Efficient Uniformly Accurate Approach to Numerical Solution and Scaled Derivatives for System of Singularly Perturbed Differential Equations

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by

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Abstract

This thesis focuses on robust computational approaches to a wide class of coupled system of singularly perturbed differential equations (SPDEs) in one and two dimensions. System of SPDEs involve the highest order spatial derivatives being multiplied by small diffusion parameters, called "singular perturbation parameters", such as ε_1 and ε_2 which can be of equal or different orders of magnitude. The solutions to these problems usually exhibit boundary layers, which can overlap, particularly in the presence of different perturbation parameters. These layers are basically narrow regions in the vicinity of the boundary of the domain, where the gradient of the solution changes rapidly as the parameters get smaller. As a result, the analytical solutions of SPDEs naturally adapt to the multi-scale nature.

System of singularly perturbed partial differential equations (PDEs) is considered to be an important class of problems which often arise in various branches of engineering, biological and chemical sciences etc. For instance, the enzyme-substrate convection-diffusion model appeared in some biochemical systems with small diffusion coefficients can be taken into consideration. Also, there are more realistic models in connection with many biological and ecological applications displaying time-lag or after-effect that are described by system of singularly perturbed delay partial differential equations (DPDEs). For instance, one can consider the diffusive Lotka-Volterra predator-prey system which include both time delay and spatial diffusion reflecting the dynamic behavior of the system based on the past history. Computational analysis of these types of PDEs and DPDEs has naturally become significant from the application point of view as well as for the multi-scale nature of their analytical solutions. On the other hand, the scaled first-order solution derivative represents a physically meaningful quantity, *e.g.*, *diffusive flux* of a substance satisfying the Fick's first law and is proportional to the concentration gradient. It generally appears in the context of diffusion phenomenon in various branches of scientific and engineering disciplines, which can be modeled as SPDEs. As the diffusion parameter becomes smaller, the corresponding solution gradient becomes significantly large, particularly within the boundary layer region. This motivates researchers for pursuing computational analysis so as to capture the scaled solution derivative efficiently by determining appropriate scaling factor depending on the boundary layer region.

The main goal of this thesis is to develop, analyze, and compute parameter-uniform numerical approximation to the solution and scaled derivatives of weakly coupled system of singularly perturbed convection-diffusion boundary-value-problems (BVPs) and time-delay initial-boundary-value-problems (IBVPs) with equal and multiple diffusion parameters in one and two dimensions. To achieve this goal, we construct a special non-uniform mesh called generalized S-mesh that is suitably adapted to the overlapping boundary layers. First of all, the piecewise-equidistant structure of the generalized S-mesh makes the implementation easier while extending for higher-dimensional problems. Moreover, for improving the accuracy of a numerical method on the generalized S-mesh, it is not necessary to increase the number of mesh-intervals; and thus, the mesh also enables to design cost-effective numerical algorithms.

In this thesis, we first develop and analyze a uniformly convergent numerical algorithm based on the generalized S-mesh for the solution and scaled derivative of a coupled system of singularly perturbed one-dimensional convection-diffusion BVPs with multiple diffusion parameters of different orders of magnitude.

Next, parameter-uniform convergence analysis of the numerical solution followed by the error analysis of the scaled discrete space derivative and the discrete time derivative is presented on the standard Shishkin mesh for a coupled system of singularly perturbed one-dimensional convection-diffusion IBVPs with different diffusion parameters. Afterwards, we extend the algorithm based on the generalized S-mesh for a coupled system of singularly perturbed one-dimensional convection-diffusion time-delay IBVPs with different diffusion parameters. Our further investigation involves parameter-uniform approximation to the solution and scaled derivatives (with respect to x and y) for a two-dimensional system of singularly perturbed convection-diffusion elliptic BVPs with equal diffusion parameters, utilizing a tensor product of two 1D generalized S-meshes. In this regard, we consider two types of elliptic PDEs: one exhibiting exponential layers along with a corner layer, and another having exponential and parabolic layers together with corner layers. Next, we extend the idea of the developed algorithm to analyze the numerical solution along with the scaled discrete space derivatives (in x and y directions) and the discrete time derivative for two-dimensional system of singularly perturbed convection-diffusion time-delay parabolic IBVPs with different diffusion parameters, utilizing a tensor product of two 1D generalized S-meshes. In this regard, we consider two types of parabolic DPDEs: one possessing overlapping regular exponential layers along with a corner layer, and another having overlapping exponential and parabolic layers along with corner layers. In addition, for each class of BVPs and IBVPs, parameter-uniform global error bounds are established in appropriate C^1-norm.