345.

[39] Z.J. Gu and Y.J. Tan, Fundamental solution method for inverse source problem of plate equation, *Applied Mathematics and Mechanics,* **33** (2012), 1513–1532.

[40] L. Hadamard, *La théorie des équations aux dérivées partielles,* Éditions scientifiques, 1964.

[41] D. Hao, L. Giang, S. Kabanikhin and M. Shishlenin, A finite difference method for the very weak solution to a Cauchy problem for an elliptic equation, *Journal of Inverse and Ill-Posed Problems,* **26** (2018), 835-857.

[42] A. Hasanov, Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach, *Journal of Mathematical Analysis and Applications,* **330**(2007), 766-779.

[43] A. Hasanov, Identification of an unknown source term in a vibrating cantilevered beam from final overdetermination, *Inverse Problems,* **25**(2009), 115015.

[44] A. Hasanov and O. Baysal, Identification of an unknown spatial load distribution in a vibrating cantilevered beam from final overdetermination, *Journal of Inverse and Ill-Posed Problems,* **23** (2015), 85–102.

[45] A. Hasanov and O. Baysal, Identification of unknown temporal and spatial load distributions in a vibrating Euler-Bernoulli beam from Dirichlet boundary measured data, *Automatica,* **71**(2016), 106–117.

[46] A. Hasanov, O. Baysal and H. Itou, Identification of an unknown shear force in a cantilever Euler–Bernoulli beam from measured boundary bending moment, *Journal of Inverse and Ill-Posed Problems*, **27**(2019), 859-876.

[47] A. Hasanov, O. Baysal and C. Sebu, Identification of an unknown shear force in the Euler-Bernoulli cantilever beam from measured boundary deflection, *Inverse Problems,* **35**(2019), 115008.

[48] A. Hasanov and A. Kawano, Identification of unknown spatial load distributions in a vibrating Euler-Bernoulli beam from limited measured data, *Inverse Problems,* **32** (2016), 055004.

[49] A. Hasanov and B.Mukanova, Relationship between representation formulas for unique regularized solutions of inverse source problem with final over determination and singular value decomposition of input-output operator, *IMA Journal of Applied Mathematics ,* **80** (2015), 676-696.

[50] A. Hasanov and V. Romanov, *Introduction to Inverse Problems for Differential Equations,* 2nd ed, Springer, 2021.

[51] A. Hasanov, V. Romanov, and O.Baysal, Unique recovery of unknown spatial load in damped Euler–Bernoulli beam equation from final time measured output, *Inverse Problems,* **37**(2021), 075005.

[52] M. Ikehata, An inverse problem for the plate in the Love-Kirchhoff theory, *IMA Journal of Applied Mathematics,* **53** (1993), 942–970.

[53] D. J. Inman, *Vibration: with Control, Measurement, and Stability,* Prentice Hall, 1989.

[54] V. Isakov, Inverse parabolic problems with the final overdetermination, *Communications on Pure and Applied Mathematics*, **44**(1991), 185-209.

[55] K. Ito and S. Nakagiri, Identifiability of stiffness and damping coefficients in Euler-Bernoulli beam equations with Kelvin-Voigt damping, *Numerical Functional Analysis and Optimization,* **18**(1997), 107-129.

[56] V.K. Ivanov, On ill-posed problems, *Mathematics of the USSR-Sbornik,* **61** (1963), 211–223.

[57] S. I. Kabanikhin, Definitions and examples of inverse and ill-posed problems, *Journal of Inverse and Ill-Posed Problems,* 16(2008), 317-357.

[58] S. I. Kabanikin, *Inverse and Ill-Posed Problems,* de Gruyter, 2011.

[59] M. Kac, Can one hear the shape of a drum? *The American Mathematical Monthly,* 73 (1966), 1–23.

[60] A. Kawano, Uniqueness in the identification of asynchronous sources and damage in vibrating beams, *Inverse Problems,* **30**(2014), 065008.

[61] A. Kirch, *An Introduction to the Mathematical Theory of Inverse Problems*, Springer, 2011.

[62] M. V. Klibanov, *Carleman Estimates and Inverse Problems in the Last Two Decades,* Springer, 2000.

[63] M. V. Klibanov, A. B. Bakushinsky and L. Beilina, Why a minimizer of the Tikhonov functional is closer to the exact solution than the first guess, *Journal of Inverse and Ill-Posed Problems,* **19**(2011), 83-105.

[64] B. Kundu and R. Ganguli, Analysis of weak solution of Euler–Bernoulli beam with axial force, *Applied Mathematics and Computation,* **298**(2017), 247–260.

[65] J.E. Lagnese and J.L Lions, *Modelling Analysis and Control of Thin Plates,* Masson, Paris, 1988.

[66] J.E. Lagnese, *Boundary Stabilization of Thin Plates,* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[67] D. Lesnic, L. Elliott and DB. Ingham, Analysis of coefficient identification problems associated to the inverse Euler-Bernoulli beam theory, *IMA Journal of Applied Mathematics,* **62**(1999), 101–116.

[68] R.Y Liang, J. Hu and F. Choy, Quantitative NDE technique for assessing damages in beam structures, *Journal of Engineering Mechanics,* **118**(1992), 1468-1487.

[69] A. E. Love, On the small force vibrations and deformations of elastic shells, *Philosophical Transactions of the Royal Society A,* **17**(1888), 491-549.

[70] C. Mahieddine and L. Rachid, Comparative study of direct and inverse problems of cracked beams, *EDP Sciences*, **149**(2018),02015.

[71] T. T. Marinov and R. S. Marinova, An inverse problem for estimation of bending stiffness in Kirchhoff-Love plates, *Computers and Mathematics with Applications,* **65** (2013), 512 – 519.

[72] B.B. Marínez, R. Denk, J. G. Ospino, J. H. Monzón and S. Rau, Long-time asymptotics for a coupled thermoelastic plate-membrane system, *Mathematical Methods in the Applied Sciences,* **44**(2021), 12881–12908.

[73] L. Meirovitch, *Fundamentals of Vibrations*, Waveland Press, Illinois, 2010.

[74] S. Nicaise and O. Zair, Determination of point sources in vibrating beams by boundary measurements: Identifiability, stability, and reconstruction results, *Electronic Journal of Differential Equations,* **20** (2004), 1-17.

[75] T. Nguyen, T. Hatano, S.G. Khan, K. Zhang, C. Edwards, R. Harniman, S.C. Burgess, M. Antognozzi, M. Miles and G. Herrmann, Estimation of the shear force in transverse dynamic force microscopy using a sliding mode observer, *AIP Advances*, **5**(2015), 097157.

[76] A.N. Norris, Dynamics of thermoelastic thin plates: a comparison of four theories, *Journal of Thermal Stresses,* **29**(2006), 169-195.

[77] A.G. Ramm, *Inverse Problems: Mathematical and Analytical Techniques with Applications to Engineering*, Springer 2005.

[78] S. S. Rao and F. F Yap, *Mechanical Vibrations,* Addison-Wesley, New York, 1995

[79] S.S. Rao, *Vibration of Continuous Systems*, John Wiley & Sons, 2007.

[80] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations,* New York, 2004.

[81] C.E. Repetto, A. Roatta and R.J. Welti, Forced vibrations of a cantilever beam, *European Journal of Physics*, **33** (2012) 1187–1195.

[82] T. Samir, K. Samir, L.T Cuong, R. V. Ravipudi, and M. Seyedali, Inverse problem for dynamic structural health monitoring based on slime mould algorithm, *Engineering with Computers,* 2021, 1-24.

[83] C.W De silva, *Vibration: Fundamentals and Practice,* CRC press, 2016.

[84] K. Sakthivel, S. Gnanavel, N. Barani Balan and K. Balachandran, Inverse problem for the reaction diffusion system by optimization method, *Applied Mathematical Modelling*, **38**(2011), 571-579.

[85] K. Sakthivel and A. Hasanov, An inverse problem for the KdV equation with Neumann boundary measured data, *Journal of Inverse and Ill-possed Problems,* **26**(2018), 133-151.

[86] K. Sakthivel, A. Hasanov and D. Anjuna, Inverse problems of identifying the unknown transverse shear force in the Euler-Bernoulli beam with Kelvin-Voigt damping, *Journal of Inverse and Ill-posed Problems,* **32**(2024), 75-106, https://doi.org/10.1515/jiip-2022-0053.

[87] S. Salsa, *Partial Differential Equations in Action: From Modelling to Theory*, Springer, 2016.

[88] L. D Teresa and E. Zuazua, Controllability of the linear system of thermoelastic plates, *Advances in Differential Equations*, **1**(1996), 369-402.

[89] A. N. Tikhonov, *On the Solution of Ill-posed Problems and the Method of Regularization,* Doklady akademii nauk, **151**(1963).

[90] A. Tikhonov and V. Arsenin, *Solutions of Ill-Posed Problems,* Geology Press, Beijing, 1979.

[91] P. S. Timoshenko and J. M. Gere, *Theory of Elastic Stability,* McGraw-Hill, New York, 1961.

[92] P. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, New York: McGraw-Hill Book Company, 1959.

[93] F. Tröltzsch, *Optimal Control of Partial Differential Equations Theory, Methods and Applications,* American Mathematical Soc., 2010.

[94] G. Ulhmann, Inverse problems: seeing the unseen, *Bulletin of Mathematical Sciences,* **4**(2014), 209-279.

[95] K. Van Bockstal, Identification of an unknown spatial load distribution in a vibrating beam or plate from the final state, *Journal of Inverse and Ill-posed Problems,* **27**(2019), 623-642.

[96] K. Van Bockstal and M.Slodička, Recovery of a space-dependent vector source in thermoelastic systems, *Inverse Problems in Science and Engineering,* **23**(2015), 956-968.

[97] F. P. Vasil'ev, *Methods for Solving Extremal Problems,* Moscow: Nauk Russian, 1981.

[98] K. G. Vinod, S. Gopalakrishnan and R. Ganguli, Free vibration and wave propagation analysis of uniform and tapered rotating beams using spectrally formulated finite elements, *International Journal of Solids and Structures*, **44** (2007), 5875-5893.

[99] B. Wu, and J. Liu, Determination of an unknown source for a thermoelastic system with a memory effect, *Inverse Problems,* **28**(2012),095012.

[100] B. Wu, and J. Liu, Conditional stability and uniqueness for determining two coefficients in a hyperbolic–parabolic system, *Inverse Problems,* **27**(2011), 075013.

[101] E. Zeidler, *Applied Functional Analysis: Main Principles and their Applications,* Springer, 1995.