Lower Dimensional Approximation of Some Problems for Thin Piezoelectric and Elastic Shells with Nonuniform Thickness

A thesis submitted

in partial fulfillment for the award of the degree of

Doctor of Philosophy

by

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Certificate

This is to certify that the thesis titled ***Lower Dimensional Approximation of Some Problems for Thin Piezoelectric and Elastic Shells with Nonuniform Thickness*** submitted by **Job Mathai**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a bona fide record of the original work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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I declare that this thesis titled ***Lower Dimensional Approximation of Some Problems for Thin Piezoelectric and Elastic Shells with Nonuniform Thickness*** submitted in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a record of the original work carried out by me under the supervision of **Dr N Sabu**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

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Abstract

Elastic or piezoelectric structures are three dimensional structres. They are very much used in various applications(eg: aerospace, biomechanics etc). Often when the thickness of the elastic or piezoelectric structure is "very small" when compare to other dimensions lower dimensional models are preferred to the actual three dimensional model.

Lower dimensional approximation of elastic and piezoelectric plates and shells with "uniform thickness" have been extensively studied. In this thesis we study the corresponding problems for "non-uniform thickness". More precisely, we study the two dimensional approximation of eigenvalue problem for piezoelectric shallow shells and flexural shells with non-uniform thickness and dynamic problem for elastic shallow shells with non-uniform thickness. We show that the solution of the three dimensional problem converge to the solutions of two dimensional model when the thickness of the shell (denoted by ϵ) goes to zero.

In the second chapter we consider eigenvalue problem for thin piezoelectric shallow shells (i.e, the curvature goes to zero as the thickness of the shell goes to zero) with non-uniform thickness. The technique used here for proving convergence rely on those used by J.Raja and N.Sabu [70] for two dimensional approximation of boundary value problem for piezoelectric shallow shells with non-uniform thickness. We first transform the problem to a domain independent of the thickness parameter ϵ and show that the scaled eigenvalues are $o(\epsilon^2)$ and the corresponding scaled eigensolutions converge to the eigensolutions of a two dimensional model. We also show that all the eigensolutions of the two dimensional problem occur this way, i.e, each eigensolution of the two dimensional model is the limit of a sequence of eigensolutions of the three dimensional problem as the thickness of the shell goes to zero.

In the third chapter we consider eigenvalue problem for flexural shells (i.e, the space of inextensional displacement is non zero) with non-uniform thickness. Here also we first transform the problem to a domain independent of ϵ and show that the eigenvalues are $o(\epsilon^2)$ and the corresponding scaled eigensolutions converge to the eigensolutions of a two dimensional model. We also show that all the eigenvalues of the limit problem are limit of sequence of eigensolutions of the three dimensional problem as the thickness of the shell goes to zero.

In the fourth chapter we consider a dynamic problem for elastic shallow shells with non-uniform thickness and we show that under suitable scalings on the applied forces and unknowns the solutions of the three dimensional model converge to the solution of two dimensional model as the thickness of the shell goes to zero.

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Nomenclature

Ω	domain in $I\!R^3$ (open, bounded, connected subset with a Lipschitz-continuous boundary, the set Ω being "locally on one side of its boundary").
$x = (x_i)$	generic point in $\overline{\Omega}$.
dx	volume element in Ω.
Γ	boundary of Ω.
(n_i)	unit normal vector along Γ.
$\Phi : \overline{\Omega} \subset I\!R^3 \longrightarrow I\!R^3$	injective and smooth enough mapping such that the three vectors $\partial_i \Phi$ are linearly independent at each point $x \in \overline{\Omega}$.
$g_i = \partial_i \Phi$	vectors of the covariant bases in the set $\Phi(\overline{\Omega})$.
g^i	vectors of the contravariant bases in the set $\Phi(\overline{\Omega})$. The vectors are defined at each $x \in \bar{\Omega}$ by the relations $g^i(x) \cdot g_j(x) = \delta^i_j$.
$g_{ij} = g^i \cdot g_j$	covariant components of the metric tensor of the $\Phi(\overline{\Omega})$.
$g = det(g_{ij})$	
$\Gamma^p_{ij} = g^p \cdot \partial_j g_i$	Christoffel symbols.
$v_{i\|\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$	covariant derivatives of a vector field $v_i g^i$ with covariant components $v_i : \overline{\Omega} \longrightarrow I\!R$.
ω	domain in $I\!R^2$ (open, bounded, connected subset with a Lipschitz-continuous boundary, the set ω being "locally on one side of its boundary").
γ or $\partial\omega$	boundary of the set ω.
$d\gamma$	length element along γ .
γ_0	measurable subset of γ with length $\gamma_0 > 0$.
$x' = (x_\alpha)$	generic point in the set ω, sometimes also denoted y.
$\partial_\alpha = \frac{\partial}{\partial x_\alpha}, \quad \partial_{\alpha\beta} = \frac{\partial^2}{\partial x_\alpha x_\beta}$	
$\Omega = \omega \times (-1, 1)$.	
$(n_i) : \partial\Omega \to I\!R^3$	unit outer normal vector along the boundary $\partial\Omega$ of Ω.

$d\Gamma$	area element along $\partial\Omega$.
$\gamma \times [-\epsilon, \epsilon]$	lateral face of the set $\bar{\Omega}^\epsilon$.
$\Gamma_0^\epsilon = \gamma_0 \times [-\epsilon, \epsilon]$	portion of the lateral face where a shell is clamped.
$\Gamma_+^\epsilon = \gamma_0 \times \epsilon$	upper face of the set $\bar{\Omega}^\epsilon$.
$\Gamma_-^\epsilon = \gamma_0 \times -\epsilon$	lower face of the set $\bar{\Omega}^\epsilon$.
$\Delta = \partial_{\alpha\alpha}$	Laplacian.
$A^{ijkl} = \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk})$.	
	contravariant components of the three- dimensional elasticity tensor.
$\hat{P}^{ijk,\epsilon}$	denote the piezoelectric tensors.
$\hat{\in}^{ij,\epsilon}$	denote the dielectric tensors.
$\mathcal{D}(\Omega)$	the space of functions in $\mathcal{C}^\infty(\Omega)$ with compact support in Ω.
$H^1(\Omega) = \{v \in L^2(\Omega);\ \partial_i v \in L^2(\Omega)\}$.	
$H_0^1(\Omega) = \{v \in L^2(\Omega);\ v = 0 \text{ on } \partial\Omega\}$.	
$H_\Gamma^1(\Omega) = \{v \in L^2(\Omega);\ v = 0 \text{ on } \partial\Gamma\}$.	

GENERAL CONVENTIONS

1. Latin indices and exponents: $i, j, p, \ldots$, take their values in the set {1, 2, 3}, unless otherwise indicated, as when they are used for indexing sequences.

2. Greek indices and exponents: $\alpha, \beta, \sigma, \ldots$ except ε, take their values in the set {1,2}.

3. The symbol "ϵ" designates a parameter that is > 0 and approaches zero.

Chapter 1

Introduction

Elastic or piezoelectric structures have wide range of applications. However for problems involving thin elatic or piezoelectric structures lower dimensional models are proposed by A.L Cauchy, von Karman etc, depending on a priori assumptions of mechanical or geometrical nature, as approximation to the actual three dimensional models. The main reason is that the lower dimensional models are more suitable for numerical computations.

But given a thin elastic or piezoelectric body with specific loading and boundary conditions how to choose between numerous lower dimensional models available? Hence before devising numerical methods to approximate a lower dimensional model we should first know whether this lower dimensional model is indeed an approximation of the given three dimensional problem. Thus one needs to justify mathematically that the solutions of the three dimensional problem converge to the solution of the two dimensional problem.

The first approach consists of directly estimating the difference between three dimensional solution and the solution of two dimensional model. For linearly elastic plates first such estimate was given by Morgenstein[63].This approach nevertheless rely on some a priori assumptions of mechanical and geometrical nature.

A second approach is by formal asymptotic method. In this method the three dimensional solution is first scaled in an appropriate manner so as to be defined on a fixed domain, then expanded as a formal series expansion in terms of the thickness parameter ϵ. The formal series expansion of the scaled three dimensional solution is then inserted into the three dimensional problem and sufficiently many factors of the successive powers of ϵ found are equated to zero until the leading term of the expansion can be computed and identified with the solution of a known lower dimensional model. This approach was used by P.G.Ciarlet([22],[23]) to derive plates and junctions in elastic structures and nonlinear plate models. P.G.Ciarlet and J.C.Paumier[21] used this method to derive Marguerre-von Karman equations for shallow shells. D.Fox, A.Raoult, J.C.Simo[32] used this method to derive nonlinear properly invariant plate theories. V.Lods and Miara [51] derived nonlinear flexural shell model and B.Miara [59] derived nonlinear membrane shell model. Asymptotic modelling of signorini problem of generalized von Karman equation for shallow shells were studied by Bensayah et al [7],[19]. They also studied the asymptotic modelling of signorini problem with coulomb friction for linearly elastostatic shallow shells and without friction of linear thin plates([6],[8]).

The scalings made in the formal asymptotic expansion method to derive two dimensional linear plate model is justified by B.Miara [57] and in nonlinear case by B.Miara [58]. In the case of linear elastic rods and shallow shells the scalings are justified by Raja and Sabu ([69], [71]).

Third approach is using asymptotic analysis. Here the basic idea is to get a bound for the solutions of the three dimensional problem in a suitable Hilbert space using Korn's type inequalities. This would imply the weak convergence of the solutions in that Hilbert space to some function and then one identifies the limit as solution of a lower dimensional problem. Using this method two dimensional models of boundary value problem for plates, shallow shells, membrane shells, flexural shells with uniform thickness were derived by Ciarlet et al ([18],[26],[27]) and the corresponding eigenvalue problem for plates was derived by Ciarlet and Kesavan [25]. Eigenvalue problem for linearly elastic shells and rods were derived by Kesavan and Sabu ([44], [45],[46]). Le Dret [47] derived the one dimensional model of rods. Y.Ji [40] has derived the two dimensional model for dynamic problem for generalized elastic membrane and L.Xiao ([89],[90]) has derived the dynamic problem for membrane and flexural shells. Rao [72] has studied asymptotic analysis for spherical shells. Bunoiu et al([14],[15]) studied junctions of rods and plates.

The boundary value problem for linearly elastic shells with non-uniform thickness was derived by Busse [17].Sabu [73] has studied the asymptotic analysis for elastic shallow shells with variable thickness. Jimbo et al [41] have studied the asymptotic behaviour of thin elastic rod with non-uniform thickness.

The error estimate between the three dimensional and two dimensional solutions for plate was derived by Destuynder ([30],[31]) and for flexural and membrane shells were derived by C.Mardare ([54],[55]). Simmonds [87] has studied the error estimates for Koiter's model.

Another approach to derive lower dimensional linear and nonlinear models is by using gamma convergence. In this method the solution(s) of the three dimensional problem is characterised as minimizer of some energy functionals and then one shows that the energy functionals gamma converge to some energy functional whose minimiser is the limit of the solutions of the three dimensional problems. This theory was developed in G. Dal Maso[29].

Using this method, Bourquin et al [13] derived linear plate model, Genevey ([37], [38]) has derived linear membrane and flexural shell model, Sabu([77],[78]) has derived one dimensional model of rod and two dimensional model of piezoelectric shallow shells. Le Dret and Raoult ([48],[49]) have derived the nonlinear membrane model for plates and shells. Friesecke et al ([33],[34]) have derived nonlinear plate models. Mora et al ([61], [62]) have shown the convergence of equilibria for thin elastic beams. Muller et al ([50], [60], [64],[65]) have derived the rod model for multiphase materials and von Karman plates. Acerbi et al [1] have studied the strain energy for elastic string.

Piezoelectricity is an electromechanical phenomenon, i.e, piezoelectric materials respond to mechanical forces and induce electric field and they induce mechanical stress or strain when subjected to electric field. They are used as sensors and actuators. They are also used in shape controlling for plane propellers as well as in manufacturing artificial organs in biomechanics. When the thickness of piezoelectric shell is very small, lower dimensional models are used as approximation. In this direction Bernadou and Haenel ([9],[10],[11]) have derived the two dimensional model for membrane and flexural shells. Piezoelectric plate models are studied by Rahmoune et al [68] and Sene [82]. N. Sabu ([74],[75],[79]) has studied the eigenvalue problem problem for shallow and flexural shells with uniform thickness and asymptotic analysis of piezoelectric shells with variable thickness was studied in [76]. Bantsuri et.al ([3]) have studied the boundary value problem for electroelasticity for a plate with thin inclusion. C. Collard and B. Miara [28] have studied the two dimensional models for geometrically nonlinear piezoelectric shells. Theory of piezoelectricity is well developed in IKeda [39] and Tzou [88]

Homogenization is an approach to study the macro behaviour of a medium by its micro properties. Homogenization of eigenvalue problem is studied by S.Kesavan([42],[43]). Homogenization of a class of nonlinear eigenvalue problem is studied by Baffico et al [2]. Bouchitte et al [12] have studied homogenization of second order energies on periodic structres. S.Ganesh et al ([35],[36]) have used blochwave method for homogenization of a class of problems. R. Mahadevan et al ([52],[53]) have studied the homogenization of some cheap control problems and and homogenization of elliptic equation in a doamin with oscillating boundary, with nonhomogenous nonlinear boundary conditions. Bunoio et al ([16]) have studied the asymptotic behaviour of bingham fluid in thin layers. An eigenvalue optimization for p-laplacian has been studied by Chorwadwala et al [20] and B. Miara [56] studied the optimal spectral approximation in plates.

Bauer et.al ([4],[5]) have studied stability of plates with circular inclusion, under tension and three dimensional problem of the axisymmetric deformation of an orthotropic sperical layer. Nazarov et al ([66],[67]) have studied thin elastic plates supported over small areas. Sachan et al [81] have studied indentation of a periodically layered planar elastic half space. Shavlakadze et al ([83]- [86]) have studied the boundary value problem for piezoelectric plates.

Chapter 2

Lower Dimensional Approximation of Eigenvalue Problem For Piezoelectric Shallow Shells with Nonuniform Thickness

2.1 Introduction

Piezoelectricity is an electromechanical phenomena: these materials generates deformation on application of electric field and conversely they induce electric field on application of mechanical deformation. Often, when the thickness of the material is very small, lower dimensional approximations of the three dimensional models are preferred, especially in numerical computations.

In this connection lower dimensional approximation of thin piezoelectric plates and shells with uniform thickness has been studied in static cases(cf: [9], [10], [11],[82]) and the corresponding eigenvalue problems has been studied for uniform thickness(cf. [74], [75], [80]). Contact problem for piezoelectric materials has been studied in ([3], [83],[84], [85] , [86]). Asymptotic analysis of static problem for piezoelectric shells with nonuniform thickness has been studied in ([70], [76]).

In this chapter we consider the eigenvalue problem for thin piezoelectric shallow shells with nonuniform thickness and study their limiting behaviour. In particular starting with the assumptions made for stationary problems we wish to derive the limiting model for vibrations of shells. We briefly outline the problem studied in this chapter and the results obtained.

We consider a bounded domain, $\Omega^\epsilon = \omega \times (-\epsilon, \epsilon),\ \ \omega \subset I\!\!R^2$ and let $x^\epsilon = (x_1, x_2, x_3^\epsilon)$ be a generic point on Ω^ϵ. Let $\phi^\epsilon : \overline{\omega} \to I\!\!R^3$ be an injective mapping and $a^3(x_1, x_2)$ denotes unit normal vector to the surface $\phi^\epsilon(\omega)$. For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to I\!R^3$ by

$$\Phi^\epsilon(x^\epsilon) = \phi^\epsilon(x_1, x_2) + x_3^\epsilon \chi(x_1, x_2) a^3(x_1, x_2) \text{ for all } x^\epsilon \in \Omega^\epsilon.$$

where $\chi \in W^{2,\infty}(\omega),\ 0 < \chi_0 < \chi(x_1, x_2)$, and $\Phi^\epsilon(\Omega^\epsilon)$ denotes the reference configuration of the shell. Note that when $\chi(x_1, x_2) = 1$, we get shell with uniform thickness ϵ.

We assume that the shell is a shallow shell, ie $\phi^\epsilon(x_1, x_2) = (x_1, x_2, \epsilon\theta(x_1, x_2))$. We then consider the eigenvalue problem and show that as the thickness of the shell goes to zero the eigensolutions of the three dimensional problem converge to the eigensolutions of two dimensional problem.

This chapter is organised as follows. In section 2.2, we describe the three dimensional problem. In section 2.3, we state the scaled problem, in section 2.4, we derive the a priori estimate for eigenvalues and in section 2.5, we study the limiting problem for shallow shells.

2.2 The Three-dimensional Problem

Let $\omega \subset I\!R^2$ be a bounded domain with a Lipschitz continuous boundary γ and let ω lie locally on one side of γ. Let $\gamma_0, \gamma_e \subset \partial\omega$ with meas$(\gamma_0) > 0$ and meas$(\gamma_e) > 0$. Let $\gamma_1 = \partial\omega \backslash \gamma_0$ and $\gamma_s = \partial\omega \backslash \gamma_e$. For each $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon),\ \ \Gamma^{\pm,\epsilon} = \omega \times \{\pm\epsilon\},\ \ \Gamma_0^\epsilon = \gamma_0 \times (-\epsilon, \epsilon),\ \ \Gamma_1^\epsilon = \gamma_1 \times (-\epsilon, \epsilon),$$

$$\Gamma_e^\epsilon = \gamma_e \times (-\epsilon, \epsilon), \Gamma_s^\epsilon = \gamma_s \times (-\epsilon, \epsilon).$$

Let $x^\epsilon = (x_1, x_2, x_3^\epsilon)$ be a generic point on Ω^ϵ and let $\partial_\alpha = \partial_\alpha^\epsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\epsilon = \frac{\partial}{\partial x_3^\epsilon}$. Let $\phi^\epsilon : \overline{\omega} \to I\!R^3$ be an injective mapping of class $C^3(\omega)$ such that the two vectors

$$a_\alpha(y) = \partial_\alpha \phi^\epsilon$$

are linearly independent for all $y \in \omega$. We define a^α to be the vectors satisfying the relation

$$a^\alpha(y).a_\beta(y) = \delta_\beta^\alpha.$$

We define

$$a^3(y) = a_3(y) = \frac{a_1 \times a_2}{|a_1 \times a_2|},$$

and

$$\left.\begin{array}{ll} a_{\alpha\beta} := a_\alpha.a_\beta & a^{\alpha\beta} := a^\alpha.a^\beta \\ b_{\alpha\beta} := a^3.\partial_\beta a_\alpha & b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha} \\ \Gamma^\sigma_{\alpha\beta} := a^\sigma.\partial_\beta a_\alpha & \end{array}\right\} \tag{2.2.1}$$

where $a_\alpha.a_\beta$ denotes usual dot product. These verify the usual symmetry relations. The area element along S is $\sqrt{a}dy$, where

$$a := det(a_{\alpha\beta}). \tag{2.2.2}$$

By the continuity of the functions defined above, there exists $a_0 > 0$ such that

$$0 < a_0 \leq a(y) \text{ for all } y \in \overline{\omega}.$$

For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to I\!R^3$ by

$$\Phi^\epsilon(x^\epsilon) = \phi^\epsilon(x_1, x_2) + x_3^\epsilon \chi(x_1, x_2) a^3(x_1, x_2) \text{ for all } x^\epsilon \in \Omega^\epsilon, \tag{2.2.3}$$

where $\chi \in W^{2,\infty}(\omega),\ 0 < \chi_0 < \chi(x_1, x_2)$. We define vectors g_i^ϵ and $g^{i,\epsilon}$ by the relations

$$g_i^\epsilon = \partial_i^\epsilon \Phi^\epsilon \text{ and } g^{j,\epsilon}.g_i^\epsilon = \delta_i^j.$$

which form the covariant and contravariant basis respectively of the tangent plane of $\Phi^\epsilon(\Omega^\epsilon)$ at $\Phi^\epsilon(x^\epsilon)$. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\epsilon = g_i^\epsilon.g_j^\epsilon \text{ and } g^{ij,\epsilon} = g^{i,\epsilon}.g^{j,\epsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon}.\partial_j^\epsilon g_i^\epsilon.$$

The volume element is given by $\sqrt{g^\epsilon}dx^\epsilon$ where

$$g^\epsilon = det(g_{ij}^\epsilon).$$

The set $\bar{\hat{\Omega}}^\varepsilon = \Phi^\varepsilon(\bar{\Omega}^\varepsilon)$ is the reference configuration of the shell and we denote a generic point of the shell by $\hat{x}^\varepsilon$. For $0 < \varepsilon \leq \varepsilon_0$, we define the sets

$$\hat{\Gamma}^{\pm,\varepsilon} = \Phi^\varepsilon(\Gamma^{\pm,\varepsilon}),\ \ \hat{\Gamma}_0^\varepsilon = \Phi^\varepsilon(\Gamma_0^\varepsilon),\ \ \hat{\Gamma}_1^\varepsilon = \Phi(\Gamma_1^\varepsilon),\ \ \hat{\Gamma}_N^\varepsilon = \hat{\Gamma}_1^{\ \varepsilon} \cup \hat{\Gamma}^{\pm\varepsilon},$$

$$\hat{\Gamma}_e^\varepsilon = \Phi^\varepsilon(\Gamma_e^\varepsilon), \hat{\Gamma}_s^\varepsilon = \Phi^\varepsilon(\Gamma_s^\varepsilon), \hat{\Gamma}_{eD}^\varepsilon = \hat{\Gamma}_e^\varepsilon \cup \hat{\Gamma}^{\pm\varepsilon}$$

We assume that the material is mechanically isotropic so that the elasticity tensor $\hat{A}^{ijkl,\varepsilon}$ is given by

$$\hat{A}^{ijkl,\varepsilon} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{2.2.4}$$

where λ and μ are Lamé constants. Clearly this tensor satisfy the symmetry relations

$$\hat{A}^{ijkl,\varepsilon} = \hat{A}^{jikl,\varepsilon} = \hat{A}^{klij,\varepsilon} \tag{2.2.5}$$

and the inequality

$$\hat{A}^{ijkl,\varepsilon}t_{ij}t_{kl} \geq C\sum_{i,j}|t_{ij}|^2 \tag{2.2.6}$$

for all symmetric tensor (t_{ij}). Let $\hat{P}^{ijk,\varepsilon}$ and $\hat{\in}^{ij,\varepsilon}$ denote the piezoelectric and dielectric tensors respectively. We assume that they are symmetric and there exists $C > 0$ such that

$$\hat{\in}^{ij,\varepsilon}t_i t_j \geq C\sum_{i}|t_i|^2 \tag{2.2.7}$$

for all $(t_i) \in I\!R^3$. Then the eigenvalue problem consists of finding $(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon, \xi^\epsilon)$, such that

$$\left.\begin{aligned} -div\hat{\sigma}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon) &= \xi^\epsilon\hat{u}^\epsilon \text{ in } \hat{\Omega}^\varepsilon,\\ \hat{\sigma}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)\nu &= 0 \text{ on } \hat{\Gamma}^\varepsilon_N,\\ \hat{u}^\varepsilon &= 0 \text{ on } \hat{\Gamma}^\varepsilon_0. \end{aligned}\right\} \tag{2.2.8}$$

$$\left.\begin{aligned} div\hat{D}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon) &= 0 \text{ in } \hat{\Omega}^\varepsilon,\\ \hat{D}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)\nu &= 0 \text{ on } \hat{\Gamma}^\varepsilon_s,\\ \hat{\varphi}^\varepsilon &= 0 \text{ on } \hat{\Gamma}^\varepsilon_{eD}. \end{aligned}\right\} \tag{2.2.9}$$

where

$$\hat{\sigma}^\varepsilon_{ij} = \hat{A}^{ijkl,\varepsilon}\hat{e}^\varepsilon_{ij} - \hat{P}^{kij,\varepsilon}\hat{E}^\varepsilon_k, \tag{2.2.10}$$

$$\hat{D}^\varepsilon_k = \hat{P}^{kij,\varepsilon}\hat{e}^\varepsilon_{ij} + \hat{\in}^{kl,\varepsilon}\hat{E}^\varepsilon_l, \tag{2.2.11}$$

$$\hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_j + \hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_i),\ \ \hat{\partial}^\varepsilon_i = \frac{\partial}{\partial\hat{x}^\varepsilon_i}\ \text{ and }\ \hat{E}^\varepsilon_k(\hat{\varphi}^\varepsilon) = -\hat{\bigtriangledown}^\varepsilon(\hat{\varphi}^\varepsilon).$$

We define the spaces

$$\hat{V}^\varepsilon = \{\hat{v}^\varepsilon \in (H^1(\hat{\Omega}^\varepsilon))^3, \hat{v}|_{\hat{\Gamma}^\varepsilon_0} = 0\}, \tag{2.2.12}$$

$$\hat{\Psi}^\varepsilon = \{\hat{\psi}^\varepsilon \in H^1(\hat{\Omega}^\varepsilon), \hat{\psi}|_{\hat{\Gamma}^\varepsilon_{eD}} = 0\}. \tag{2.2.13}$$

The variational form of the system (2.2.8)-(2.2.9) is to find $(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon, \xi^\epsilon) \in \hat{V}^\varepsilon \times \hat{\Psi}^\varepsilon \times \mathbb{R}$ such that

$$\hat{a}^\varepsilon_\chi((\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon), (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon)) = \xi^\epsilon \hat{l}^\varepsilon(\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) \text{ for all } (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) \in \hat{V}^\varepsilon \times \hat{\Psi}^\varepsilon \tag{2.2.14}$$

where

$$\begin{aligned} \hat{a}^\varepsilon_\chi((\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon), (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon)) &= \int_{\hat{\Omega}^\varepsilon} \hat{A}^{ijkl,\varepsilon} \hat{e}^\varepsilon_{kl}(\hat{u}^\varepsilon) \hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon) d\hat{x}^\varepsilon + \int_{\hat{\Omega}^\varepsilon} \hat{\in}^{ij,\varepsilon} \hat{\partial}^\varepsilon_i \hat{\varphi}^\varepsilon \hat{\partial}^\varepsilon_j \hat{\psi}^\varepsilon d\hat{x}^\varepsilon \\ &+ \int_{\hat{\Omega}^\varepsilon} \hat{P}^{mij,\varepsilon} \left(\hat{\partial}^\varepsilon_m \hat{\varphi}^\varepsilon \hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon) - \hat{\partial}^\varepsilon_m \hat{\psi}^\varepsilon \hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon) \right) d\hat{x}^\varepsilon \end{aligned} \tag{2.2.15}$$

$$\hat{l}^\varepsilon(\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{u}^\varepsilon \hat{v}^\varepsilon d\hat{x}^\varepsilon \tag{2.2.16}$$

Since the mappings $\Phi^\varepsilon : \overline{\Omega}^\varepsilon \to \overline{\hat{\Omega}}^\varepsilon$ are assumed to be $\mathcal{C}^1$ diffeomorphism, the correspondence that associates with every vector $\hat{v}^\varepsilon = (\hat{v}^\varepsilon_i) \in \hat{V}^\varepsilon$ (note that $(\hat{v}^\varepsilon_i)$ are the components of the vector $\hat{v}^\varepsilon = \hat{v}^\varepsilon_i \hat{e}^i$, where $(\hat{e}^i)^3_{i=1}$ is the standard basis in $I\!R^3$) the vector $v^\varepsilon = (v^\varepsilon_i)$ defined by

$$\hat{v}^\varepsilon_i(\hat{x}^\varepsilon)\hat{e}^i = v^\varepsilon_i(x^\varepsilon) g^i(x^\varepsilon)$$

induces a bijection between the spaces $\hat{V}^\varepsilon$ and V^ε, where

$$V^\varepsilon = \{v^\varepsilon \in (H^1(\Omega^\varepsilon))^3 | v^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_0\}. \tag{2.2.17}$$

Then we have (cf. [24])

$$\hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i(\hat{x}^\varepsilon) = (\partial^\varepsilon_l v^\varepsilon_k - \Gamma^{q,\varepsilon}_{lk} v^\varepsilon_q)(g^{k,\varepsilon})_i (g^{l,\varepsilon})_j, \tag{2.2.18}$$

$$\hat{e}_{ij}(\hat{v}^\varepsilon)(\hat{x}^\varepsilon) = e^\varepsilon_{k||l}(v^\varepsilon)(g^{k,\varepsilon})_i (g^{l,\varepsilon})_j, \tag{2.2.19}$$

where

$$e^\varepsilon_{i||j}(v^\varepsilon) = \frac{1}{2}(\partial^\varepsilon_i v^\varepsilon_j + \partial^\varepsilon_j v^\varepsilon_i) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p. \tag{2.2.20}$$

Also with any scalar function $\hat{\varphi}^\varepsilon \in \hat{\Psi}^\varepsilon$, the correspondence $\hat{\varphi}^\varepsilon(\hat{x}^\varepsilon) = \varphi^\varepsilon(x^\varepsilon)$ induces a bijection between the spaces $\hat{\Psi}^\varepsilon$ and Ψ^ε where

$$\Psi^\varepsilon = \{\psi^\varepsilon \in H^1(\Omega^\varepsilon) | \psi^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{eD}\}. \tag{2.2.21}$$

Then

$$\hat{\partial}_j \hat{\varphi}^\varepsilon = \hat{\partial}_j \varphi^\varepsilon(x^\varepsilon) = \hat{\partial}_j \varphi^\varepsilon((\Phi^\varepsilon)^{-1}(\hat{x}^\varepsilon)) = \partial_l \varphi^\varepsilon(x^\varepsilon)(g^l(x^\varepsilon))_j. \tag{2.2.22}$$

Then the variational problem consists of finding $(u^\varepsilon, \varphi^\varepsilon, \xi^\epsilon), (u^\varepsilon, \varphi^\varepsilon) \neq (0,0)$ such that

$$a_\chi^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) = \xi^\epsilon l^\varepsilon(v^\varepsilon, \psi^\varepsilon) \text{ for all } (v^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times \Psi^\varepsilon \tag{2.2.23}$$

where

$$\begin{aligned} a_\chi^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) &= \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(v^\varepsilon) e^\varepsilon_{i||j}(v^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} \in^{ij,\varepsilon} \partial_i^\varepsilon \varphi^\varepsilon \partial_j^\varepsilon \psi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \\ &+ \int_{\Omega^\varepsilon} P^{mij,\varepsilon} \left(\partial_m^\varepsilon \varphi^\varepsilon e^\varepsilon_{i||j}(v^\varepsilon) - \partial_m^\varepsilon \psi^\varepsilon e^\varepsilon_{i||j}(u^\varepsilon) \right) \sqrt{g^\varepsilon} dx^\varepsilon, \end{aligned} \tag{2.2.24}$$

$$l^\varepsilon(v^\varepsilon, \psi^\varepsilon) = \int_{\Omega^\varepsilon} u^\varepsilon v^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon, \tag{2.2.25}$$

.

$$A^{ijkl,\varepsilon} = \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{2.2.26}$$

$$P^{pqr,\varepsilon} = \hat{P}^{ijk,\varepsilon}.(g^{p,\varepsilon})_i (g^{q,\varepsilon})_j (g^{r,\varepsilon})_k, \tag{2.2.27}$$

$$\in^{pq,\varepsilon} = \hat{\in}^{ij,\varepsilon} (g^{p,\varepsilon})_i (g^{q,\varepsilon})_j. \tag{2.2.28}$$

It can be shown that there exists a constant $C > 0$ such that for all symmetric tensors (t_{ij})

$$A^{ijkl,\varepsilon} t_{kl} t_{ij} \geq C \sum_{i,j=1}^{3} (t_{ij})^2. \tag{2.2.29}$$

Using (2.2.7) and that $(g^{j,\varepsilon})$ forms contravariant basis, it follows that for any vector $(t_i) \in \mathbb{R}^3$

$$\in^{kl,\varepsilon} t_k t_l \geq C \sum_{j=1}^{3} t_j^2. \tag{2.2.30}$$

Moreover from the symmetry of $\hat{A}^{ijkl,\varepsilon},\ \hat{P}^{ijk,\varepsilon},\ \hat{\in}^{ij,\varepsilon}$ we have the symmetries

$$A^{ijkl,\varepsilon} = A^{klij,\varepsilon} = A^{jikl,\varepsilon}, \in^{kl,\varepsilon} = \in^{lk,\varepsilon}, P^{ijk,\varepsilon} = P^{kij,\varepsilon}. \tag{2.2.31}$$

Using (2.2.29) and (2.2.30) we have

$$a_{\chi}^{\varepsilon}((u^{\varepsilon},\varphi^{\varepsilon}),(u^{\varepsilon},\varphi^{\varepsilon})) = \int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e^{\varepsilon}_{k||l}(u^{\varepsilon}) e^{\varepsilon}_{i||j}(u^{\varepsilon})\sqrt{g^{\varepsilon}}dx^{\varepsilon} + \int_{\Omega^{\varepsilon}} \in^{ij,\varepsilon} \partial^{\varepsilon}_i \varphi^{\varepsilon} \partial^{\varepsilon}_j \varphi^{\varepsilon} \sqrt{g^{\varepsilon}}dx^{\varepsilon}$$
$$\geq C(||u^{\varepsilon}||^2_{1,\Omega^{\varepsilon}} + ||\varphi^{\varepsilon}||^2_{1,\Omega^{\varepsilon}}). \tag{2.2.32}$$

Clearly the bilinear form associated with the left-hand side of (2.2.23) is elliptic. Hence by Lax-Milgram theorem, given $f^{\epsilon} \in (H^{-1}(\Omega^{\epsilon}))^3$ and $h^{\epsilon} \in H^{-1}(\Omega^{\epsilon})$, there exists a unique $(u^{\epsilon},\varphi^{\epsilon})$ such that

$$a^{\epsilon}_{\chi}((u^{\epsilon},\varphi^{\epsilon}),(v^{\epsilon},\psi^{\epsilon})) = \langle (f^{\epsilon},h^{\epsilon}),(v^{\epsilon},\psi^{\epsilon})\rangle. \tag{2.2.33}$$

In particular, for each $f^{\epsilon} \in (L^2(\Omega^{\epsilon}))^3$, there exists a unique solution $(u^{\epsilon},\varphi^{\epsilon})$ such that

$$a^{\epsilon}_{\chi}((u^{\epsilon},\varphi^{\epsilon}),(v^{\epsilon},\psi^{\epsilon})) = \langle f^{\epsilon},v^{\epsilon}\rangle_{0,\Omega^{\epsilon}}. \tag{2.2.34}$$

This is equivalent to the following equations.

$$\int_{\Omega^{\epsilon}} A^{ijkl,\epsilon} e^{\epsilon}_{k||l}(u^{\epsilon}) e^{\epsilon}_{i||j}(v^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon} + \int_{\Omega^{\epsilon}} P^{mij,\epsilon}\partial^{\epsilon}_m(\varphi^{\epsilon}) e^{\epsilon}_{i||j}(v^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon}$$
$$= \int_{\Omega^{\epsilon}} f^{\epsilon}v^{\epsilon}\sqrt{g^{\epsilon}}dx^{\epsilon}\ \forall\ v^{\epsilon} \in V^{\epsilon} \tag{2.2.35}$$

and

$$\int_{\Omega^{\epsilon}} \in^{ij,\epsilon}\partial^{\epsilon}_i\varphi^{\epsilon}\partial^{\epsilon}_j\psi^{\epsilon}\sqrt{g^{\epsilon}}dx^{\epsilon} = \int_{\Omega^{\epsilon}} P^{mij,\epsilon}\partial^{\epsilon}_m\psi^{\epsilon} e^{\epsilon}_{i||j}(u^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon}\ \forall\ \psi^{\epsilon} \in \Psi^{\epsilon}. \tag{2.2.36}$$

For each $h^{\epsilon} \in V^{\epsilon}$, it follows from (2.2.30) that there exists a unique $T^{\epsilon}_{\chi}(h^{\epsilon}) \in \Psi^{\epsilon}$ such that

$$\int_{\Omega^{\epsilon}} \in^{ij,\epsilon}\partial^{\epsilon}_i T^{\epsilon}_{\chi}(h^{\epsilon})\partial^{\epsilon}_j\psi^{\epsilon}\sqrt{g^{\epsilon}}dx^{\epsilon} = \int_{\Omega^{\epsilon}} P^{mij,\epsilon}\partial^{\epsilon}_m\psi^{\epsilon} e^{\epsilon}_{i||j}(h^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon}\ \forall\ \psi^{\epsilon} \in \Psi^{\epsilon}. \tag{2.2.37}$$

and that the map $T^{\epsilon}_{\chi} : V^{\epsilon} \to \Psi^{\epsilon}$ is continuous. In particular $\varphi^{\epsilon} = T^{\epsilon}_{\chi}(u^{\epsilon})$ and the equations (2.2.35)-(2.2.36) becomes

$$\int_{\Omega^{\epsilon}} A^{ijkl,\epsilon} e^{\epsilon}_{k||l}(u^{\epsilon}) e^{\epsilon}_{i||j}(v^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon} + \int_{\Omega^{\epsilon}} P^{mij,\epsilon}\partial^{\epsilon}_m(T^{\epsilon}_{\chi}(u^{\epsilon})) e^{\epsilon}_{i||j}(v^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon}$$
$$= \int_{\Omega^{\epsilon}} f^{\epsilon}v^{\epsilon}\sqrt{g^{\epsilon}}dx^{\epsilon}\ \forall\ v^{\epsilon} \in V^{\epsilon} \tag{2.2.38}$$

$$\int_{\Omega^{\epsilon}} \in^{ij,\epsilon}\partial^{\epsilon}_i(T^{\epsilon}_{\chi}(u^{\epsilon}))\partial^{\epsilon}_j\psi^{\epsilon}\sqrt{g^{\epsilon}}dx^{\epsilon} = \int_{\Omega^{\epsilon}} P^{mij,\epsilon}\partial^{\epsilon}_m\psi^{\epsilon} e^{\epsilon}_{i||j}(u^{\epsilon})\sqrt{g^{\epsilon}}dx^{\epsilon}\ \forall\ \psi^{\epsilon} \in \Psi^{\epsilon}. \tag{2.2.39}$$

For each $h^\epsilon \in (L^2(\Omega^\epsilon))^3$, it follows from (2.2.39) and (2.2.37) that there exists a unique $G_\chi^\epsilon(h^\epsilon) \in V^\epsilon$ such that

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k||l}(G^\epsilon_\chi(h^\epsilon)) e^\epsilon_{i||j}(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon_\chi(G^\epsilon_\chi(h^\epsilon)))e^\epsilon_{i||j}(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon$$
$$= \int_{\Omega^\epsilon} h^\epsilon v^\epsilon \sqrt{g^\epsilon}dx^\epsilon \;\forall\; v^\epsilon \in V^\epsilon \tag{2.2.40}$$

and that $G^\epsilon_\chi : (L^2(\Omega^\epsilon))^3 \to V^\epsilon$ is continuous. Then the eigenvalue problem consists in finding pairs $(\xi^\epsilon, u^\epsilon) \in \mathcal{R} \times V(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k||l}(u^\epsilon) e^\epsilon_{i||j}(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon_\chi(u^\epsilon))e^\epsilon_{i||j}(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon$$
$$= \xi^\epsilon \int_{\Omega^\epsilon} u^\epsilon v^\epsilon \sqrt{g^\epsilon}dx^\epsilon \;\forall\; v^\epsilon \in V^\epsilon \tag{2.2.41}$$

$$\int_{\Omega^\epsilon} \in^{ij,\epsilon}\partial^\epsilon_i(T^\epsilon_\chi(u^\epsilon))\partial^\epsilon_j\psi^\epsilon\sqrt{g^\epsilon}dx^\epsilon = \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m\psi^\epsilon e^\epsilon_{i||j}(u^\epsilon)\sqrt{g^\epsilon}dx^\epsilon \;\forall\; \psi^\epsilon \in \Psi^\epsilon. \tag{2.2.42}$$

By classical arguments, we can show that there exists a sequence of eigenvalues

$$0 < \xi^{1,\epsilon} \le \xi^{2,\epsilon} \le ... \le \xi^{l,\epsilon} \le ...\infty \tag{2.2.43}$$

and we can choose a corresponding eigenfunctions $\{u^{l,\epsilon}\}$ such that

$$\int_{\Omega^\epsilon} u_i^{l,\epsilon} u_i^{m,\epsilon}\sqrt{g}dx^\epsilon = \delta_{lm}. \tag{2.2.44}$$

The sequence forms an orthonormal basis in the weighted space

$$L^2(g_\epsilon, \Omega^\epsilon) = \{u^\epsilon | \int_{\Omega^\epsilon} u_i^\epsilon u_i^\epsilon \sqrt{g^\epsilon}dx^\epsilon < \infty\}. \tag{2.2.45}$$

These eigenvalues can be charecterised as

$$\xi^{m,\epsilon} = \min_{W^\epsilon \in V^\epsilon_m} \max_{v^\epsilon \in W^\epsilon} R^\epsilon(v^\epsilon) \tag{2.2.46}$$

where V^ϵ_m denote the collection of all $m-$dimensional subspaces of V^ϵ and

$$R^\epsilon(v^\epsilon) = \frac{\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k||l}(v^\epsilon) e^\epsilon_{i||j}(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon_\chi(v^\epsilon))e^\epsilon_{i||j}(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon}{\int_{\Omega^\epsilon} v^\epsilon v^\epsilon \sqrt{g^\epsilon}dx^\epsilon} \tag{2.2.47}$$

2.3 The Scaled Problem

We now perform a change of variable so that the domain no longer depends on ϵ. With $x = (x_1, x_2, x_3) \in \Omega = \omega \times (-1, 1)$, we associate $x^\epsilon = (x_1, x_2, \epsilon x_3) \in \Omega^\epsilon$. Let

$$\Gamma_0 = \gamma_0 \times (-1, 1), \quad \Gamma_1 = \gamma_1 \times (-1, 1), \quad \Gamma^\pm = \omega \times \{\pm 1\}, \quad \Gamma_e = \gamma_e \times (-1, 1)$$

$$\Gamma_s = \gamma_s \times (-1, 1), \quad \Gamma_N = \Gamma_1 \cup \Gamma^+ \cup \Gamma^-, \quad \Gamma_{eD} = \Gamma^+ \cup \Gamma^- \cup \Gamma_e.$$

With the functions $\Gamma^{p,\epsilon}, g^\epsilon, A^{ijkl,\epsilon}, P^{ijk,\epsilon}, \in^{ij,\epsilon}: \Omega^\epsilon \to I\!R$, we associate the functions $\Gamma^p(\epsilon)$, $g^\epsilon, A^{ijkl}(\epsilon), P^{ijk}(\epsilon), \in^{ij}(\epsilon) : \Omega \to I\!R$ defined by

$$\Gamma^p(\epsilon)(x) := \Gamma^{p,\epsilon}(x^\epsilon), \;\; g(\epsilon)(x) = g^\epsilon(x^\epsilon), \;\; A^{ijkl}(\epsilon)(x) = A^{ijkl,\epsilon}(x^\epsilon), \tag{2.3.1}$$

$$P^{ijk}(\epsilon)(x) = P^{ijk,\epsilon}(x^\epsilon), \;\; \in^{ij}(\epsilon)(x) = \in^{ij,\epsilon}(x^\epsilon) \tag{2.3.2}$$

Since the shell is a shallow shell there exists a function $\theta \in C^3(\omega)$ such that

$$\phi^\epsilon(x_1, x_2) = (x_1, x_2, \epsilon\theta(x_1, x_2)), \text{ for all } (x_1, x_2) \in \omega \tag{2.3.3}$$

In this case, we make the following scalings on the eigensolutions.

$$u_\alpha^{m,\epsilon}(x^\epsilon) = \epsilon^2 u_\alpha^m(\epsilon)(x), \quad v_\alpha(x^\epsilon) = \epsilon^2 v_\alpha(x), \tag{2.3.4}$$

$$u_3^{m,\epsilon}(x^\epsilon) = \epsilon u_3^m(\epsilon)(x), \quad v_3(x^\epsilon) = \epsilon v_3(x), \tag{2.3.5}$$

$$T^\epsilon(u^{m,\epsilon}(x^\epsilon)) = \epsilon^3 T(\epsilon)(u^m(\epsilon)(x)), \quad T^\epsilon(v(x^\epsilon)) = \epsilon^3 T(\epsilon)(v(x)), \tag{2.3.6}$$

$$\xi^{m,\epsilon} = \epsilon^2 \xi^m(\epsilon). \tag{2.3.7}$$

With the tensors $e^\epsilon_{i||j}$, we associate the tensors $e_{i||j}(\epsilon)$ through the relation

$$e^\epsilon_{i||j}(v^\epsilon)(x^\epsilon) = \epsilon^2 e_{i||j}(\epsilon; v)(x). \tag{2.3.8}$$

We define the spaces

$$V(\Omega) = \{v \in (H^1(\Omega))^3, v|_{\Gamma_0} = 0\}, \tag{2.3.9}$$

$$\Psi(\Omega) = \{\psi \in H^1(\Omega), \psi|_{\Gamma_{eD}} = 0\}. \tag{2.3.10}$$

We denote $\varphi^m(\epsilon) = T_\chi(\epsilon)(u^m(\epsilon))$. Then the variational equations (2.2.41)-(2.2.43) become

$$\begin{aligned} &\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon, u^m(\epsilon))e_{i||j}(\epsilon, v)\sqrt{g(\epsilon)}dx + \int_\Omega P^{3kl}\partial_3\varphi^m(\epsilon)e_{k||l}(\epsilon, v)\sqrt{g(\epsilon)}dx \\ &+\epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon)e_{k||l}(\epsilon, v)\sqrt{g(\epsilon)}dx \\ &= \xi^m(\epsilon)\int_\Omega [\epsilon^2 u_\alpha^m(\epsilon)v_\alpha + u_3^m(\epsilon)v_3]\sqrt{g(\epsilon)}dx \text{ for all } v \in V(\Omega). \end{aligned} \tag{2.3.11}$$

$$\begin{aligned} &\int_\Omega \in^{33}(\epsilon)\partial_3\varphi^m(\epsilon)\partial_3\psi\sqrt{g(\epsilon)}dx + \epsilon\int_\Omega [\in^{3\alpha}(\epsilon)(\partial_\alpha\varphi^m(\epsilon)\partial_3\psi + \partial_3\varphi^m(\epsilon)\partial_\alpha\psi)]\sqrt{g(\epsilon)}dx \\ &+\epsilon^2\int_\Omega \in^{\alpha\beta}(\epsilon)\partial_\alpha\varphi^m(\epsilon)\partial_\beta\psi\sqrt{g(\epsilon)}dx \\ &= \int_\Omega P^{3kl}(\epsilon)\partial_3\psi e_{k||l}(\epsilon, u^m(\epsilon))\sqrt{g(\epsilon)}dx \\ &+\epsilon\int_\Omega [P^{\alpha kl}(\epsilon)\partial_\alpha\psi e_{k||l}(\epsilon, u^m(\epsilon))]\sqrt{g(\epsilon)}dx \text{ for all } \psi \in \Psi(\Omega), \end{aligned} \tag{2.3.12}$$

$$\int_\Omega [\epsilon^2 u_\alpha^m(\epsilon)u_\alpha^n(\epsilon) + u_3^m(\epsilon)u_3^n(\epsilon)]\sqrt{g(\epsilon)}dx = \delta_{mn}. \tag{2.3.13}$$

Based on the above scalings, we have the following lemma.

Lemma 2.3.1. The functions $e_{\alpha||\beta}(\epsilon; v)$ defined in (2.3.8) are of the form

$$\left.\begin{aligned} e_{\alpha||\beta}(\epsilon; v) &= \tilde{e}_{\alpha\beta}(v) + \epsilon^2 e^\sharp_{\alpha||\beta}(\epsilon; v), \\ e_{\alpha||3}(\epsilon; v) &= \tfrac{1}{\epsilon}\{\tilde{e}_{\alpha 3}(v) + \epsilon^2 e^\sharp_{\alpha||3}(\epsilon; v)\}, \\ e_{3||3}(\epsilon; v) &= \tfrac{1}{\epsilon^2}\tilde{e}_{33}(v), \end{aligned}\right\} \tag{2.3.14}$$

where

$$\left.\begin{aligned} \tilde{e}_{\alpha\beta}(v) &= \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \tfrac{v_3}{\chi}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}\chi) \\ \tilde{e}_{\alpha 3}(v) &= \tfrac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \\ \tilde{e}_{33}(v) &= \partial_3 v_3, \end{aligned}\right\} \tag{2.3.15}$$

Also there exists constant C such that

$$\left.\begin{array}{c}\sup_{0<\epsilon\leq\epsilon_0}\max_{\alpha,j}||e^{\sharp}_{\alpha,j}(\epsilon;v)||_{0,\Omega}\leq C||v||_{1,\Omega} \text{ for all } v\in V,\\ \sup_{0<\epsilon\leq\epsilon_0}\max_{x\in\Omega}|g(\epsilon)(x)-\chi^2|\leq C\epsilon^2,\\ \sup_{0<\epsilon\leq\epsilon_0}\max_{x\in\Omega}|A^{ijkl}(\epsilon)-A^{ijkl}|\leq C\epsilon^2,\end{array}\right\} \tag{2.3.16}$$

where

$$\left.\begin{array}{c}A^{\alpha\beta\gamma\tau}=\lambda\delta^{\alpha\beta}\delta^{\gamma\tau}+\mu(\delta^{\alpha\gamma}\delta^{\beta\tau}+\delta^{\alpha\tau}\delta^{\beta\gamma})\\ A^{\alpha\beta\gamma3}(0)=0,\quad A^{\alpha\beta33}(0)=\frac{1}{\chi^2}\lambda\delta^{\alpha\beta},\quad A^{\alpha3\gamma3}(0)=\frac{1}{\chi^2}\mu\delta^{\alpha\gamma}\\ A^{\alpha333}(0)=0,\quad A^{3333}(0)=\frac{1}{\chi^4}(\lambda+2\mu),\end{array}\right\} \tag{2.3.17}$$

$$A^{ijkl}(\epsilon)t_{kl}t_{ij}\geq Ct_{ij}t_{ij}, \tag{2.3.18}$$

for $0<\epsilon\leq\epsilon_0$ and for all symmetric tensors (t_{ij}).

Proof. Using the assumption (2.3.3) we have

$$g_\alpha(\epsilon)=\begin{pmatrix}\delta_{\alpha1}-\epsilon^2x_3[\chi\partial_{\alpha1}\theta+\partial_1\theta\partial_\alpha\chi]+O(\epsilon^2)\\ \delta_{\alpha2}-\epsilon^2x_3[\chi\partial_{\alpha2}\theta+\partial_2\theta\partial_\alpha\chi]+O(\epsilon^2)\\ \epsilon[\partial_\alpha\theta+x_3\partial_\alpha\chi]+O(\epsilon^3)\end{pmatrix} \tag{2.3.19}$$

$$g_3(\epsilon)=\begin{pmatrix}-\epsilon\chi\partial_1\theta+O(\epsilon^3)\\ -\epsilon\chi\partial_2\theta+O(\epsilon^3)\\ \chi+O(\epsilon^2)\end{pmatrix}, \tag{2.3.20}$$

$$g^\alpha(\epsilon)=\begin{pmatrix}\delta_{\alpha1}+O(\epsilon^2)\\ \delta_{\alpha2}+O(\epsilon^2)\\ \epsilon\partial_\alpha\theta+O(\epsilon^2)\end{pmatrix}, g^3(\epsilon)=\begin{pmatrix}-\epsilon\partial_1\theta+O(\epsilon^3)\\ -\epsilon\partial_2\theta+O(\epsilon^3)\\ 1+O(\epsilon^2)\end{pmatrix}, \tag{2.3.21}$$

$$g_{\alpha\beta}(\epsilon)=\delta_{\alpha\beta}+\epsilon^2[\partial_\alpha\theta\partial_\beta\theta-2x_3[\partial_{\alpha\beta}\theta+\partial_\alpha\theta\partial_\beta\chi]+O(\epsilon^4) \tag{2.3.22}$$

$$g_{\alpha3}(\epsilon)=O(\epsilon), g_{33}(\epsilon)=\chi^2+O(\epsilon^2), \tag{2.3.23}$$

$$\Gamma^\sigma_{\alpha\beta}(\epsilon)=O(\epsilon^2),\ \ \Gamma^3_{\alpha\beta}(\epsilon)=\frac{\epsilon}{\chi}[\partial_{\alpha\beta}\theta+x_3\partial_{\alpha\beta}\chi]+O(\epsilon^3),\ \ \Gamma^\sigma_{\alpha3}=O(\epsilon). \tag{2.3.24}$$

The announced results follows from the above relations. □

Lemma 2.3.2. Let $\theta \in C^3(\overline{\omega})$ be a given function and let the functions $\tilde{e}_{ij}(v)$ be defined as in (2.3.15). Then there exists a constant C such that

$$\|v\|_{1,\Omega} \leq C \left\{ \sum_{i,j} \|\tilde{e}_{ij}(v)\|^2 \right\}^{\frac{1}{2}} \ \forall\, v \in V(\Omega) \tag{2.3.25}$$

Proof. See the proof of Lemma 4.2 in [73]. □

We assume that there exists P^{kij} and $\in^{ij}$ such that

$$\sup_{0<\epsilon\leq\epsilon_0} \max_{x\in\Omega} |P^{kij}(\epsilon) - P^{kij}| \leq C\epsilon \tag{2.3.26}$$

$$\sup_{0<\epsilon\leq\epsilon_0} \max_{x\in\Omega} | \in^{ij}(\epsilon) - \in^{ij} | \leq C\epsilon \tag{2.3.27}$$

2.4 A priori estimates

In this section, we show that for each positive integer m, the scaled eigenvalues $\{\xi^m(\epsilon)\}$ are bounded uniformly with respect to ϵ.

Let $\varphi \in H_0^2(\omega)$. Then

$$v_\varphi := (-x_3\partial_1\varphi, -x_3\partial_2\varphi, \varphi) \in V(\Omega) \tag{2.4.1}$$

and

$$\tilde{e}_{\alpha\beta}(v_\varphi) = -x_3\partial_{\alpha\beta}\varphi - \frac{\varphi}{\chi}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}\chi), \quad \tilde{e}_{i3}(v_\varphi) = 0. \tag{2.4.2}$$

Hence

$$e_{\alpha||\beta}(\epsilon, v_\varphi) = -x_3\partial_{\alpha\beta}\varphi - \frac{\varphi}{\chi}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}\chi) + O(\epsilon^2), \tag{2.4.3}$$

$$e_{\alpha||3}(\epsilon, v_\varphi) = O(\epsilon),\ e_{3||3}(\epsilon, v_\varphi) = 0. \tag{2.4.4}$$

Lemma 2.4.1. There exists a constant $C > 0$ such that

$$|\epsilon\partial_\alpha(T_\chi(\epsilon)(v_\varphi))|_{0,\Omega} \leq C|\varphi|_{2,\omega}. \tag{2.4.5}$$

$$|\partial_3(T_\chi(\epsilon)(v_\varphi))|_{0,\Omega} \leq C|\varphi|_{2,\omega}. \tag{2.4.6}$$

Proof. With the scalings (2.3.3)-(2.3.7), the variational equation (2.2.37) posed on the domain Ω reads as follows. For each $h \in (H^1(\Omega))^3$, there exists a unique solution

$T_\chi(\epsilon)(h) \in (H^1(\Omega))^3$ such that

$$\begin{aligned}
&\int_\Omega \in^{33}(\epsilon)\partial_3 T_\chi(\epsilon)(h)\partial_3\psi\sqrt{g(\epsilon)}dx \\
&+\epsilon\int_\Omega[\in^{\alpha 3}(\epsilon)(\partial_\alpha T_\chi(\epsilon)(h)\partial_3\psi+\partial_3 T_\chi(\epsilon)(h)\partial_\alpha\psi)]\sqrt{g(\epsilon)}dx \\
&+\epsilon^2\int_\Omega \in^{\alpha\beta}(\epsilon)\partial_\alpha T_\chi(\epsilon)(h)\partial_\beta\psi\sqrt{g(\epsilon)}dx \\
&=\int_\Omega P^{3kl}(\epsilon)\partial_3\psi e_{k||l}(\epsilon,h)\sqrt{g(\epsilon)}dx+\epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\psi e_{k||l}(\epsilon,h)\sqrt{g(\epsilon)}dx. \qquad (2.4.7)
\end{aligned}$$

for all $\psi \in \Psi$.Taking $h = v_\varphi$ and $\psi = T_\chi(\epsilon)(v_\varphi)$ in the above equation, we have

$$\begin{aligned}
&\int_\Omega \in^{33}(\epsilon)\partial_3 T_\chi(\epsilon)(v_\varphi)\partial_3 T_\chi(\epsilon)(v_\varphi)\sqrt{g(\epsilon)}dx \\
&+\epsilon\int_\Omega[\in^{\alpha 3}(\epsilon)(\partial_\alpha T_\chi(\epsilon)(v_\varphi)\partial_3 T_\chi(\epsilon)(v_\varphi)+\partial_3 T_\chi(\epsilon)(v_\varphi)\partial_\alpha T_\chi(\epsilon)(v_\varphi))]\sqrt{g(\epsilon)}dx \\
&+\epsilon^2\int_\Omega \in^{\alpha\beta}(\epsilon)\partial_\alpha T_\chi(\epsilon)(v_\varphi)\partial_\beta T_\chi(\epsilon)(v_\varphi)\sqrt{g(\epsilon)}dx \\
&=\int_\Omega P^{3kl}(\epsilon)\partial_3 T_\chi(\epsilon)(v_\varphi)e_{k||l}(\epsilon,v_\varphi)\sqrt{g(\epsilon)}dx \\
&+\epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha T_\chi(\epsilon)(v_\varphi)e_{k||l}(\epsilon,v_\varphi)\sqrt{g(\epsilon)}dx. \qquad (2.4.8)
\end{aligned}$$

Using the relations (2.2.30) and (2.4.2)-(2.4.4), it follows that there exists a constant $C > 0$ such that

$$\begin{aligned}
&|\partial_3(T_\chi(\epsilon)(v_\varphi))|^2_{0,\Omega}+|\epsilon\partial_\alpha(T_\chi(\epsilon)(v_\varphi))|^2_{0,\Omega} \\
&\le C\{|\partial_3 T_\chi(\epsilon)(v_\varphi)|_{0,\Omega}|\varphi|_{2,\omega}+|\epsilon\partial_\alpha T_\chi(\epsilon)(v_\varphi)|_{0,\Omega}|\varphi|_{2,\omega}\} \qquad (2.4.9)
\end{aligned}$$

and hence the result follows. □

Theorem 2.4.2. For each positive integer m, there exists a constant $C(m) > 0$ such that

$$\xi^m(\epsilon) \le C(m) \qquad (2.4.10)$$

where the constant $C(m)$ depends only on m.

Proof. Let V_m denote the collection of all m-dimensional subspaces of V. Then the m^{th}

eigenvalue can be characterised by

$$\xi^m(\epsilon) = \min_{W \in V_m} \max_{v \in W} \frac{N_\chi(\epsilon)(v,v))}{D(\epsilon)(v,v)} \tag{2.4.11}$$

where

$$\begin{aligned} N_\chi(\epsilon)(v,v) = &\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon,v) e_{i||j}(\epsilon,v)\sqrt{g(\epsilon)}dx + \int_\Omega P^{3kl}(\epsilon)\partial_3 T_\chi(\epsilon)(v) e_{k||l}(\epsilon,v)\sqrt{g(\epsilon)}dx \\ &+ \epsilon \int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha T_\chi(\epsilon)(v) e_{k||l}(\epsilon,v)\sqrt{g(\epsilon)}dx. \qquad (2.4.12) \\ D(\epsilon)(v,v) = &\int_\Omega \left[\epsilon^2 v_\alpha v_\alpha + v_3 v_3\right]\sqrt{g(\epsilon)}dx. \qquad (2.4.13) \end{aligned}$$

Let W_m denote the collection of all $m-$ dimensional subspaces of $H_0^2(\omega)$. Let $W \in W_m$. For $\varphi \in W$, define

$$v_\varphi = (-x_1\partial_2\varphi, -x_2\partial_3\varphi, \varphi) \tag{2.4.14}$$

and

$$U = \{v_\varphi : \varphi \in W\}. \tag{2.4.15}$$

It follows that $U \in V_m$. Hence

$$\xi^m(\epsilon) \le \min_{U \in V_m} \max_{\varphi \in W - \{0\}} \frac{N_\chi(\epsilon)(v_\varphi, v_\varphi)}{D(\epsilon)(v_\varphi, v_\varphi)}. \tag{2.4.16}$$

From the definition of $A^{ijkl}(\epsilon)$ we have

$$\int_\Omega A^{ijkl}(\epsilon) e_{i||j}(\epsilon, v_\varphi) e_{k||l}(\epsilon, v_\varphi)\sqrt{g(\epsilon)}dx \le C\Sigma_{i,j}||e_{i||j}(\epsilon, v_\varphi)||^2_{0,\Omega}. \tag{2.4.17}$$

But

$$\begin{aligned} ||e_{\alpha||\beta}(\epsilon, v_\varphi)||^2_{0,\Omega} &\le C||\tilde{e}_{\alpha\beta}(v_\varphi)||^2 + C\epsilon^2||e^\sharp_{\alpha||\beta}(\epsilon; v_\varphi)||^2_{0,\Omega} \\ &\le C||\triangle\varphi||^2_{0,\omega}, \qquad (2.4.18) \\ ||e_{\alpha||3}(\epsilon, v_\varphi)||^2_{0,\Omega} &\le C||\tilde{e}_{\alpha 3}(v_\varphi)||^2 + \le C\epsilon^2||e^\sharp_{\alpha||3}(\epsilon; v_\varphi)||^2_{0,\Omega} \\ &\le C||\triangle\varphi||^2_{0,\omega}, \qquad (2.4.19) \\ ||e_{3||3}(\epsilon, v_\varphi)||^2_{0,\Omega} &= 0. \qquad (2.4.20) \end{aligned}$$

It follows from the definition of v_φ that there exists a constant C such that

$$\int_\Omega [\epsilon^2 (v_\varphi)^2_\alpha + (v_\varphi)^2_3]\sqrt{g(\epsilon)}dx \geq C \int_\omega \varphi^2 d\omega. \tag{2.4.21}$$

Combining relations (2.4.5)-(2.4.6) and (2.4.16)-(2.4.20) we get

$$\xi^m(\epsilon) \leq C \min_{U\in W_m} \max_{\varphi\in W-\{0\}} \frac{\int_\omega |\triangle\varphi|^2 d\omega}{\int_\omega \varphi^2 d\omega}. \tag{2.4.22}$$

But the expression on the right hand side of the above relation gives the m-th eigenvalue of the two dimensional problem

$$\left.\begin{array}{c} \triangle^2 u = \lambda u \;\; in \;\; \omega \\ u = \partial_\nu u = 0 \;\; on \;\; \partial\omega \end{array}\right\} \tag{2.4.23}$$

This completes the proof by setting $C(m) = C\lambda^m$. □

2.5 Limit Problem

Theorem 2.5.1. a) For each positive integer m, there exists $u^m \in H^1(\Omega), \varphi^m \in L^2(\Omega)$ and $\xi^m \in I\!R$ such that

$$u^m(\epsilon) \to u^m \text{ strongly in } (H^1(\Omega))^3, \quad \varphi^m(\epsilon) \to \varphi^m \text{ strongly in } L^2(\Omega), \tag{2.5.1}$$

$$(\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \to (0, 0, \partial_3\varphi^m) \text{ strongly in } (L^2(\Omega))^3, \tag{2.5.2}$$

$$\xi^m(\epsilon) \to \xi^m. \tag{2.5.3}$$

b) Define the spaces

$$V_H(\omega) = \{(\eta_\alpha) \in (H^1(\omega))^2; \eta_\alpha = 0 \text{ on } \gamma_0\}, \tag{2.5.4}$$

$$V_3(\omega) = \{\eta_3 \in H^2(\omega); \eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}, \tag{2.5.5}$$

$$V_{KL} = \left\{v \in H^1(\Omega) | v = \eta_\alpha - x_3\partial_\alpha\eta_3, (\eta_i) \in V_H(\omega) \times V_3(\omega)\right\}. \tag{2.5.6}$$

Then there exists $(\zeta_\alpha^m, \zeta_3^m) \in V_H \times V_3(\omega)$ such that

$$u_\alpha^m = \zeta_\alpha^m - x_3 \partial_\alpha \zeta_3^m \text{ and } u_3^m = \zeta_3^m. \tag{2.5.7}$$

$$\varphi^m = (1 - x^2) \frac{p^{3\alpha\beta}}{p^{33}} \partial_{\alpha\beta} \zeta_3^m, \tag{2.5.8}$$

and $(\zeta^m, \xi^m) \in V_H \times V_3 \times I\!R$ satisfies

$$\begin{aligned} -\int_\omega m_{\alpha\beta}(\zeta^m) \partial_{\alpha\beta} \eta_3 \chi d\omega + \int_\omega [n_{\alpha\beta}^\theta(\zeta^m) \partial_{\alpha\beta} \theta + m_{\alpha\beta}(\zeta^m) \partial_{\alpha\beta} \chi] \eta_3 \chi d\omega \\ + \frac{2}{3} \int_\omega \frac{p^{3\alpha\beta} p^{3\rho\tau}}{p^{33}} \partial_{\rho\tau} \zeta_3^m \partial_{\alpha\beta} \eta_3 \chi d\omega = \xi^m \int_\omega \zeta_3^m \eta_3 \chi d\omega \ \ \forall \, \eta_3 \in V_3, \end{aligned} \tag{2.5.9}$$

$$\int_\omega n_{\alpha\beta}^\theta \partial_\beta \eta_\alpha \chi d\omega = 0 \ \ \forall \, \eta_\alpha \in V_H, \tag{2.5.10}$$

where

$$m_{\alpha\beta}(\zeta) = -\left\{ \frac{4\lambda\mu}{3(\lambda + 4\mu)} \triangle \zeta_3 \delta_{\alpha\beta} + \frac{4\mu}{3} \left(\partial_{\alpha\beta} \zeta_3 + \zeta_3 \frac{\partial_{\alpha\beta} \chi}{\chi} \right) \right\}, \tag{2.5.11}$$

$$n_{\alpha\beta}^\theta(\zeta) = \frac{4\lambda\mu}{\lambda + 2\mu} \tilde{e}_{\sigma\sigma}(\zeta) \delta_{\alpha\beta} + 4\mu \tilde{e}_{\alpha\beta}(\zeta), \tag{2.5.12}$$

$$p^{33} = \frac{1}{\mu} P^{3\alpha 3} P^{3\alpha 3} + \frac{1}{\lambda + 2\mu} P^{333} P^{333} + \in^{33}, \tag{2.5.13}$$

$$p^{3\alpha\beta} = P^{3\alpha\beta} - \frac{\lambda}{\lambda + 2\mu} P^{333} \delta^{\alpha\beta}. \tag{2.5.14}$$

Proof. For the sake of clarity, the proof is divided into several steps.
Step (i): Define the vector $\tilde{\varphi}_i^m(\epsilon)$ and the tensor $\tilde{K}^m(\epsilon) = (\tilde{K}_{ij}^m(\epsilon))$ by

$$\tilde{\varphi}_i^m(\epsilon) = (\epsilon \partial_1 \varphi^m(\epsilon), \epsilon \partial_2 \varphi^m(\epsilon), \partial_3 \varphi^m(\epsilon)). \tag{2.5.15}$$

,

$$\tilde{K}_{\alpha\beta}^m(\epsilon) = \tilde{e}_{\alpha\beta}(u^m(\epsilon)), \tilde{K}_{\alpha 3}^m(\epsilon) = \frac{1}{\epsilon} \tilde{e}_{\alpha 3}(u^m(\epsilon)), \tilde{K}_{33}^m(\epsilon) = \frac{1}{\epsilon^2} \tilde{e}_{33}(u^m(\epsilon)). \tag{2.5.16}$$

Claim: There exists constant $C > 0$ and ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$,

$$||u^m(\epsilon)||_{1,\Omega} \leq C, |\tilde{K}^m_{ij}(\epsilon)|_{0,\Omega} \leq C, |\tilde{\varphi}^m_i(\epsilon)| \leq C \tag{2.5.17}$$

for all $0 < \epsilon \leq \epsilon_0$. Letting $(v, \psi) = (u^m(\epsilon), \varphi^m(\epsilon))$ in (2.3.11), we have

$$\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon, u^m(\epsilon)) e_{i||j}(\epsilon, u^m(\epsilon)) \sqrt{g(\epsilon)} dx + \int_\Omega \in^{ij}(\epsilon) \tilde{\varphi}^m_i(\epsilon) \tilde{\varphi}^m_j(\epsilon) \sqrt{g(\epsilon)} dx$$
$$= \xi^m(\epsilon) \int_\Omega [\epsilon^2 u^m_\alpha(\epsilon).u^m_\alpha(\epsilon) + u^m_3(\epsilon) u^m_3(\epsilon)] \sqrt{g(\epsilon)} dx. \tag{2.5.18}$$

Also, using the coerciveness properties (2.2.29),(2.2.30), and the inequality (2.3.25) we have for $0 < \epsilon < \{\min \epsilon_0, 1\}$,

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon, u^m(\varepsilon)) e_{i||j}(\varepsilon, u^m(\varepsilon)) \sqrt{g(\varepsilon)} dx + \int_\Omega \in^{ij}(\varepsilon) \tilde{\varphi}^m_i(\varepsilon) \tilde{\varphi}^m_j(\varepsilon) \sqrt{g(\varepsilon)} dx \\
&\geq C \sum_{i,j} \left\| e_{i||j}(\varepsilon, u^m(\varepsilon)) \right\|^2_{0,\Omega} + C \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&= C \sum_{\alpha,\beta} \left\| \tilde{e}_{\alpha\beta}(u^m(\varepsilon)) + \varepsilon^2 e^\sharp_{\alpha\beta}(\varepsilon, u^m(\varepsilon)) \right\|^2_{0,\Omega} \\
&+ C_1 \sum_\alpha \left\| \frac{1}{\varepsilon} \tilde{e}_{\alpha 3}(u^m(\varepsilon)) + \varepsilon e^\sharp_{\alpha 3}(\varepsilon, u^m(\varepsilon)) \right\|^2_{0,\Omega} \\
&+ C \left\| \frac{1}{\varepsilon^2} \tilde{e}_{33}(u^m(\varepsilon)) \right\|^2_{0,\Omega} + C \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&\geq C \left\{ \frac{1}{2} \sum_{i,j} \|\tilde{e}_{ij}(u^m(\varepsilon))|^2_{0,\Omega} - 3\varepsilon^3 C_2 \|u^m(\varepsilon)\|^2_{1,\Omega} \right\} + C \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&\geq C_3 \|u^m(\varepsilon)\|^2_{1,\Omega} + C_4 \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&\geq C_5 \left(\|u^m(\varepsilon)\|^2_{0\Omega} + \|\tilde{\varphi}^m(\varepsilon)\|^2_{0,\Omega} \right).
\end{aligned} \tag{2.5.19}$$

Combining equations (2.5.19) with the relations (2.3.14) we get the relation (2.5.17).
Step (ii): Claim: There exists functions $\varphi^m \in L^2(\Omega)$ such that

$$(\epsilon\partial_1 \varphi^m(\epsilon), \epsilon\partial_2 \varphi^m(\epsilon), \partial_3 \varphi^m(\epsilon)) \rightharpoonup (0, 0, \partial_3 \varphi^m) \quad \text{weakly in} \quad (L^2(\Omega))^3 \quad \text{as } \epsilon \to 0. \tag{2.5.20}$$

Since $\tilde{\varphi}^m(\epsilon) = (\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon))$ is bounded, there exists $\tilde{\varphi}^m$ such that

$$\tilde{\varphi}^m(\epsilon) = (\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \rightharpoonup \tilde{\varphi}^m = (\tilde{\varphi}_1^m, \tilde{\varphi}_2^m, \tilde{\varphi}_3^m) \text{ weakly in } (L^2(\Omega))^3. \tag{2.5.21}$$

Now

$$\varphi^m(\epsilon)(x_1, x_2, x_3) = \int_{-1}^{x_3} \partial_3\varphi^m(\epsilon)(x_1, x_2, x_3)ds \tag{2.5.22}$$

and hence

$$\|\varphi^m(\epsilon)\|_{0,\Omega} \leq \sqrt{2}\|\partial_3\varphi^m(\epsilon)\|_{0,\Omega} \leq C. \tag{2.5.23}$$

Hence there exists φ^m in $L^2(\Omega)$ such that $\varphi^m(\epsilon) \rightharpoonup \varphi^m$ weakly and therefore

$$(\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \rightharpoonup (0, 0, \partial_3\varphi^m) \quad \text{weakly in} \quad (L^2(\Omega))^3 \quad \text{as } \epsilon \to 0. \tag{2.5.24}$$

Step (iii): From step (i) it follows that there exists a subsequence, indexed by ϵ for notational convenience, and functions $u^m \in V(\Omega)$ and $\tilde{K}_{ij}^m \in (L^2(\Omega))^9$ such that

$$u^m(\epsilon) \rightharpoonup u^m \text{ weakly in } H^1(\Omega), \quad \tilde{K}^m(\epsilon) \rightharpoonup \tilde{K}^m \text{ weakly in } L^2(\Omega), \quad \text{as } \epsilon \to 0, \tag{2.5.25}$$

Claim: There exist functions $(\zeta_\alpha^m) \in H^1(\omega)$ and $\zeta_3^m \in H^2(\omega)$ satisfying $\zeta_i^m = \partial_\nu\zeta_3^m = 0$ on γ_0 such that

$$u_\alpha^m = \zeta_\alpha^m - x_3\partial_\alpha\zeta_3^m, \quad u_3^m = \zeta_3^m, \tag{2.5.26}$$

and

$$\tilde{K}_{\alpha\beta}^m = \tilde{e}_{\alpha\beta}(u^m), \;\; \tilde{K}_{\alpha 3}^m = -\frac{\chi}{\mu}P^{3\alpha 3}\partial_3\varphi^m, \;\; \tilde{K}_{33}^m = -\frac{\chi^2}{\lambda + 2\mu}(\chi^2 P^{333}\partial_3\varphi^m + \lambda\tilde{K}_{\beta\beta}^m). \tag{2.5.27}$$

Since $u(\varepsilon) \rightharpoonup u$ weakly in $H^1(\Omega)$, the definition (2.3.15) of the functions $\tilde{e}_{\alpha\beta}(v)$ shows that the function $\tilde{K}_{\alpha\beta}(\varepsilon) = \tilde{e}_{\alpha\beta}(u(\varepsilon))$ converges weakly in $L^2(\Omega)$ to the function $\tilde{e}_{\alpha\beta}(u)$.

We next recall the following result(cf. [22]). Let $w \in L^2(\Omega)$ be given then

$$\int_\Omega w\partial_3 v dx = 0 \text{ for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0, \text{ then } w = 0. \tag{2.5.28}$$

The equation (2.3.11) - (2.3.13) can be written as

$$\int_\Omega \Big(\Big\{ \Big[A^{\alpha\beta\sigma\tau}(0) + \varepsilon^2 A_\sharp^{\alpha\beta\sigma\tau}(\varepsilon)\Big] \Big[\tilde{K}_{\sigma\tau}^m(\varepsilon) + \varepsilon^2 e_{\sigma\tau}^\sharp(\varepsilon; u(\varepsilon))\Big] + \Big[A^{\alpha\beta 33}(0) + \varepsilon^2 A_\sharp^{\alpha\beta 33}(\varepsilon)\Big] \tilde{k}_{33}^m(\varepsilon) \Big\}$$

$$\left\{\frac{1}{2}\partial_\alpha v_\beta + \frac{1}{2}\partial_\beta v_\alpha - \frac{v_3}{\chi}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}\chi) + \varepsilon^2 e^\sharp_{\alpha\beta}(\varepsilon; v)\right\}$$
$$+ \{4[A^{\alpha 3\sigma 3}(0) + \varepsilon^2 A^{\alpha 3\sigma 3}_\sharp(\varepsilon)][\tilde{K}^m(\varepsilon)_{\sigma 3} + \varepsilon e^\sharp_{\sigma 3}(\varepsilon; u(\varepsilon))]\}$$
$$\left\{\frac{1}{2\varepsilon}\partial_\alpha v_3 + \frac{1}{2\varepsilon}\partial_3 v_\alpha + \varepsilon e^\sharp_{\alpha 3}(\varepsilon; v)\right\} + \left\{\left[A^{33\sigma\tau}(0) + \varepsilon^2 A^{33\sigma\tau}_\sharp(\varepsilon)\right]\left[\tilde{K}^m_{\sigma\tau}(\varepsilon) + \varepsilon^2 e^\sharp_{\sigma\tau}(\varepsilon; u(\varepsilon))\right]\right.$$
$$\left.+ \left[A^{3333}(0) + \varepsilon^2 A^{3333}_\sharp(\varepsilon)\right]\tilde{K}^m_{33}(\varepsilon)\right\}\left\{\frac{1}{\varepsilon^2}\partial_3 v_3\right\}\Bigg)\sqrt{\chi^2 + \varepsilon^2 g^\sharp(\varepsilon)}\, dx$$
$$+ \int_\Omega \in^{33}(\varepsilon)\partial_3\varphi^m(\varepsilon)\partial_3\psi\sqrt{g(\varepsilon)}dx + \int_\Omega P^{3kl}\left[\partial_3\varphi^m(\varepsilon)e_{k||l}(\varepsilon, v) - \partial_3\psi e_{k||l}(\varepsilon, u(\varepsilon))\right]\sqrt{g(\varepsilon)}dx$$
$$+ \varepsilon\int_\Omega \in^{3\alpha}(\varepsilon)\left[\partial_\alpha\varphi^m(\varepsilon)\partial_3\psi + \partial_3\varphi^m(\varepsilon)\partial_\alpha\psi)\right]\sqrt{g(\varepsilon)}dx + \varepsilon^2\int_\Omega \in^{\alpha\beta}(\varepsilon)\partial_\alpha\varphi^m(\varepsilon)\partial_\beta\psi\sqrt{g(\varepsilon)}dx$$
$$+ \varepsilon\int_\Omega \left[P^{\alpha kl}(\varepsilon)\partial_\alpha\varphi^m(\varepsilon)e_{k||l}(\varepsilon, u(\varepsilon)) - \partial_\alpha\psi e_{k||l}(\varepsilon, v)\right]\sqrt{g(\varepsilon)}dx$$
$$= \xi^m(\epsilon)\int_\Omega u^m_i v_i\sqrt{\chi^2 + \varepsilon^2 g^\sharp(\varepsilon)}\, dx\ \forall\, v \in V(\Omega). \tag{2.5.29}$$

Multiplying the above equation by ε^2, taking $v_\alpha = 0$ and letting $\varepsilon \to 0$, we get

$$\int_\Omega \left[\frac{\lambda}{\chi^2}\tilde{K}_{\sigma\sigma} + \frac{(\lambda + 2\mu)}{\chi^4}\tilde{K}_{33} + P^{333}\partial_3\varphi\right]\partial_3 v_3\chi dx = 0\ \forall\, v_3 \in H^1(\Omega), v_3 = 0 \text{ in } \Gamma_0. \tag{2.5.30}$$

which implies $\chi^2\lambda\tilde{K}_{\sigma\sigma} + (\lambda + 2\mu)\tilde{K}_{33} + \chi^4 P^{333}\partial_3\varphi = 0$ and hence the third relation in (2.5.27) follows. Again, multiplying equation (2.5.29) by ε, taking $v_3 = 0$ and letting $\varepsilon \to 0$, we get

$$\int_\Omega \left[\frac{\mu}{\chi}\tilde{K}_{\alpha 3} + P^{3\alpha 3}\partial_3\varphi\right]\partial_3 v_\alpha dx = 0\ \forall\, v_\alpha \in (H^1(\Omega))^2, v_\alpha = 0 \text{ in } \Gamma_0. \tag{2.5.31}$$

which implies $(\mu\tilde{K}_{\alpha 3} + \chi P^{3\alpha 3}\partial_3\varphi) = 0$ and hence the second relation in (2.5.27) follows.
Step iv: The function φ^m is of the form,

$$\varphi^m = (1 - x^2)\frac{p^{3\alpha\beta}}{p^{33}}\partial_{\alpha\beta}\xi^m_3. \tag{2.5.32}$$

Letting $\epsilon \to 0$ in (2.3.12), we get

$$\int_\Omega \left(P^{3kl}\tilde{K}^m_{kl} - \in^{33}\partial_3\varphi^m\right)\partial_3\psi\chi dw = 0\ \forall\, \psi \in \Psi \tag{2.5.33}$$

This implies

$$\partial_3(P^{3kl}\tilde{K}^m_{kl} - \in^{33}\partial_3\varphi^m) = 0 \text{ in } \mathcal{D}'(\Omega) \tag{2.5.34}$$

Hence

$$P^{3kl}\tilde{K}^m_{kl} - \in^{33} \partial_3\varphi^m = d^1 \text{ with } d^1 \in \mathcal{D}'(\omega). \tag{2.5.35}$$

Using the expression for $\tilde{K}^m_{ij}$ given by (2.5.27) the above rquation reduces to

$$p^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - p^{33}\partial_3\varphi^m = d^1 \tag{2.5.36}$$

which implies

$$\partial_3\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}\tilde{K}^m_{\alpha\beta} - \frac{d^1}{p^{33}} \tag{2.5.37}$$

Then

$$\partial_3\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}\left[\hat{e}_{\alpha\beta}(\zeta) - x_3\left(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta}\chi}{\chi}\zeta_3\right)\right] - \frac{1}{p^{33}}d^1. \tag{2.5.38}$$

which gives

$$\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}\left[x_3\hat{e}_{\alpha\beta}(\zeta) - x_3^2\left(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta}\chi}{\chi}\zeta_3\right)\right] - \frac{x_3}{p^{33}}d^1 + d^0. \tag{2.5.39}$$

Since φ satisfies the boundary conditions $\varphi|_{\Gamma^+} = 0,\ \ \varphi|_{\Gamma^-} = 0$, it follows that φ^m is of the form (2.5.32).

Step (v): The function (ζ_i^m) satisfies (2.5.9)- (2.5.10).

Taking $v \in V_{KL}$ and letting $\epsilon \to 0$ in equation (2.3.11) we have

$$\int_\Omega A^{\alpha\beta kl}\tilde{K}^m_{kl}\tilde{K}^m_{\alpha\beta}(v)\chi dx + \int_\Omega \in^{33} \partial_3\varphi^m\partial_3\psi\chi dx + \int_\Omega P^{3\alpha\beta}\partial_3\varphi^m\tilde{K}^m_{\alpha\beta}(v)\chi dx$$
$$- \int_\Omega P^{3kl}\partial_3\psi\tilde{K}^m_{kl}\chi dx = \xi^m\int_\Omega u_3^m v_3\chi dx. \tag{2.5.40}$$

Replacing u^m and $\tilde{K}^m_{ij}$ by the expressions obtained in (2.5.26) and (2.5.27), and taking v of the form

$$v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3 \text{ and } v_3 = \eta_3 \tag{2.5.41}$$

with $(\eta_i) \in V_H \times V_3$, it is verified that equation (2.5.40) coincide with equation (2.5.9)-(2.5.10). It can be proved as in [76] that the convergences $u^m(\epsilon) \rightharpoonup u^m$ in $H^1(\Omega)$ and $\varphi^m(\epsilon) \rightharpoonup \varphi^m$ in $L^2(\Omega)$ are strong. □

Lemma 2.5.2. For a given $\zeta_3 \in H_0^2(\omega)$ there exists a unique vector $\zeta_\alpha \in (H_0^1(\omega))^2$ such that

$$\int_\omega n^\theta_{\alpha\beta}(\zeta)\partial_\beta\eta_\alpha\chi d\omega = 0 \ \forall\ (\eta_\alpha) \in (H_0^1(\omega))^2 \tag{2.5.42}$$

Proof. The equation (2.5.42) can be written as

$$\int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu}e_{\rho\rho}(\zeta)\delta_{\alpha\beta}+2\mu e_{\alpha\beta}(\zeta)\right]\partial_\beta\eta_\alpha\chi d\omega=\int_\omega\left[\frac{2\lambda\mu}{\lambda+2\mu}(\partial_{\sigma\sigma}\theta)\delta_{\alpha\beta}+\mu\partial_{\alpha\beta}\theta\zeta_3\right]\partial_\beta\eta_\alpha\chi d\omega.$$

Clearly, the bilinear form

$$\tilde{b}(\zeta_\alpha,\eta_\alpha)=\int_\omega[\frac{2\lambda\mu}{\lambda+2\mu}e_{\rho\rho}(\zeta)\delta_{\alpha\beta}+2\mu e_{\alpha\beta}(\zeta)]\partial_\beta\eta_\alpha\chi d\omega \tag{2.5.43}$$

is V_H elliptic and symmetric. Hence by Lax- Milgram lemma given $f_\alpha\in(H^{-1}(\omega))^2$ there exists a unique $\zeta_\alpha\in(H_0^1(\omega))^2$ such that $\tilde{b}(\zeta_\alpha,\eta_\alpha)=\langle f_\alpha,\eta_\alpha\rangle$. The result follows by setting

$$\langle\zeta_\alpha,\eta_\alpha\rangle=\int_\omega[\frac{2\lambda\mu}{\lambda+2\mu}(\partial_{\sigma\sigma}\theta)\delta_{\alpha\beta}+\mu\partial_{\alpha\beta}\theta\zeta_3]\partial_\beta\eta_\alpha\chi d\omega \tag{2.5.44}$$

□

Thus, given $\zeta_3\in V_3$, we denote by $T_\chi\zeta_3$ the vector (ζ_α,ζ_3), where (ζ_α) is the solution of (2.5.43). In particular, $T_\chi\zeta_3^m=(\zeta_\alpha^m,\zeta_3^m)$. Substituting this in equation (2.5.9), we have

$$b(\zeta_3^m,\eta_3)=\xi^m\int_\omega\zeta^m\eta_3\chi d\omega \text{ for all } \eta_3\in V_3 \tag{2.5.45}$$

where

$$\begin{aligned} b(\zeta_3,\eta_3) &= -\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3\chi d\omega+\int_\omega\left[n^\theta_{\alpha\beta}(T\zeta_3)\partial_{\alpha\beta}\theta+m_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\chi\right]\eta_3\chi d\omega \\ &+ \frac{2}{3}\int_\omega\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\partial_{\alpha\beta}\eta_3\chi d\omega. \end{aligned} \tag{2.5.46}$$

The bilinear form $b(.,.)$ defined by (2.5.46) is V_H-elliptic and symmetric (cf. [73]) Hence there exists sequence of eigensolutions for the problem (2.5.45) . The injection $H_0^2(\omega)\hookrightarrow L^2(\omega)$ is compact and so we have a sequence of eigenvalues tending to infinity and eigenvectors which form an orthonormal basis of $L^2(\omega)$.

Theorem 2.5.3. Let $\xi^l(\epsilon)\to\xi^l$ and let $u^l(\epsilon)\to u^l$ in V. Then ξ^l is the l - th eigenvalue of the problem(2.5.46) and $\{u_3{}^l\}$ is *an orthogonal basis for* $L^2(\omega)$. *Thus, all the eigenvalues and eigenvectors of the limit problem are obtained as limits of* $\{(\xi^l(\epsilon),u^l(\epsilon))\}_{l=1}^\infty$.

Proof. From (2.2.43) we have

$$0<\xi^1(\epsilon)\le\xi^2(\epsilon)\le\cdots\le\xi^l(\epsilon)\le\xi^{l+1}(\epsilon)\le\cdots\to\infty \text{ as } l\to\infty$$

and since $b(.,.)$ is elliptic, it follows that

$$0 < \xi^1 \leq \xi^2 \leq \cdots \leq \xi^l \leq \xi^{l+1} \leq \cdots \to \infty \text{ as } l \to$$

Further passing to the limit in relation (2.3.13) gives

$$\int_\Omega {u_3}^l {u_3}^m \chi dx = \delta_{lm}$$

. That is

$$\int_\omega {u_3}^l {u_3}^m \chi \, d\omega = \frac{1}{2}\delta_{lm}. \tag{2.5.47}$$

Claim: There are no other eigenvalues of the limit problem.
Assume the contrary. Let $\xi \in$ be an eigenvalue such that $\xi \neq \xi^l$ for all l. Then there exists an eigenfunction ζ_3 such that

$$\int_\omega {\zeta_3}^2 \chi d\omega = \frac{1}{2} \quad \text{and} \quad \int_\omega \zeta_3 \zeta_3^l \chi \, d\omega = 0. \tag{2.5.48}$$

For each $\epsilon > 0$, let $w(\epsilon) \in V$ be the unique solution of the problem

$$N_\chi(\epsilon)(w(\epsilon), v) = \xi \int_\Omega \zeta_3 v_3 \chi dx \tag{2.5.49}$$

for all $v \in V$. Then proceeding as in Theorem 2.5.1, we can show that $w(\epsilon) \to w$ in V and that $w_\alpha = z_\alpha - x_3 \partial_\alpha z_3$ and that $w_3 = z_3 \in {H_0}^2(\omega)$. Further, if $z = (z_\alpha, z_3)$, then $z = T_\chi z_3$ and z_3 will be the solution of

$$b(z_3, \eta_3) = \xi \int_\omega \zeta_3 \eta_3 \chi d\omega \tag{2.5.50}$$

for all $\eta_3 \in {H_0}^2(\omega)$. By the uniqueness of the solution if follows that $z_3 = \zeta_3$. Since the sequence $\{\xi^l\}$ is unbounded, we can choose l such that

$$\xi < \xi^l. \tag{2.5.51}$$

Consider the vector

$$v(\epsilon) = w(\epsilon) - \sum_{k=1}^{l} D(\epsilon)(w(\epsilon), u^k(\epsilon)) u^k(\epsilon).$$

Since $D(\epsilon)(v(\epsilon), u^k(\epsilon)) = 0$ for $1 \le k \le l$, it follows from the variational characterization of the eigenvalues, that

$$\xi^{l+1}(\epsilon) \quad \le \quad \frac{N_\chi \epsilon(v(\epsilon), v(\epsilon))}{D(\epsilon)(v(\epsilon), v(\epsilon))}. \tag{2.5.52}$$

Now

$$\begin{aligned} N_\chi(\epsilon)(w(\epsilon), w(\epsilon)) &= \xi \int_\Omega \zeta_3 \omega_3(\epsilon) \sqrt{g(\epsilon)} dx \\ &\to 2\xi \int_\omega \zeta_3^2 \chi d\omega. \end{aligned} \tag{2.5.53}$$

$$\begin{aligned} N_\chi(\epsilon)(w(\epsilon), u^k(\epsilon)) &= \xi^k(\epsilon) D(\epsilon)(w(k), u^k(\epsilon)) \\ &\to 0 \end{aligned} \tag{2.5.54}$$

$$N_\chi(\epsilon)(u^k(\epsilon), u^m(\epsilon)) = \xi^k(\epsilon)\delta_{km} = 0 \quad \text{for } k \ne m. \tag{2.5.55}$$

$$\begin{aligned} D(\epsilon)(w(k), u^k(\epsilon)) &= \int_\Omega \left[\epsilon^2 w_\alpha(\epsilon) u_\alpha^k(\epsilon) + w_3(\epsilon) u_3^k(\epsilon)\right] \sqrt{g(\epsilon)} dx \\ &\to 2 \int_\omega \zeta_3 \zeta_3^k \chi d\omega = 0 \end{aligned} \tag{2.5.56}$$

Thus we get

$$N_\chi(\epsilon)(v(\epsilon), v(\epsilon)) \to 2\xi \int_\omega \zeta_3^2 d\omega. \tag{2.5.57}$$

Also

$$v(\epsilon) - w(\epsilon) \to 0 \text{ in } V(\Omega) \tag{2.5.58}$$

Hence

$$\lim_{\epsilon \to 0} D(\epsilon)(v(\epsilon), v(\epsilon))) = 2 \int_\omega \zeta_3^2 d\omega \tag{2.5.59}$$

Passing to the limit in (2.5.52) we get

$$\xi^{l+1} \le \xi$$

which is a contradiction. □

2.6 Conclusion

Starting with the three dimensional model of eigenvalue problem for piezoelectric shallow shell with non-uniform thickness, we obtained a two dimensional model involving the nonuniform function χ.

One difference between the two dimensional static model and eigenvalue problem is that in the latter case it is possible to express it involving only the third component of the eigenvector.

Chapter 3

Lower Dimensional Approximation of Eigenvalue Problem For Piezoelectric Flexural Shells with Nonuniform Thickness

3.1 Introduction

In this chapter, we study the limiting behaviour of eigensolutions, describing the vibrations of a thin piezoelectric flexural shell (ie. the space of inextensional displacements is nontrivial), clamped along a portion of its lateral surface, as thickness of the shell approaches to zero.
For all $\eta = (\eta_i) \in (H^1(\omega))^2 \times H^2(\omega)$, define

$$\gamma_{\alpha\beta}(\eta) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - \frac{1}{\chi} b_{\alpha\beta}\eta_3. \tag{3.1.1}$$

Define the space of inextensional displacements by

$$V_F(\omega) = \{\eta = (\eta_i) \in (H^1(\omega))^2 \times H^2(\omega) | \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\eta) = 0 \text{ in } \omega\}. \tag{3.1.2}$$

We assume henceforth that $V_F(\omega)$ is infinite dimensional. For instance if the middle surface is a plate or if it is flat in a small region $\omega' \subset \omega$ so that $b_{\alpha\beta=0}$ in that region and hence functions of the form $(0, 0, \varphi) \in V_F(\omega)$ where $\phi \in \mathcal{D}(\omega)$ and hence the space is infinite dimensional.

We show that the eigenvalues are of order $o(\epsilon^2)$ and the corresponding scaled eigensolutions converge to the eigensolutions of the limit problem occur this way.

This chapter is organized as follows. In section 3.2 we transform the problem to a scaled domain, in section 3.3 we derive a priori estimates for eigenvalues and in section 3.4 we study the limiting behaviour of the eigensolutions.

3.2 The Scaled Three Dimensional Problem

We now make a change of variable so that the domain no longer depends on ϵ. With $x = (x_1, x_2, x_3) \in \Omega = \omega \times (-1, 1)$ we associate $x^\epsilon = (x_1, x_2, \epsilon x_3) \in \omega \times (-\epsilon, \epsilon)$. We make the following assumptions on the unknowns.

$$u_i^{m,\epsilon}(x^\epsilon) = u_i^m(\epsilon)(x), \quad v_i(x^\epsilon) = v_i(x), \tag{3.2.3}$$

$$\xi^{m,\epsilon} = \epsilon^2 \xi^m(\epsilon). \tag{3.2.4}$$

Then the eigenvalue problem (2.2.41)-(2.2.42) becomes: find $(u^m(\epsilon), \xi^m(\epsilon)) \in V(\Omega) \times R$ such that

$$\begin{aligned}&\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(u^m(\epsilon)) e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}dx + \frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3\varphi^m(\epsilon) e_{k||l}(\epsilon)(v)\sqrt{g(\epsilon)}dx \\ &+ \int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon) e_{k||l}(\epsilon)(v)\sqrt{g(\epsilon)}dx = \epsilon^2\xi^m(\epsilon)\int_\Omega u_i^m(\epsilon) v_i\sqrt{g(\epsilon)}dx \;\forall\, v \in V.\end{aligned} \tag{3.2.5}$$

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_\Omega \in^{33}(\epsilon)\partial_3\varphi^m(\epsilon)\partial_3\psi\sqrt{g(\epsilon)}dx + \frac{1}{\epsilon}\int_\Omega [\in^{3\alpha}(\epsilon)(\partial_\alpha\varphi^m(\epsilon)\partial_3\psi + \partial_3\varphi^m(\epsilon)\partial_\alpha\psi)]\sqrt{g(\epsilon)}dx \\ &+ \int_\Omega \in^{\alpha\beta}(\epsilon)\partial_\alpha\varphi^m(\epsilon)\partial_\beta\psi\sqrt{g(\epsilon)}dx = \frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3\psi e_{k||l}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}dx \\ &+ \int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\psi e_{k||l}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}dx \;\; \forall\psi \in \Psi,\end{aligned} \tag{3.2.6}$$

$$\int_\Omega u_i^m(\epsilon) u_i^n(\epsilon)\sqrt{g(\epsilon)}dx = \delta_{mn}. \tag{3.2.7}$$

For $v \in V(\Omega)$, define

$$\begin{aligned}\rho_{\alpha\beta}(v) &= v_{3|\alpha\beta} + \chi b_\alpha^\sigma v_{\sigma|\beta} + \chi b_\beta^\sigma v_{\sigma|\alpha} + \chi b_{\alpha|\beta}^\sigma v_\sigma - \left(c_{\alpha\beta} + \frac{1}{\chi} e_{|\alpha\beta}\right) v_3 \\ &+ \frac{2}{\chi^2}\partial_\alpha\chi\partial_\beta\chi v_3 - \frac{1}{\chi}\partial_\sigma\chi\partial_\beta v_3 - \frac{1}{\chi}\partial_\beta\chi\partial_\alpha v_3\end{aligned} \tag{3.2.8}$$

where

$$v_{\alpha|\beta} = \partial_\alpha v_\beta - \Gamma^\rho_{\sigma\alpha} v_\rho, \quad v_{|\alpha\beta} = \partial_{\alpha\beta} v - \Gamma^\rho_{\alpha\beta}\partial_\rho v. \tag{3.2.9}$$

and

$$\begin{aligned} e^1_{\alpha||\beta}(\epsilon)(v) &= \frac{1}{2\epsilon}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{1}{\epsilon}\Gamma^\sigma_{\alpha\beta} v_\sigma - \frac{1}{\epsilon\chi} b_{\alpha\beta} v_3 \\ &+ x_3(\chi b^\sigma_{\beta|\alpha} + \partial_\beta \chi b^\sigma_\alpha + \partial_\alpha \chi b^\sigma_3) v_\sigma + x_3(c_{\alpha\beta} - \frac{1}{\chi} e_{|\alpha\beta}) v_3. \end{aligned} \tag{3.2.10}$$

We need the following lemma (cf. lemma 5.1 in [17]) for proving a priori estimates and to identify the limit problem.

Lemma 3.2.1. The functions $\Gamma^\sigma_{\alpha\beta}(\epsilon), g(\epsilon), e_{\alpha||\beta}(\epsilon)$ satisfy the following relations.

$$\left\|\Gamma^\sigma_{\alpha\beta}(\epsilon) - \Gamma^\sigma_{\alpha\beta}\right\|_{0,\infty} + \left\|\Gamma^3_{\alpha\beta}(\epsilon) - \frac{1}{\chi} b_{\alpha\beta}\right\|_{0,\infty} + \|\Gamma^\sigma_{\alpha 3}(\epsilon) + \chi b^\sigma_\alpha\|_{0,\infty} + \left\|\Gamma^3_{\alpha 3}(\epsilon) - \frac{1}{\chi}\partial_\alpha \chi\right\|_{0,\infty} \le C\epsilon, \tag{3.2.11}$$

$$||g(\epsilon) - \chi^2 a||_{0,\infty} \le C\epsilon, \tag{3.2.12}$$

$$||A^{ijkl}(\epsilon) - A^{ijkl}(0)||_{0,\infty} \le C\epsilon, \tag{3.2.13}$$

with

$$A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),\ A^{\alpha\beta\sigma 3}(0) = 0,$$

$$A^{\alpha\beta 33}(0) = \tfrac{1}{\chi^2}\lambda a^{\alpha\beta},\ \ A^{\alpha 3\sigma 3}(0) = \tfrac{1}{\chi^2}\mu a^{\alpha\sigma},$$

$$\left.\begin{array}{c} ||\Gamma^\sigma_{\alpha\beta}(\epsilon) - \{\Gamma^\sigma_{\alpha\beta} + \epsilon x_3[-\chi b^\sigma_\beta|\alpha) - \partial_\beta \chi b^\sigma_\alpha - \partial_\alpha \chi b^\sigma_\beta]\}||_{0,\infty,\Omega} \le C\epsilon^2, \\ ||\Gamma^3_{\alpha\beta}(\epsilon) - \{\frac{1}{\chi} b_{\alpha\beta} + \epsilon x_3[\frac{1}{\chi} e_{|\alpha\beta} - c_{\alpha\beta}]\}||_{0,\infty,\Omega} \le C\epsilon^2, \\ ||\Gamma^\sigma_{\alpha 3}(\epsilon) - \{\chi b^\sigma_\alpha - \epsilon x_3 \chi^2 b^\sigma_\tau b^\tau_\alpha\}||_{0,\infty,\Omega} \le C\epsilon^2, \\ ||\Gamma^3_{\alpha 3}(\epsilon) - \{\frac{1}{\chi}\partial_\alpha \chi + \epsilon x_3 \partial_\beta \chi b^\beta_\alpha\}||_{0,\infty,\Omega} \le C\epsilon^2 \end{array}\right\} \tag{3.2.14}$$

$$||\frac{1}{\epsilon} e_{\alpha||\beta}(\epsilon)(v) - e^1_{\alpha||\beta}(\epsilon)(v)||_{0,\Omega} \le C\epsilon \Sigma_i ||v_i||_{0,\Omega}, \tag{3.2.15}$$

$$||\frac{1}{\epsilon}\partial_3 e_{\alpha||\beta}(\epsilon)(v) + \rho_{\alpha\beta}(v)||_{-1,\Omega} \le C\{\Sigma_i ||e_{i||3}(\epsilon)(v)||_{0,\Omega} + \epsilon \Sigma_\alpha ||v_\alpha||_{0,\Omega} + \epsilon ||v_3||_{1,\Omega}\}. \tag{3.2.16}$$

□

Lemma 3.2.2. Let $(u(\epsilon))_{\epsilon>0}$ be a sequence of functions $(u(\epsilon)) \in V(\Omega)$ such that

$$u(\epsilon) \to u \text{ weakly in } \left(H^1(\Omega)\right)^3 \text{ and } u(\epsilon) \to u \text{ weakly in } \left(L^2(\Omega)\right)^3, \tag{3.2.17}$$

$$\frac{1}{\epsilon} e_{i\|j}(\epsilon)(u(\epsilon)) \to e^1_{i\|j} \text{ weakly in } L^2(\Omega), \tag{3.2.18}$$

as $\epsilon \to 0$. Then

$$\mathbf{u} = (u_i) \text{ is independent of the variable } x_3, \tag{3.2.19}$$

$$\overline{\mathbf{u}} = (\overline{u_i}) = \frac{1}{2}\int_{-1}^{1} u dx_3 \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), \overline{u}_i = \partial_v \overline{u}_3 = 0 \text{ on } \gamma, \tag{3.2.20}$$

$$\gamma_{\alpha\beta}(u) = 0, \tag{3.2.21}$$

$$\rho_{\alpha\beta}(u) \in L^2(\Omega) \text{ and } \rho_{\alpha\beta}(u) = -\partial_3 e^1_{\alpha\|\beta}. \tag{3.2.22}$$

If in addition, there exists a function $\Psi_{\alpha\beta} \in H^{-1}(\Omega)$ such that

$$\rho_{\alpha\beta}(u(\epsilon)) \to \Psi_{\alpha\beta} \text{ in } H^{-1}(\Omega) \text{ as } \epsilon \to 0, \tag{3.2.23}$$

then

$$u(\epsilon) \to u \text{ strongly in } (H^1(\Omega))^3 \text{ as } \epsilon \to 0, \tag{3.2.24}$$

$$\rho_{\alpha\beta}(u) = \Psi_{\alpha\beta} \text{ and } \Psi_{\alpha\beta} \in L^2(\Omega). \tag{3.2.25}$$

□

Proof. See the proof of Lemma 5.3 in [17]. □

Lemma 3.2.3. For all $\eta = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, we let $\gamma_{\alpha\beta}(\eta)$ and $\rho_{\alpha\beta}(\eta)$ be defined as in (3.1.1) and (3.2.8) We assume that

$$\gamma_{\alpha\beta}(\eta) = \rho_{\alpha\beta}(\eta) = 0 \text{ in } L^2(\omega), \tag{3.2.26}$$

$$\eta_i = \partial_v \eta_3 = 0 \text{ on } \gamma_0 \subset \gamma = \partial\omega \text{ with length } \gamma_0 > 0. \tag{3.2.27}$$

Then $\eta = 0$. □

Proof. See the proof of Lemma 5.4 of [17]. □

3.3 A priori estimates

In this section we show that for each positive integer m the scaled eigenvalues $\xi^m(\epsilon)$ are bounded uniformly with respect to ϵ. Let $\eta = (\eta_i) \in V_F(\omega)$. Define $\upsilon_\epsilon(\eta) \in V(\Omega)$ by

$$(\upsilon_\epsilon(\eta))_\alpha = \eta_\alpha - \epsilon x_3 \left(\partial_\alpha \eta_3 + 2\chi b^\sigma_\alpha \eta_\sigma - \frac{2}{\chi}\partial_\alpha \chi \eta_3\right), \tag{3.3.1}$$

$$(\upsilon_\epsilon(\eta))_3 = \eta_3. \tag{3.3.2}$$

Setting $\theta_\alpha = \partial_\alpha \eta_3 + 2\chi b^\sigma_\alpha \eta_\sigma - \frac{2}{\chi}\partial_\alpha \chi \eta_3$ we can write $(\upsilon_\epsilon(\eta))_\alpha = \eta_\alpha - \epsilon x_3 \theta_\alpha$. Since $\gamma_{\alpha\beta}(\eta) = 0$ we have

$$\begin{aligned} e^1_{\alpha||\beta}(\epsilon)(v(\epsilon)) \;&=\; -x_3\{\frac{1}{2}(\partial_\alpha \theta_\beta) - \Gamma^\sigma_{\alpha\beta}\theta - (\chi b^\sigma_{\beta|\alpha} + \partial_\beta \chi b^{\sigma\alpha} + \partial_\alpha \chi b^\sigma_\beta)\eta_\sigma - (c_{\alpha\beta} - \frac{1}{\chi}e_{|\alpha\beta})\eta_3 \\ &\quad -\epsilon x_3^2(\chi b^\sigma_{\beta|\alpha} + \partial_\beta \chi b^\sigma_\alpha + \partial_\alpha \chi b^\sigma_\beta)\theta_\sigma. \end{aligned} \tag{3.3.3}$$

But

$$\begin{aligned} \frac{1}{2}(\partial_\alpha \theta_\beta) \;&-\; \Gamma^\sigma_{\alpha\beta}\theta_\sigma - (\chi b^\sigma_{\beta|\alpha} + \partial_\beta \chi b^{\sigma\alpha} + \partial_\alpha \chi b^\sigma_\beta)\eta_\sigma - (c_{\alpha\beta} - \frac{1}{\chi}e_{|\alpha\beta})\eta_3 \\ &= \partial_{\alpha\beta}\eta_3 + \chi b^\sigma_\alpha \partial_\beta \eta_\alpha + \chi b^\sigma_\beta \partial_\beta \eta_\alpha + \chi(\partial_\beta b^\sigma_\alpha + \partial_\alpha b^\sigma_\beta - 2\Gamma^\tau_{\alpha\beta} b^\sigma_\tau)\eta_\sigma \\ &- \Gamma^\sigma_{\alpha\beta}\partial_\sigma \eta_3 + \frac{2}{\chi}\Gamma^\sigma_{\alpha\beta}\partial_\sigma \eta_3 - \chi b^\sigma_{\beta|\alpha}\eta_\sigma - c_{\alpha\beta}\eta_3 + \frac{1}{\chi}e_{|\alpha\beta}\eta_3 \\ &+ \frac{2}{\chi^2}\partial_\alpha \chi \partial_\beta \eta_3 - \frac{2}{\chi}\partial_{\alpha\beta}\chi \eta_3 \frac{1}{\chi}\partial_\alpha \chi \partial_\beta \chi \eta_3 - \frac{1}{\chi}\partial_\beta \chi \partial_\alpha \eta_3 \\ &= \eta_{3|\alpha\beta} + \chi b^\sigma_\alpha \eta_{\sigma|\beta} + \chi b^\sigma_\beta \eta_{\sigma|\alpha} + \chi b^\sigma_{\alpha|\beta}\eta_\sigma - \left(c_{\alpha\beta} + \frac{1}{\chi}e_{|\alpha\beta}\right)\eta_3 \\ &+ \frac{2}{\chi^2}\partial_\alpha \chi \partial_\beta \chi \eta_3 - \frac{1}{\chi}\partial_\sigma \chi \partial_\beta \eta_3 - \frac{1}{\chi}\partial_\beta \chi \partial_\alpha \eta_3 \\ &= \rho_{\alpha\beta}(\eta). \end{aligned} \tag{3.3.4}$$

Thus

$$e^1_{\alpha||\beta}(\epsilon)(v(\epsilon)) = -x_3 \rho_{\alpha\beta}(\eta) - \epsilon x_3^2(\chi b^\sigma_{\beta|\alpha} + \partial_\beta \chi b^\sigma_\alpha + \partial_\alpha \chi b^\sigma_\beta)\theta_\sigma. \tag{3.3.5}$$

Hence

$$\epsilon^{-1} e_{\alpha||\beta}(\epsilon)(\upsilon_\epsilon(\eta)) \to -x_3 \rho_{\alpha\beta}(\eta) \text{ in } L^2(\Omega) \text{ as } \epsilon \to 0, \tag{3.3.6}$$

Also

$$\epsilon^{-1}e_{\alpha||3}(\epsilon)(v(\epsilon)) = \frac{1}{\epsilon}[(\Gamma^{\sigma}_{\alpha 3}(\epsilon)+\epsilon b^{\sigma}_{\alpha})\eta_\alpha+(\Gamma^{3}_{\alpha 3}(\epsilon)-\frac{1}{\chi}\partial_\alpha\chi)\eta_3-\epsilon x_3\Gamma^{\sigma}_{\alpha 3}(\epsilon)\theta_\sigma] \leq C. \quad (3.3.7)$$

Lemma 3.3.1. Let $\eta \in V_F(\omega)$. Then there exists a constant $C > 0$ such that

$$\frac{1}{\epsilon^2}|\partial_3(T_\chi(\epsilon)(v_\epsilon(\eta)))|_{0,\Omega} \leq C||\rho_{\alpha\beta}(\eta)||_{0,\omega}, \quad (3.3.8)$$

$$\frac{1}{\epsilon}|\partial_\alpha(T_\chi(\epsilon)(v_\epsilon(\eta)))|_{0,\Omega} \leq C||\rho_{\alpha\beta}(\eta)||_{0,\omega}. \quad (3.3.9)$$

Proof. The variational equation (3.2.5)- (3.2.6) posed on the domain Ω reads as follows. For each $h \in (H^1(\Omega))^3$, there exists a unique solution $T_\chi(\epsilon)(h) \in (H^1(\Omega))^3$ such that

$$\begin{aligned}
&\frac{1}{\epsilon^2}\int_\Omega \in^{33}(\epsilon)\partial_3 T_\chi(\epsilon)(h)\partial_3\psi\sqrt{g(\epsilon)}dx \\
&+\frac{1}{\epsilon}\int_\Omega [\in^{\alpha 3}(\epsilon)(\partial_\alpha T_\chi(\epsilon)(h)\partial_3\psi + \partial_3 T_\chi(\epsilon)(h)\partial_\alpha\psi)]\sqrt{g(\epsilon)}dx \\
&+\int_\Omega \in^{\alpha\beta}(\epsilon)\partial_\alpha T_\chi(\epsilon)(h)\partial_\beta\psi\sqrt{g(\epsilon)}dx \\
&= \frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3\psi e_{k||l}(\epsilon)(h)\sqrt{g(\epsilon)}dx \\
&= +\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\psi e_{k||l}(\epsilon)(h)\sqrt{g(\epsilon)}dx \forall\psi \in \Psi. \quad (3.3.10)
\end{aligned}$$

Taking $h = v_\epsilon(\eta)$ and $\psi = T_\chi(\epsilon)(v_\epsilon(\eta))$ in the above equation, we have

$$\begin{aligned}
&\frac{1}{\epsilon^2}\int_\Omega \in^{33}(\epsilon)\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&+\frac{1}{\epsilon}\int_\Omega [\in^{\alpha 3}(\epsilon)(\partial_\alpha T_\chi(\epsilon)(v_\epsilon(\eta))\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta)) + \partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))\partial_\alpha T_\chi(\epsilon)(v_\epsilon(\eta))]\sqrt{g(\epsilon)}dx \\
&+\int_\Omega \in^{\alpha\beta}(\epsilon)\partial_\alpha T_\chi(\epsilon)(v_\epsilon(\eta))\partial_\beta T_\chi(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&= \frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))e_{k||l}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&+\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha T_\chi(\epsilon)(v_\epsilon(\eta))e_{k||l}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx. \quad (3.3.11)
\end{aligned}$$

Using the relations (2.2.30) and (3.3.3)-(3.3.7) it follows that there exists a constant $C > 0$

such that

$$\frac{1}{\epsilon^2}|\partial_3(T_\chi(\epsilon)(v_\epsilon(\eta)))|^2_{0,\Omega} + |\partial_\alpha(T_\chi(\epsilon)(v_\epsilon(\eta)))|^2_{0,\Omega}$$
$$\leq C\{|\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))|_{0,\Omega}||\rho_{\alpha\beta}(\eta)||_{0,\omega} + |\epsilon\partial_\alpha T_\chi(\epsilon)(v_\epsilon(\eta))|_{0,\Omega}||\rho_{\alpha\beta}(\eta)||_{0,\omega}\} \qquad (3.3.12)$$

and hence the result follows. □

Theorem 3.3.2. Assume that $V_F(\omega)$ is an infinite dimensional subspace of $V(\Omega)$. Then for each $l \geq 1$, the sequence $\xi^l(\epsilon)$ is bounded uniformly with respect to ϵ.

Proof. Let $\mathcal{V}_m$ denote the collection of all m-dimensional subspaces of V. Then

$$\xi^m(\epsilon) = \min_{W\in\mathcal{V}_m}\max_{v\in W}\frac{\tilde{N}_\chi(\epsilon)(v,v)}{\tilde{D}(\epsilon)(v,v)} \qquad (3.3.13)$$

where

$$\begin{aligned}\tilde{N}_\chi(\epsilon)(v,v) &= \frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(v)e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}dx \\ &\quad+\frac{1}{\epsilon^3}\int_\Omega P^{3kl}(\epsilon)\partial_3 T_\chi(\epsilon)(v)e_{k||l}(\epsilon)(v)\sqrt{g(\epsilon)}dx \\ &\quad+\frac{1}{\epsilon^2}\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha T_\chi(\epsilon)(v)e_{k||l}(\epsilon)(v)\sqrt{g(\epsilon)}dx, \qquad (3.3.14)\\ \tilde{D}(\epsilon)(v,v) &= \int_\Omega v_i v_i\sqrt{g(\epsilon)}dx. \qquad (3.3.15)\end{aligned}$$

Let $\mathcal{W}_m$ denotes the collection of all m-dimensional subspaces of $V_F(\omega)$. Consider the map $S_\epsilon : V_F(\omega) \to V(\Omega)$ defined by

$$S_\epsilon(\eta) = v_\epsilon(\eta). \qquad (3.3.16)$$

For sufficiently small ϵ, S_ϵ is one-one. Thus if $W \in \mathcal{W}_m$, then $S_\epsilon(W) \in \mathcal{V}_m$. Consequently, we have

$$\xi^m(\epsilon) \leq \min_{W\in\mathcal{W}_m}\max_{\eta\in W\setminus\{0\}}\frac{\tilde{N}_\chi(\epsilon)(v_\epsilon(\eta),v_\epsilon(\eta))}{\tilde{D}(\epsilon)(v_\epsilon(\eta),v_\epsilon(\eta))}. \qquad (3.3.17)$$

On one hand

$$\begin{aligned}\int_\Omega (v_\epsilon(\eta))_i(v_\epsilon(\eta))_i\sqrt{g(\epsilon)}dx &\geq \chi\sqrt{a_0}\int_\Omega (v_\epsilon(\eta))_i(v_\epsilon(\eta))_i dx \\ &\geq 2\chi\sqrt{a_0}\int_\omega \eta_i\eta_i d\omega. \qquad (3.3.18)\end{aligned}$$

On the other hand, using the symmetries of $A^{ijkl}(\epsilon)$, the fact that $A^{\alpha\beta\sigma 3}(\epsilon) = A^{\alpha 333}(\epsilon) = 0$, and the relations (3.3.3)-(3.3.7) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(v_\epsilon(\eta))e_{i||j}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&\qquad \le C\left\{\int_\Omega A^{\alpha\beta\sigma\tau}(\epsilon)\left[\frac{1}{\epsilon}e_{\sigma\tau}(\epsilon)(v_\epsilon(\eta))\right]\left[\frac{1}{\epsilon}e_{\alpha||\beta}(\epsilon)(v_\epsilon(\eta))\right]dx\right. \\
&\qquad \left. +4\int_\Omega A^{\alpha 3\sigma 3}(\epsilon)\left[\frac{1}{\epsilon}e_{\sigma||3}(\epsilon)(v_\epsilon(\eta))\right]\left[\frac{1}{\epsilon}e_{\alpha||3}(\epsilon)(v_\epsilon(\eta))\right]dx\right\} \\
&\qquad \le C\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\eta)||^2_{0,\omega}
\end{aligned} \tag{3.3.19}$$

for $\epsilon \le 1$. Also, from the relations (3.3.6)-(3.3.9) it follows that

$$\begin{aligned}
&\frac{1}{\epsilon^3}\int_\Omega P^{3kl}(\epsilon)\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))e_{k||l}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&\qquad = \int_\Omega P^{3kl}(\epsilon)\left\{\frac{1}{\epsilon^2}\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))\right\}\left\{\frac{1}{\epsilon}e_{k||l}(\epsilon)(v_\epsilon(\eta))\right\}\sqrt{g(\epsilon)}dx \\
&\qquad \le C\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\eta)||^2_{0,\omega}
\end{aligned} \tag{3.3.20}$$

and

$$\begin{aligned}
&\frac{1}{\epsilon^2}\int_\Omega P^{\alpha kl}(\epsilon)\partial_3 T_\chi(\epsilon)(v_\epsilon(\eta))e_{k||l}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&\qquad = \int_\Omega P^{\alpha kl}(\epsilon)\left\{\frac{1}{\epsilon}\partial_\alpha T_\chi(\epsilon)(v_\epsilon(\eta))\right\}\left\{\frac{1}{\epsilon}e_{k||l}(\epsilon)(v_\epsilon(\eta))\right\}\sqrt{g(\epsilon)}dx \\
&\qquad \le C\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\eta)||^2_{0,\omega}.
\end{aligned} \tag{3.3.21}$$

Hence

$$\frac{\tilde{N}_\chi(\epsilon)(v_\epsilon(\eta), v_\epsilon(\eta))}{\tilde{D}(\epsilon)(v_\epsilon(\eta), v_\epsilon(\eta))} \le C\frac{\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\eta)||^2_{0,\omega}}{\sum_i ||\eta_i||^2_{0,\omega}}. \tag{3.3.22}$$

Let us define the two-dimensional elasticity tensor $a^{\alpha\beta\sigma\tau}$ by

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}). \tag{3.3.23}$$

It is known that (cf. [24]), there exists $C > 0$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\eta)\rho_{\sigma\tau}(\eta)\chi\sqrt{a}dy \geq C\sum_\alpha ||\rho_{\alpha\beta}(\eta)||^2_{0,\omega} \tag{3.3.24}$$

for all $\eta \in V_F(\omega)$. Thus, we have

$$\frac{\tilde{N}_\chi(\epsilon)(v_\epsilon(\eta), v_\epsilon(\eta))}{\tilde{D}(\epsilon)(v_\epsilon(\eta), v_\epsilon(\eta))} \leq C\frac{\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\eta)\rho_{\sigma\tau}(\eta)\chi\sqrt{a}d\omega}{\int_\omega \eta_i\eta_i\chi\sqrt{a}d\omega} \tag{3.3.25}$$

and hence, from (3.3.17) and (3.3.25) it follows that

$$\xi^m(\epsilon) \leq C\Lambda^m. \tag{3.3.26}$$

where Λ^m is the m^{th}-eigenvalue of the two-dimensional problem: Find $(\Lambda, \zeta) \in I\!R \times V_F(\omega)\backslash\{0\}$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\zeta)\rho_{\alpha\beta}(\eta)\chi\sqrt{a}d\omega = \Lambda\int_\omega \eta_i\zeta_i\chi\sqrt{a}d\omega \tag{3.3.27}$$

for all $\eta \in V_F(\omega)$. This completes the proof. □

3.4 Limit Problem

In this section we show that the scaled eigensolutions of the three dimensional problem converge to the eigensolutions of the two dimensional problem.

Theorem 3.4.1. Assume that the space $V_F(\omega)$ is infinite dimensional. Then a) For each integer $m \geq 1$, there exists a subsequence (still indexed by ϵ) such that

$$u^m(\epsilon) \to u^m \text{ strongly in } (H^1(\Omega))^3, \tag{3.4.1}$$

$$(\frac{1}{\epsilon}\partial_1\varphi^m(\epsilon), \frac{1}{\epsilon}\partial_2\varphi^m(\epsilon), \frac{1}{\epsilon^2}\partial_3\varphi^m(\epsilon)) \to (0, 0, \partial_3\varphi^m) \text{ strongly in } (L^2(\Omega))^3, \tag{3.4.2}$$

$$\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}(1 - x_3^2)\rho_{\alpha\beta}(u^m). \tag{3.4.3}$$

$$\xi^m(\epsilon) \to \xi^m. \tag{3.4.4}$$

b) u^m is independent of x_3, $\overline{u}^m = \frac{1}{2}\int_{-1}^{1} u^m dx \in V_F(\omega)$ and the pair $(\xi^m, \overline{u}^m)$ solves the two-dimensional eigenvalue problem for piezoelectric flexural shell, viz; find $(\xi, \zeta) \in I\!R \times V_F(\omega)\backslash\{0\}$ such that

$$\frac{1}{6}\int_\omega C^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\zeta)\rho_{\alpha\beta}(\eta)\chi\sqrt{a}dy = \xi\int_\omega \zeta_i\eta_i\chi\sqrt{a}dy \text{ for all } \eta = \eta_i \in V_F(\omega) \qquad (3.4.5)$$

where

$$C^{\alpha\beta\sigma\tau} = \left(a^{\alpha\beta\sigma\tau} + \frac{4p^{3\alpha\beta}p^{3\sigma\tau}}{p^{33}}\right), \qquad (3.4.6)$$

$$p^{3\alpha\beta} = P^{3\alpha\beta} - \frac{\lambda}{\lambda+2\mu}P^{333}a^{\alpha\beta}, \qquad (3.4.7)$$

$$p^{33} = \frac{1}{\mu}P^{3\alpha3}P^{3\alpha3} + \frac{1}{\lambda+2\mu}P^{333}P^{333} + \in^{33}. \qquad (3.4.8)$$

Proof. :The proof is divided into several steps.

Step (i): There exists constant C and $\epsilon_0 > 0$ such that

$$||u^m(\epsilon)||_{1,\Omega} \leq C, \qquad (3.4.9)$$

$$|\frac{1}{\epsilon}\partial_1\varphi^m(\epsilon)|_0^2 + |\frac{1}{\epsilon}\partial_2\varphi^m(\epsilon)|_0^2 + |\frac{1}{\epsilon^2}\partial_3\varphi^m(\epsilon)|_0^2 \leq C \qquad (3.4.10)$$

for all $0 < \epsilon \leq \epsilon_0$. Define the vector

$$\tilde{\varphi}^m(\epsilon) = (\frac{1}{\epsilon}\partial_1\varphi^m(\epsilon), \frac{1}{\epsilon}\partial_2\varphi^m(\epsilon), \frac{1}{\epsilon^2}\partial_3\varphi^m(\epsilon)). \qquad (3.4.11)$$

Letting $(v, \psi) = (u^m(\epsilon), \varphi^m(\epsilon))$ in (3.2.5), we have

$$\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(u^m(\epsilon))e_{i||j}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}dx + \frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3\varphi^m(\epsilon)e_{k||l}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}dx$$
$$+\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon)e_{k||l}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}dx = \epsilon^2\xi^m(\epsilon)\int_\Omega u_i^m(\epsilon).u_i^m(\epsilon)\sqrt{g(\epsilon)}dx. \qquad (3.4.12)$$

Using (3.2.6), the above equation becomes

$$\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(u^m(\epsilon))e_{i||j}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}dx + \int_\Omega \in^{ij}(\epsilon)\tilde{\varphi}_i^m(\epsilon)\tilde{\varphi}_j^m(\epsilon)\sqrt{g(\epsilon)}dx$$
$$= \xi^m(\epsilon)\int_\Omega u_i^m(\epsilon).u_i^m(\epsilon)\sqrt{g(\epsilon)}dx \qquad (3.4.13)$$

Using the generalized Korn's inequality (cf.[17]),

$$\|v\|_{1,\Omega}^2 \leq \Sigma_{i,j} || \frac{1}{\epsilon} e_{i||j}(\epsilon)(u)||^2 \ \forall \ v \in V \tag{3.4.14}$$

the relation (2.2.29), the orthogonality relation (3.2.7) and the boundedness of the eigenvalues, we have

$$\begin{aligned}
C\{||u^m(\epsilon)||_{1,\Omega}^2 + ||\tilde{\varphi}_i^m(\epsilon)||_{0,\Omega}^2\} &\leq \sum_{i,j} || \frac{1}{\epsilon} e_{i||j}(\epsilon)(u^m(\epsilon))||_{0,\Omega}^2 + ||\tilde{\varphi}_i^m(\epsilon)||_{0,\Omega}^2\} \\
&\leq \frac{C}{\epsilon^2} \int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(u^m(\epsilon)) e_{i||j}(\epsilon)(u^m(\epsilon)) \sqrt{g(\epsilon)} dx \\
&+ \int_\Omega \in^{ij} (\epsilon) \tilde{\varphi}_i^m(\epsilon) \tilde{\varphi}_j^m(\epsilon) \sqrt{g(\epsilon)} dx \\
&= C\xi^m(\epsilon) \int_\Omega u_i^m(\epsilon).u_i^m(\epsilon) \sqrt{g(\epsilon)} dx \\
&\leq C\xi^m(\epsilon).
\end{aligned} \tag{3.4.15}$$

Step (ii): From step (i) it follows that there exists a subsequence $(\tilde{\varphi}^m(\epsilon))$ and $\tilde{\varphi}^m \in (L^2(\Omega))^3$ such that

$$\tilde{\varphi}^m = \left(\frac{1}{\epsilon} \partial_1 \varphi^m(\epsilon), \frac{1}{\epsilon} \partial_2 \varphi^m(\epsilon), \frac{1}{\epsilon^2} \partial_3 \varphi^m(\epsilon) \right) \to (\tilde{\varphi}_1^m, \tilde{\varphi}_2^m, \tilde{\varphi}_3^m) \text{ weakly in } (L^2(\Omega))^3.$$

Now,

$$\frac{1}{\epsilon^2} \varphi^m(\epsilon)(x_1, x_2, x_3) = \int_{-1}^{x_3} \frac{1}{\epsilon^2} \partial_3 \varphi^m(\epsilon)(x_1, x_2, x_3) ds$$

This implies

$$\left\| \frac{1}{\epsilon^2} \varphi^m(\epsilon) \right\| \leq \sqrt{2} \left\| \frac{1}{\epsilon^2} \partial_3 \varphi^m(\epsilon) \right\| \leq C.$$

Hence there exists a $\varphi^m \in L^2(\Omega)$ such that

$$\left(\frac{1}{\epsilon} \partial_1 \varphi^m(\epsilon), \frac{1}{\epsilon} \partial_2 \varphi^m(\epsilon), \frac{1}{\epsilon^2} \partial_3 \varphi^m(\epsilon) \right) \to (0, 0, \partial_3 \varphi^m) \text{ weakly in } (L^2(\Omega))^3.$$

Step (iii): It follows from Step (i) that $u^m(\epsilon) \rightharpoonup u^m$ weakly in $H^1(\Omega)$ (hence strongly in $L^2(\Omega)$), $\frac{1}{\epsilon} e_{i||j}(\epsilon)(u^m(\epsilon)) \rightharpoonup e_{i||j}^{1,m}$ weakly in $L^2(\Omega)$ and $\varphi^m(\epsilon) \to \varphi^m$. Then it follows from lemma (3.2.2) that u^m is independent of x_3, $\gamma_{\alpha\beta}(u^m) = 0, i.e. \overline{u}^m \in V_F(\omega)$ and the limit functions $e^{1,m}_{i||j}$ are related to the limit function u^m by

$$-\partial_3 e^{1,m}_{\alpha||\beta} = \rho_{\alpha\beta}(u^m). \tag{3.4.16}$$

Step (iv): The limit functions $e^1_{i||j}$ are related to the limit functions $u = (u_i)$ by

$$e^{1,m}_{\alpha||3} = -\frac{1}{4\mu} P^{3\alpha 3}\partial_3\varphi^m \tag{3.4.17}$$

$$e^{1,m}_{3||3} = -\frac{1}{\lambda + 2\mu}(P^{333}\partial_3\varphi^m + \lambda\chi^2 a^{\alpha\beta} e^{1,m}_{\alpha\beta}) \tag{3.4.18}$$

Let $\upsilon = (\upsilon_i)$ be an arbitrary function in the space $V(\Omega)$. Then

$$\epsilon e_{\alpha||\beta}(\epsilon)(\upsilon) \to 0 \text{ strongly in } L^2(\Omega), \tag{3.4.19}$$

$$\epsilon e_{\alpha||3}(\epsilon)(\upsilon) \to \frac{1}{2}\partial_3\upsilon_\alpha \text{ strongly in } L^2(\Omega), \tag{3.4.20}$$

$$\epsilon e_{3||3}(\epsilon)(\upsilon) = \partial_3\upsilon_3 \text{ for all } \epsilon > 0. \tag{3.4.21}$$

Equation (3.2.5) can be written as

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon) e_{k|||l}(\epsilon) e_{i|||j}(\epsilon)(\upsilon)\sqrt{g(\epsilon)}dx \\
&= \int_\Omega \left(A^{\alpha\beta\rho\sigma}(\epsilon)\left[\frac{1}{\epsilon} e_{\alpha|||\beta}(\epsilon)\right]\left[\epsilon e_{\rho|||\sigma}(\epsilon)(\upsilon)\right] + 2A^{\alpha\beta\rho 3}(\epsilon)\left[\frac{1}{\epsilon} e_{\alpha|||\beta}(\epsilon)\right]\left[\epsilon e_{\rho|||3}(\epsilon)(\upsilon)\right]\right)\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left(A^{\alpha\beta 33}(\epsilon)\left[\frac{1}{\epsilon} e_{\alpha|||\beta}(\epsilon)\right]\left[\epsilon e_{3|||3}(\epsilon)(\upsilon)\right] + 2A^{\alpha 3\rho\sigma}(\epsilon)\left[\frac{1}{\epsilon} e_{\alpha|||3}(\epsilon)\right]\left[\epsilon e_{\rho|||\sigma}(\epsilon)(\upsilon)\right]\right)\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left(4A^{\alpha 3\rho 3}(\epsilon)\left[\frac{1}{\epsilon} e_{\alpha|||3}(\epsilon)\right]\left[\epsilon e_{\rho|||3}(\epsilon)(\upsilon)\right] + 2A^{\alpha 333}(\epsilon)\left[\frac{1}{\epsilon} e_{\alpha|||3}(\epsilon)\right]\left[\epsilon e_{3|||3}(\epsilon)(\upsilon)\right]\right)\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega \left(A^{33\rho\sigma}(\epsilon)\left[\frac{1}{\epsilon} e_{3|||3}(\epsilon)\right]\left[\epsilon e_{\rho|||\sigma}(\epsilon)(\upsilon)\right] + 2A^{33\rho 3}(\epsilon)\left[\frac{1}{\epsilon} e_{3|||3}(\epsilon)\right]\left[\epsilon e_{\rho|||3}(\epsilon)(\upsilon)\right]\right)\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega (A^{3333}(\epsilon)\left[\frac{1}{\epsilon} e_{3|||3}(\epsilon)\right]\left[\epsilon e_{3|||3}(\epsilon)(\upsilon)\right]\sqrt{g(\epsilon)}dx + \frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3\varphi^m(\epsilon) e_{k||l}(\epsilon)(\upsilon)\sqrt{g(\epsilon)}dx \\
&+ \int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon) e_{k||l}(\epsilon)(\upsilon)\sqrt{g(\epsilon)}dx = \epsilon^2\xi^m(\epsilon)\int_\Omega u^m_i \upsilon_i\sqrt{\chi^2 + \epsilon^2 g(\epsilon)}.dx
\end{aligned} \tag{3.4.22}$$

Keep $v \in V$ fixed in and let $\epsilon \to 0$. we obtain

$$\int_\Omega \left\{ 2\mu a^{\alpha\sigma}\ e^{1,m}_{\sigma||3}\partial_3 v_\alpha + \left[\lambda a^{\sigma\tau}\ e^{1,m}_{\sigma||\tau} + (\lambda+2\mu)e^{1,m}_{3||3}\right]\partial_3 v_3 \right\}\chi\sqrt{a}\,dx$$
$$+ \int_\Omega \left\{ \frac{P^{3\alpha 3}}{2}\partial_3\varphi^m\partial_3 v_\alpha + P^{333}\partial_3\varphi^m\partial_3 v_3 \right\}\chi\sqrt{a}\,dx \quad = \quad 0 \qquad (3.4.23)$$

Letting v vary in V gives relations (3.4.17) - (3.4.18).

Step (v): The function φ^m is of the form

$$\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}(1 - x_3^2)\rho_{\alpha\beta}(u^m). \qquad (3.4.24)$$

Passing to the limit as $\epsilon \to 0$ in (3.2.6), we get

$$\int_\Omega \epsilon^{33}\partial_3\varphi^m\partial_3\psi\chi\sqrt{a}\,dx - \int_\Omega P^{3kl}\partial_3\psi e^{1,m}_{k||l}\chi\sqrt{a}dx \quad = \quad 0\ \forall\,\psi \in \Psi. \qquad (3.4.25)$$

$$\text{i.e,}\quad \int_\Omega \left(P^{3kl}e^{1,m}_{k||l} - \epsilon^{33}\partial_3\varphi^m\right)\partial_3\psi\chi\sqrt{a}dx \quad = \quad 0\ \forall\,\psi \in \Psi. \qquad (3.4.26)$$

This is equivalent to

$$\partial_3\left(P^{3kl}e^{1,m}_{k||l} - \epsilon^{33}\partial_3\varphi^m\right) \quad = \quad 0 \text{ in } D'(\Omega) \qquad (3.4.27)$$

which implies that $\left(P^{3kl}e^{1,m}_{k||l} - \epsilon^{33}\partial_3\varphi^m\right) = d^1$ with $d^1 \in D'(\Omega)$. Then

$$\partial_3\varphi^m = \frac{1}{\in^{33}}\left(p^{3kl}e^{1,m}_{k||l} - d^1\right) \qquad (3.4.28)$$

Using the expression for $e^{1,m}_{k||l}$, we have

$$\partial_3\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}e^{1,m}_{\alpha||\beta}(u^m) - \frac{1}{p^{33}}d^1, \qquad (3.4.29)$$

Which gives

$$\varphi^m \quad = \quad \frac{p^{3\alpha\beta}}{p^{33}}\left[x_3 e^{1,m}_{\alpha||\beta}(u^m)\right] - x_3\frac{1}{p^{33}}d^1 + d^0. \qquad (3.4.30)$$

Since the function u_i^m is independent of x_3 (cf. step (ii)), relation (3.4.16) implies

$$e^{1,m}_{\alpha||\beta} \quad = \quad \theta_{\alpha\beta} - x_3\rho_{\alpha\beta}(u^m), \theta_{\alpha\beta} \in L^2(\Omega) \qquad (3.4.31)$$

Hence

$$\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}\left[x_3\theta_{\alpha\beta} - x_3\rho_{\alpha\beta}(u^m)\right] - \frac{x_3}{p^{33}}d_1 + d_0. \tag{3.4.32}$$

Using $\varphi_{|\Gamma^-} = \varphi_{|\Gamma^+} = 0$ it follows that φ^m is of the form (3.4.24)

Step (vi): Keep the function $\eta \in V_F(\omega)$ fixed, let $v = v_\epsilon(\eta)$ in the variational equation (3.2.5), where $(v_\epsilon(\eta))$ is defined in (3.3.1)-(3.3.2) and let $\epsilon \to 0$. Using the relations (3.2.11) and (3.2.14), we get

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\Bigg(\int_\Omega A^{ijkl}(\epsilon)e^m_{k||l}(\epsilon)e_{i||j}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\
&+\frac{1}{\epsilon}\int_\Omega P^{3kl}(\epsilon)\partial_3\varphi^m(\epsilon)e_{k||l}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx + \int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon)e_{k||l}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx\Bigg) \\
&\quad = \int_\Omega \{A^{\alpha\beta\sigma\tau}(0)e^{1,m}_{\sigma||\tau} + A^{\alpha\beta 33}(0)e^{1,m}_{3||3}\}\{-x_3\rho_{\alpha\beta}(\eta)\}\chi\sqrt{a}dx \\
&\quad + \int_\Omega P^{3\alpha\beta}\partial_3\varphi^m\{-x_3\rho_{\alpha\beta}(\eta)\}\chi\sqrt{a}dx \\
&\quad = \lim_{\epsilon\to 0}\int_\Omega u^m_i(\epsilon)v_i(\epsilon)\sqrt{g(\epsilon)}dx = \frac{1}{2}\int_\omega u^m_i\eta_i\chi\sqrt{a}dy.
\end{aligned} \tag{3.4.33}$$

Replacing $e^{1,m}_{i||j}$ and φ^m by their values found in (3.4.16)-(3.4.18) ,it can be verified that equation (3.4.33) coincides with (3.2.5).

The strong convergence of $u^m(\epsilon)$ to u^m in $H^1(\Omega)$ and $(\frac{1}{\epsilon}\partial_1\varphi^m(\epsilon), \frac{1}{\epsilon}\partial_2\varphi^m(\epsilon), \frac{1}{\epsilon^2}\partial_3\varphi^m(\epsilon))$ to $(0, 0, \partial_3\varphi^m)$ in $L^2(\Omega)$ can be proved as in [74]. □

Lemma 3.4.2. Let $(\xi^m, \overline{u}^m), m \geq 1$, be the eigensolutions of problem (3.4.5) found as limits of the subsequence $(\xi^m(\epsilon), u^m(\epsilon))_{\epsilon>0}, m \geq 1$ of eigensolutions, orthonormalized as in (3.2.7) of problem (3.2.5). Then the sequence $(\xi^m)_{m=1}^\infty$ comprises all the eigenvalues, counting multiplicities, of problem (3.4.5) and the associated sequence $(\overline{u}^m)_{m=1}^\infty$ of eigenfunctions forms a complete orthonormal set in the space $V_F(\omega)$.

Proof. The proof is similar to the proof of theorem 2.5.3. □

3.5 Conclusion

We considered the eigenvalue problem for piezoelectric flexural shells with non-uniform thickness and we have shown that if dim$(V_F)(\omega) = \infty$, the eigensolutions of the three dimensional problem converge to eigensolutions of two dimensional flexural shell involving the non-uniform function χ.

If $\dim(V_F)(\omega)$ is finite, say N, then one can prove that only the first N eigenvalues are of $o(\epsilon^2)$ and the corresponding eigensolutions converge to N eigensolutions of two dimensional flexural problem.

To the best of our knowledge, we donot know of any example of shells for which $V_F(\omega)$ is finite dimensional.

Chapter 4

Asymptotic Analysis of Dynamic Problem for Shallow Shells with Nonuniform Thickness

4.1 Introduction

In this chapter we consider the dynamic problem, concerning propagation of vibration, for shallow shells with nonuniform thickness. We then transfer the problem to a domain independent of ϵ by making suitable scalings on the unknowns and data and show that the scaled solutions converge to the solution of a two dimensional dynamic model.

This chapter is organised as follows. Section 4.2 describes the three dimensional problem. In section 4.3 we study the existence and uniqueness of the three dimensional problem. In section 4.4 we transfer the problem to a scaled domain and in section 4.5 we study the asymptotic behaviour of the scaled solutions.

4.2 The Three-dimensional Problem

Let $\omega \subset I\!\!R^2$ be a bounded domain with a Lipschitz continuous boundary γ and let ω lie locally on one side of γ. For each $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \;\; \Gamma^{\pm,\epsilon} = \omega \times \{\pm\epsilon\}, \;\; \Gamma^\epsilon = \gamma \times (-\epsilon, \epsilon)$$

Let $x^\epsilon = (x_1, x_2, x_3^\epsilon)$ be a generic point on Ω^ϵ and let $\partial_\alpha = \partial_\alpha^\epsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\epsilon = \frac{\partial}{\partial x_3^\epsilon}$. We assume that for each ϵ, we are given a function $\theta^\epsilon : \omega \to I\!\!R$ of class C^3. We then define the map $\varphi^\epsilon : \omega \to I\!R^3$ by

$$\varphi^\epsilon(x_1, x_2) = (x_1, x_2, \theta^\epsilon(x_1, x_2)) \text{ for all } (x_1, x_2) \in \omega. \tag{4.2.1}$$

At each point of the surface $S^\epsilon = \varphi^\epsilon(\omega)$, we define the normal vector

$$a^\epsilon = (|\partial_1\theta^\epsilon|^2 + |\partial_2\theta^\epsilon|^2 + 1)^{-\frac{1}{2}}(-\partial_1\theta^\epsilon, -\partial_2\theta^\epsilon, 1). \tag{4.2.2}$$

For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to I\!R^3$ by

$$\Phi^\epsilon(x^\epsilon) = \varphi^\epsilon(x_1, x_2) + x_3^\epsilon \chi(x_1, x_2) a^\epsilon(x_1, x_2) \text{ for all } x^\epsilon \in \Omega^\epsilon. \tag{4.2.3}$$

where $\chi \in W^{2,\infty},\ 0 < \chi_0 < \chi(x_1, x_2)$ denotes the thickness function. We define the space

$$V(\hat{\Omega}^\epsilon) = \{\hat{v}^\epsilon \in (H^1(\hat{\Omega}^\epsilon))^3;\ \hat{v}^\epsilon|_{\hat{\Gamma}^\epsilon} = 0\}. \tag{4.2.4}$$

For $\hat{v}^\epsilon \in V(\hat{\Omega}^\epsilon)$, we define

$$\hat{e}_{ij}(\hat{v}^\epsilon) = \frac{1}{2}\left(\frac{\partial \hat{u}_i^\epsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\epsilon}{\partial x_i}\right). \tag{4.2.5}$$

Then the variational form of the dynamic problem is to find $\hat{u}^\epsilon(t) \in V(\hat{\Omega}^\epsilon)$ almost everywhere (a.e) $\ \forall\, t \in [0, T]$ such that

$$\rho^\epsilon \int_{\hat{\Omega}^\epsilon} \ddot{\hat{u}}^\epsilon \hat{v}^\epsilon d\hat{x}^\epsilon + \int_{\hat{\Omega}^\epsilon} \hat{A}^{ijkl,\epsilon} e_{kl}^\epsilon(\hat{u}^\epsilon) e_{ij}^\epsilon(\hat{v}^\epsilon) d\hat{x}^\epsilon = \int_{\hat{\Omega}^\epsilon} \hat{f}^\epsilon.\hat{v}^\epsilon d\hat{x}^\epsilon, \quad \forall\ \hat{v}^\epsilon \in V(\hat{\Omega}^\epsilon),\ \ 0 < t < T \tag{4.2.6}$$

and

$$\hat{u}^\epsilon(0, \hat{x}^\epsilon) = \hat{\phi}^\epsilon, \quad \dot{\hat{u}}^\epsilon(0, \hat{x}^\epsilon) = \hat{\psi}^\epsilon \tag{4.2.7}$$

$$\hat{A}^{ijkl,\epsilon} = \lambda^\epsilon \delta^{ij}\delta^{kl} + \mu^\epsilon(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{4.2.8}$$

where $\dot{\hat{u}}^\epsilon$ and $\ddot{\hat{u}}^\epsilon$ denote $\dfrac{d\hat{u}^\epsilon}{dt}$ and $\dfrac{d^2\hat{u}^\epsilon}{dt^2}$ respectively and ρ^ϵ denotes the density of the material.

4.3 Existence and uniqueness of solutions to the three dimensional equation

Theorem 4.3.1. Assume that $\hat{f}^\epsilon \in W^{1,\infty}(0,T;L^2(\hat{\Omega}^\epsilon))$, $\hat{\phi}^\epsilon \in V(\hat{\Omega}^\epsilon)$ and $\hat{\psi}^\epsilon \in L^2(\hat{\Omega}^\epsilon)$. Then there exists a unique solution to the problem (4.2.6) - (4.2.7).

Proof. : Since $V(\hat{\Omega}^\epsilon)$ is separable, we can choose a basis $\{\hat{w}_i^\epsilon\}_{i=1}^\infty \in V(\hat{\Omega}^\epsilon)$. We define the approximate solution $\hat{u}^{\epsilon,m}(t,\hat{x}^\epsilon)$ of order m of the problem (4.2.6)- (4.2.7) in the following way.

$$\hat{u}^{\epsilon,m}(t,\hat{x}^\epsilon) = \sum_{p=1}^{m} \alpha_p^{\epsilon,m}(t)\hat{w}_p^\epsilon(\hat{x}^\epsilon) \tag{4.3.1}$$

where $\alpha_p^{\epsilon,m}(t)$ are determined by

$$\rho^\epsilon \int_{\hat{\Omega}^\epsilon} \ddot{\hat{u}}^{\epsilon,m}\hat{w}_p^\epsilon d\hat{x}^\epsilon + \int_{\hat{\Omega}^\epsilon} \hat{A}^{ijkl}(\epsilon)e_{kl}^\epsilon(\hat{u}^{\epsilon,m})e_{ij}^\epsilon(\hat{w}_p^\epsilon)d\hat{x}^\epsilon = \int_{\hat{\Omega}^\epsilon} \hat{f}^\epsilon.\hat{w}_p^\epsilon d\hat{x}^\epsilon \quad \forall\, 0 \le t \le T \tag{4.3.2}$$

for $p = 1,2,...,m$.

$$\hat{u}^{\epsilon,m}(0,\hat{x}^\epsilon) = \hat{u}_0^{\epsilon,m}(\hat{x}^\epsilon), \quad \dot{\hat{u}}^{\epsilon,m}(0,\hat{x}^\epsilon) = \hat{u}_1^{\epsilon,m}(\hat{x}^\epsilon), \tag{4.3.3}$$

where, as $m \to \infty$

$$\hat{u}_0^{\epsilon,m}(\hat{x}^\epsilon) = \sum_{p=1}^{m} \alpha_p^{\epsilon,m}(0)\hat{w}_p^\epsilon(\hat{x}^\epsilon) \to \hat{\phi}^\epsilon(\hat{x}^\epsilon) \ \ strongly \ \ in \ \ V(\hat{\Omega}^\epsilon), \tag{4.3.4}$$

$$\hat{u}_1^{\epsilon,m}(\hat{x}^\epsilon) = \sum_{p=1}^{m} \dot{\alpha}_p^{\epsilon,m}(0)\hat{w}_p^\epsilon(\hat{x}^\epsilon) \to \hat{\psi}^\epsilon(\hat{x}^\epsilon) \ \ strongly \ \ in \ \ L^2(\hat{\Omega}^\epsilon). \tag{4.3.5}$$

From the theory of ordinary differential equations, it follows that there exists a unique solution $\{\alpha_p^m(t), p = 1,2,..,m\}$ on $[0,T]$ to the equations (4.3.2)- (4.3.3). Multiplying both sides of (4.3.2) by $\dot{\alpha}_p^{\epsilon,m}$ and summing up from $p = 1,2,...,m$, we get

$$\rho^\epsilon \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}^\epsilon} (\dot{\hat{u}}^{\epsilon,m})^2 d\hat{x}^\epsilon + \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}^\epsilon} A^{ijkl}(\epsilon)e_{kl}^\epsilon(\hat{u}^{\epsilon,m})e_{ij}^\epsilon(\hat{u}^{\epsilon,m})d\hat{x}^\epsilon = \int_{\hat{\Omega}^\epsilon} \hat{f}^\epsilon.\dot{\hat{u}}^{\epsilon,m}d\hat{x}^\epsilon. \tag{4.3.6}$$

Integrating the above inequality from 0 to $t, \;\; t \in [0,T]$, we get

$$\begin{aligned}
\rho^\epsilon \frac{1}{2} \int_{\hat{\Omega}^\epsilon} (\dot{\hat{u}}^{\epsilon,m})^2 d\hat{x}^\epsilon \;\; &+ \;\; \frac{1}{2} \int_{\hat{\Omega}^\epsilon} \hat{A}^{ijkl}(\epsilon) e^\epsilon_{kl}(\hat{u}^{\epsilon,m}) e^\epsilon_{ij}(\hat{u}^{\epsilon,m}) d\hat{x}^\epsilon \\
&= \;\; \frac{1}{2} \int_{\hat{\Omega}^\epsilon} (\dot{\hat{u}}_1^{\epsilon,m})^2 d\hat{x}^\epsilon + \frac{1}{2} \int_{\hat{\Omega}^\epsilon} A^{ijkl}(\epsilon) e^\epsilon_{kl}(\hat{u}_0^{\epsilon,m}) e^\epsilon_{ij}(\hat{u}_0^{\epsilon,m}) d\hat{x}^\epsilon \\
&+ \;\; \int_0^t \int_{\hat{\Omega}^\epsilon} \hat{f}^\epsilon \dot{\hat{u}}^{\epsilon,m} d\hat{x}^\epsilon dt. \qquad (4.3.7)
\end{aligned}$$

Using the Korn's inequality

$$||\hat{u}^{\epsilon,m}||^2_{1,\hat{\Omega}^\epsilon} \leq C_1 ||e_{ij}(\hat{u}^{\epsilon,m})||^2_{0,\hat{\Omega}^\epsilon} \; \forall \hat{u}^{\epsilon,m} \in V(\hat{\Omega}^\epsilon), \qquad (4.3.8)$$

the coerciveness of $\hat{A}^{ijkl,\epsilon}$ and the boundedness of the function $\dot{\hat{u}}_0^{\epsilon,m}, \hat{u}_0^{\epsilon,m}, \hat{f}^\epsilon$ in $L^2(\hat{\Omega}^\epsilon)$ and $V(\hat{\Omega}^\epsilon)$ respectively, we get

$$||\dot{\hat{u}}^{\epsilon,m}||^2_{0,\hat{\Omega}^\epsilon} + ||\hat{u}^{\epsilon,m}||^2_{1,\hat{\Omega}^\epsilon} \leq C_2 \left(1 + \int_0^t \int_{\hat{\Omega}^\epsilon} (\dot{\hat{u}}^{\epsilon,m})^2 d\hat{x}^\epsilon dt \right). \qquad (4.3.9)$$

Hence by Gronwall's inequality we get

$$||\dot{\hat{u}}^{\epsilon,m}||^2_{0,\hat{\Omega}^\epsilon} + ||\hat{u}^{\epsilon,m}||^2_{1,\hat{\Omega}^\epsilon} \leq C_3. \qquad (4.3.10)$$

Differentiating both sides of (4.3.2) with respect to t we have

$$\rho^\epsilon \int_{\hat{\Omega}^\epsilon} \dddot{\hat{u}}^{\epsilon,m} \hat{w}^\epsilon_p d\hat{x}^\epsilon + \int_{\hat{\Omega}} \hat{A}^{ijkl}(\epsilon) e^\epsilon_{kl}(\dot{\hat{u}}^{\epsilon,m}) e^\epsilon_{ij}(\hat{w}^\epsilon_p) d\hat{x}^\epsilon = \int_{\hat{\Omega}^\epsilon} \dot{\hat{f}}^\epsilon . \hat{w}^\epsilon_p d\hat{x}^\epsilon \quad \forall \, 0 \leq t \leq T. \qquad (4.3.11)$$

Multiplying by $\ddot{\alpha}^m_p$ on both sides of (4.3.11) and summing from $p = 1, 2, \ldots, m$ we get in a similar way,

$$||\ddot{\hat{u}}^{\epsilon,m}||^2_{0,\hat{\Omega}^\epsilon} + ||\dot{\hat{u}}^{\epsilon,m}||^2_{1,\hat{\Omega}^\epsilon} \leq C, \quad 0 < t < T \qquad (4.3.12)$$

From (4.3.10) and (4.3.11), we notice that there exists a subsequence $(\hat{u}^{\epsilon,m})$ and a function $\hat{u}^\epsilon(t)$ such that as $m \to \infty$

$$\hat{u}^{\epsilon,m} \to \hat{u}^\epsilon \;\; weak^* \; in \;\; L^\infty(0,T; V(\hat{\Omega}^\epsilon)) \qquad (4.3.13)$$

$$\dot{\hat{u}}^{\epsilon,m} \to \dot{\hat{u}}^\epsilon \;\; weak^* \; in \;\; L^\infty(0,T; V(\hat{\Omega}^\epsilon)) \qquad (4.3.14)$$

$$\ddot{\hat{u}}^{\epsilon,m} \to \ddot{\hat{u}}^{\epsilon} \ \ weak^* \ in \ \ L^{\infty}(0,T;L^2(\hat{\Omega}^{\epsilon})) \tag{4.3.15}$$

Letting $m \to \infty$ in (4.3.2), we get

$$\rho^{\epsilon}\int_{\hat{\Omega}^{\epsilon}} \ddot{\hat{u}}^{\epsilon}\hat{w}_p^{\epsilon}d\hat{x}^{\epsilon} + \int_{\hat{\Omega}^{\epsilon}} \hat{A}^{ijkl}(\epsilon)e_{kl}^{\epsilon}(\hat{u}^{\epsilon})e_{ij}^{\epsilon}(\hat{w}_p^{\epsilon})d\hat{x}^{\epsilon} = \int_{\hat{\Omega}^{\epsilon}} \hat{f}^{\epsilon}.\hat{w}_p^{\epsilon}d\hat{x}^{\epsilon} \tag{4.3.16}$$

Since $\{\hat{w}_p^{\epsilon}\}$ is a basis for $V(\hat{\Omega}^{\epsilon})$ we get

$$\rho^{\epsilon}\int_{\hat{\Omega}^{\epsilon}} \ddot{\hat{u}}^{\epsilon}v^{\epsilon}d\hat{x}^{\epsilon} + \int_{\hat{\Omega}^{\epsilon}} \hat{A}^{ijkl}(\epsilon)e_{kl}^{\epsilon}(\hat{u}^{\epsilon})e_{ij}^{\epsilon}(\hat{v}^{\epsilon})d\hat{x}^{\epsilon} = \int_{\hat{\Omega}^{\epsilon}} \hat{f}^{\epsilon}.\hat{v}^{\epsilon}d\hat{x}^{\epsilon} \quad \forall \hat{v}^{\epsilon} \in V(\hat{\Omega}^{\epsilon}) \tag{4.3.17}$$

Since $\hat{u}^{\epsilon}(t) \in L^{\infty}(0,T;V(\hat{\Omega}^{\epsilon}))$, $\dot{\hat{u}}^{\epsilon}(t) \in L^{\infty}(0,T;V(\hat{\Omega}^{\epsilon}))$ and $\ddot{\hat{u}}^{\epsilon}(t) \in L^{\infty}(0,T;L^2(\hat{\Omega}^{\epsilon}))$, after an eventual modification on a set of measure zero, $\hat{u}^{\epsilon}(t) \in C([0,T];V(\hat{\Omega}^{\epsilon}))$, $\dot{\hat{u}}^{\epsilon}(t) \in C([0,T];L^2(\hat{\Omega}^{\epsilon}))$. The relation (4.3.12) implies

$$||\ddot{\hat{u}}^{m,\epsilon}(t)||_{V^*(\hat{\Omega}^{\epsilon})} \leq C \tag{4.3.18}$$

where $V^*(\hat{\Omega}^{\epsilon})$ denotes the dual of $V(\hat{\Omega}^{\epsilon})$. This together with (4.3.10) implies that the sequence of functions $(\hat{u}^{m,\epsilon}(t))_{m=1}^{\infty}$ and $(\dot{\hat{u}}^{m,\epsilon}(t))_{m=1}^{\infty}$ are equicontinuous in $C([0,T];L^2(\hat{\Omega}^{\epsilon}))$ and $C([0,T];V^*(\hat{\Omega}^{\epsilon}))$ respectively. Hence by Arzela-Ascoli theorem, there exists a subsequence such that as $m \to \infty$,

$$\hat{u}^{\epsilon,m} \to \hat{u}^{\epsilon} \text{ strongly in } C([0,T];L^2(\hat{\Omega}^{\epsilon})),$$

$$\dot{\hat{u}}^{\epsilon,m} \to \dot{\hat{u}}^{\epsilon} \text{ strongly in } C([0,T];V^*(\hat{\Omega}^{\epsilon})).$$

By (4.3.4), (4.3.5) and the above convergences, we deduce that as $t \to 0$

$$\hat{u}^{\epsilon}(t,\hat{x}^{\epsilon}) \to \hat{\phi}^{\epsilon}(\hat{x}^{\epsilon}) \text{ in } L^2(\hat{\Omega}^{\epsilon}),$$

$$\dot{\hat{u}}^{\epsilon}(t,\hat{x}^{\epsilon}) \to \hat{\psi}^{\epsilon}(\hat{x}^{\epsilon}) \text{ in } V^*(\hat{\Omega}^{\epsilon}).$$

Thus $\hat{u}^{\epsilon}(t,x^{\epsilon})$ is a solution of problem (4.2.6) - (4.2.7). □

4.4 The Scaled Problem

To study the asymptotic behaviour of the solution as the thickness of the shell goes to zero, we first transform the problem (4.2.6) - (4.2.7) to $\Omega^{\epsilon} = \omega \times (-\epsilon,\ \epsilon)$ and then to a domain $\Omega = \omega \times (-1,\ 1)$ which is independent of ϵ.

Since the mappings $\Phi^\epsilon : \Omega^\epsilon \to \hat{\Omega}^\epsilon$ are assumed to be diffeomorphisms, the correspondence

$$v_i^\epsilon(t, x^\epsilon) g^{i,\epsilon} = \hat{v}_i^\epsilon(t, \hat{x}^\epsilon)\hat{e}^i$$

where $\{\hat{e}^i\}_{i=1}^3$ denotes basis vectors, induces a bijection between $V(\hat{\Omega}^\epsilon)$ and $V(\Omega^\epsilon)$ where

$$V(\Omega^\epsilon) = \{v^\epsilon \in (H^1(\Omega^\epsilon))^3; v^\epsilon = 0 \text{ on } \Gamma_0^\epsilon\}.$$

Then we have (cf.[18])

$$\hat{\partial}_j^\epsilon \hat{v}_i^\epsilon = v_{k||l}^\epsilon (g^{k,\epsilon})_i (g^{l,\epsilon})_j, \quad v_{k||l}^\epsilon = \partial_l^\epsilon v_k^\epsilon - \Gamma_{lk}^{q,\epsilon}(x^\epsilon) v_q^\epsilon,$$

$$\hat{e}_{ij}^\epsilon(\hat{v}^\epsilon) = e_{k||l}^\epsilon(v^\epsilon)(g^{k,\epsilon})_i (g^{l,\epsilon})_j, \quad e_{i||j}^\epsilon(v^\epsilon) = e_{ij}^\epsilon(v^\epsilon) - \Gamma_{ij}^{p,\epsilon} v_p^\epsilon.$$

We define

$$A^{ijkl,\epsilon} = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu(g^{ik,\epsilon} g^{kl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}).$$

Then the problem (4.2.6) posed over Ω^ϵ becomes: find $u^\epsilon(t) \in V(\Omega^\epsilon)$ a.e $\forall t \in [0, T]$ such that

$$\rho^\epsilon \int_{\Omega^\epsilon} \ddot{u}_i^\epsilon v_j^\epsilon g^{ij}(\epsilon)\sqrt{g^\epsilon} dx^\epsilon + \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k||l}^\epsilon(u^\epsilon) e_{i||j}^\epsilon(v^\epsilon)\sqrt{g^\epsilon} dx^\epsilon = \int_{\Omega^\epsilon} f^{i,\epsilon} v_i^\epsilon \sqrt{g^\epsilon} dx^\epsilon \quad \forall v^\epsilon \in V(\Omega^\epsilon). \tag{4.4.1}$$

and

$$u^\epsilon(0) = \phi^\epsilon, \quad \dot{u^\epsilon}(0) = \psi^\epsilon. \tag{4.4.2}$$

To transform the above problem from the domain Ω^ϵ to the domain $\Omega = \omega \times (-1, 1)$ independent of ϵ, we make the following scalings.

$$u_\alpha^\epsilon(t, x^\epsilon) = \epsilon^2 u_\alpha(\epsilon)(t, x), \quad v_\alpha(t, x^\epsilon) = \epsilon^2 v_\alpha(t, x), \tag{4.4.3}$$

$$u_3^\epsilon(t, x^\epsilon) = \epsilon u_3(\epsilon)(t, x), \quad v_3(t, x^\epsilon) = \epsilon v_3(t, x) \tag{4.4.4}$$

With the applied body forces f^ϵ, and the initial conditions ϕ^ϵ and ψ^ϵ, we associate the functions $f(\epsilon), \phi(\epsilon)$ and $\psi(\epsilon)$ through the relation

$$f_i^\epsilon(t, x^\epsilon) = \epsilon^3 f_i(\epsilon)(t, x), \quad \rho^\epsilon = \epsilon^2 \rho, \tag{4.4.5}$$

$$\phi_\alpha^\epsilon = \epsilon^2 \phi_\alpha(\epsilon), \quad \phi_3^\epsilon = \epsilon \phi_3(\epsilon), \tag{4.4.6}$$

$$\psi^{\epsilon}_{\alpha} = \epsilon^2 \psi_{\alpha}(\epsilon), \quad \psi^{\epsilon}_{3} = \epsilon \psi_3(\epsilon) \tag{4.4.7}$$

Note that the scalings made on the applied forces are different from the scaling made on the applied forces in the static case (cf.[18]).

With the tensors $e^{\epsilon}_{i||j}$, we associate the tensors $e_{i||j}(\epsilon)$ through the relation

$$e^{\epsilon}_{i||j}(v^{\epsilon})(x^{\epsilon}) = \epsilon^2 e_{i||j}(\epsilon; v)(x). \tag{4.4.8}$$

We define the space

$$V = \{v \in (H^1(\Omega))^3; v|_{\Gamma_0} = 0\}. \tag{4.4.9}$$

Assumption: We assume that the shell is a shallow shell; i.e. there exists a function $\theta \in C^3(\omega)$ such that $\theta^{\epsilon} = \epsilon\theta$

$$i.e. \quad \varphi^{\epsilon}(x_1, x_2) = (x_1, x_2, \epsilon\theta(x_1, x_2)), \text{ for all } (x_1, x_2) \in \omega. \tag{4.4.10}$$

Then the scaled solution $u(\epsilon)(t, x)$ satisfies

$$\begin{aligned}
&\rho \left[\int_{\Omega} \epsilon^2 \ddot{u}_{\alpha}(\epsilon) v_{\beta} g^{\alpha\beta}(\epsilon) \sqrt{g(\epsilon)} dx + \int_{\Omega} \epsilon \ddot{u}_{\alpha}(\epsilon) v_3 g^{\alpha 3}(\epsilon) \sqrt{g(\epsilon)} dx \right. \\
&\left. + \int_{\Omega} \epsilon \ddot{u}_3(\epsilon) v_{\beta} g^{3\beta}(\epsilon) \sqrt{g(\epsilon)} dx + \int_{\Omega} \ddot{u}_3(\epsilon) v_3 g^{33}(\epsilon) dx \right] \\
&+ \int_{\Omega} A^{ijkl}(\epsilon) e_{k||l}(\epsilon; u(\epsilon)) e_{i||j}(\epsilon; v)) \sqrt{g(\epsilon)} dx \\
&= \int_{\Omega} \epsilon f_{\alpha}.v_{\alpha}(\epsilon) \sqrt{g(\epsilon)} dx + \int_{\Omega} f_3 v_3 \sqrt{g(\epsilon)} dx \;\; \forall \;\; v \in V
\end{aligned} \tag{4.4.11}$$

$$u(\epsilon)(0, x) = \phi(\epsilon), \quad \dot{u}(\epsilon)(0, x) = \psi(\epsilon) \tag{4.4.12}$$

Then the functions $e_{i||j}(\epsilon, v)$ defined in (4.4.8) are of the form(cf. lemma 2.3.1)

$$\left. \begin{aligned}
e_{\alpha||\beta}(\epsilon; v) &= \tilde{e}_{\alpha\beta}(v) + \epsilon^2 e^{\sharp}_{\alpha||\beta}(\epsilon; v), \\
e_{\alpha||3}(\epsilon; v) &= \tfrac{1}{\epsilon}\{\tilde{e}_{\alpha 3}(v) + \epsilon^2 e^{\sharp}_{\alpha||3}(\epsilon; v)\}, \\
e_{3||3}(\epsilon; v) &= \tfrac{1}{\epsilon^2} \tilde{e}_{33}(v),
\end{aligned} \right\} \tag{4.4.13}$$

where

$$\left.\begin{array}{c}\tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{v_3}{\chi}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}\chi)\\ \tilde{e}_{\alpha 3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha),\\ \tilde{e}_{33}(v) = \partial_3 v_3,\end{array}\right\} \tag{4.4.14}$$

Also there exists constant C such that

$$\left.\begin{array}{c}\sup_{0<\epsilon\leq\epsilon_0}\max_{\alpha,j}||e^\sharp_{\alpha,j}(\epsilon;v)||_{0,\Omega} \leq C||v||_{1,\Omega} \text{ for all } v \in V,\\ \sup_{0<\epsilon\leq\epsilon_0}\max_{x\in\Omega}|g(\epsilon)(x) - \chi^2| \leq C\epsilon^2,\\ \sup_{0<\epsilon\leq\epsilon_0}\max_{x\in\Omega}|A^{ijkl}(\epsilon) - A^{ijkl}| \leq C\epsilon^2,\end{array}\right\} \tag{4.4.15}$$

where

$$\left.\begin{array}{c}A^{\alpha\beta\gamma\tau} = \lambda\delta^{\alpha\beta}\delta^{\gamma\tau} + \mu(\delta^{\alpha\gamma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\gamma})\\ A^{\alpha\beta\gamma 3}(0) = 0, \quad A^{\alpha\beta 33}(0) = \frac{1}{\chi^2}\lambda\delta^{\alpha\beta}, \quad A^{\alpha 3\gamma 3}(0) = \frac{1}{\chi^2}\mu\delta^{\alpha\gamma}\\ A^{\alpha 333}(0) = 0, \quad A^{3333}(0) = \frac{1}{\chi^4}(\lambda + 2\mu),\end{array}\right\} \tag{4.4.16}$$

$$A^{ijkl}(\epsilon)t_{kl}t_{ij} \geq Ct_{ij}t_{ij}, \tag{4.4.17}$$

for $0 < \epsilon \leq \epsilon_0$ and for all symmetric tensors (t_{ij}).

4.5 Asymptotic Analysis

In this section we show that the solution of the three dimensional dynamic problem converges to the solution of two dimensional dynamic problem.

Theorem 4.5.1. Assume that the scaled initial data $\{\phi(\epsilon), \psi(\epsilon)\}_{\epsilon>0} \in V(\Omega) \times [L^2(\Omega)]^3$ of the problem (4.4.11)- (4.4.12) satisfy

$$\phi(\epsilon) \to \phi \quad \text{strongly in} \quad V(\Omega), \tag{4.5.1}$$

$$\psi(\epsilon) \to \psi \quad \text{strongly in} \quad L^2(\Omega), \tag{4.5.2}$$

$$f_i(\epsilon) \to f_i \quad \text{strongly in} \quad W^{1,\infty}(0,T;L^2(\Omega)). \tag{4.5.3}$$

Then there exists a subsequence $(u(\epsilon))_{\epsilon>0}$ (still indexed by ϵ for notational convenience) and a function $u \in L^\infty(0,T;V(\Omega))$ satisfying

$$\left.\begin{aligned} u(\epsilon) \to u \; weak^* \; in \; L^\infty(0,T;V(\Omega)), \\ \dot{u}_3(\epsilon) \to \dot{u}_3 \; weak^* \; in \; L^\infty(0,T;L^2(\Omega)), \\ \dot{u}_\alpha(\epsilon) \to 0 \; weak^* \; in \; L^\infty(0,T;L^2(\Omega)). \end{aligned}\right\} \tag{4.5.4}$$

Also the limit function $u = \{u_\alpha, u_3\}$ is a Kirchhoff-Love displacement, that is

$$u_\alpha = \zeta_\alpha - x_3 \partial_\alpha \zeta_3, \;\; u_3 = \zeta_3, \;\; \zeta_i \text{ is independent of } x_3. \tag{4.5.5}$$

and $\zeta = (\zeta_\alpha, \zeta_3)$ satisfies

$$\begin{aligned} &\rho \int_\omega \ddot{u}_3 \eta_3 \chi d\omega - \int_\omega m_{\alpha\beta}(\zeta_3) \partial_{\alpha\beta} \eta_3 \chi d\omega - \int_\omega (n_{\alpha\beta}(\zeta) \partial_{\alpha\beta} \theta \eta_3 + m_{\alpha\beta}(\zeta_3) \partial_{\alpha\beta} \chi) \eta_3 \chi d\omega \\ &= \int_\omega f_3 \eta_3 \chi d\omega \quad \forall \eta_3 \in H_0^2(\omega), \end{aligned} \tag{4.5.6}$$

$$\int_\omega n_{\alpha\beta}(\zeta) \partial_\alpha \eta_\beta \chi d\omega = 0 \forall (\eta_\alpha) \in \left(H_0^1(\omega)\right)^2 \tag{4.5.7}$$

where

$$m_{\alpha\beta}(\zeta_3) = -\frac{2\lambda\mu}{3(\lambda+2\mu)}(\Delta\zeta_3 + \zeta_3 \frac{\Delta\chi}{\chi})\delta_{\alpha\beta} + \frac{4\mu}{3}(\partial_{\alpha\beta}\zeta_3 + \zeta_3 \frac{\partial_{\alpha\beta}\chi}{\chi}), \tag{4.5.8}$$

$$n_{\alpha\beta}(\zeta) = \frac{2\lambda\mu}{\lambda+2\mu} \hat{e}_{\sigma\sigma}(\zeta)\delta_{\alpha\beta} + 2\mu \hat{e}_{\alpha\beta}(\zeta), \tag{4.5.9}$$

where

$$\hat{e}_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha) - \zeta_3 \frac{\partial_{\alpha\beta}\theta}{\chi} = \frac{1}{2}\int_{-1}^{1} \tilde{e}_{\alpha\beta}(\zeta) dx_3 \tag{4.5.10}$$

$$u(0,x) = \phi, \;\; \dot{u}(0,x) = \psi \tag{4.5.11}$$

and $\{\phi, \psi\}$ is the weak limit of $\{\phi(\epsilon), \psi(\epsilon)\}_{\epsilon>0}$ in $V(\Omega) \times L^2(\Omega)$.

Proof. Taking $v = \dot{u}(\epsilon)(t,x)$ in (4.4.11), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho\left[\epsilon^2\dot{u}_\alpha(\epsilon)\dot{u}_\beta(\epsilon)g^{\alpha\beta}(\epsilon) + 2\epsilon\dot{u}_\alpha(\epsilon)\dot{u}_3(\epsilon)g^{\alpha 3}(\epsilon) + \dot{u}_3(\epsilon)\dot{u}_3(\epsilon)g^{33}(\epsilon)\right]\sqrt{g(\epsilon)}dx$$
$$+\frac{1}{2}\frac{d}{dt}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon;u(\epsilon))e_{i||j}(\epsilon;u(\epsilon))\sqrt{g(\epsilon)}dx$$
$$= \int_\Omega \epsilon f_\alpha(\epsilon)\dot{u}_\alpha(\epsilon)\sqrt{g(\epsilon)}dx + \int_\Omega f_3(\epsilon)\dot{u}_3(\epsilon)\sqrt{g(\epsilon)}dx \tag{4.5.12}$$

Using the positive definiteness of $(g^{ij}(\epsilon))$ and integrating from 0 to t, $0 < t \le T$, we get

$$\frac{1}{2}\int_\Omega (\epsilon\dot{u_\alpha}(\epsilon))^2\sqrt{g(\epsilon)}dx + \frac{1}{2}\int_\Omega (\dot{u}_3(\epsilon))^2\sqrt{g(\epsilon)}dx$$
$$+\frac{1}{2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon;u(\epsilon))e_{i||j}(\epsilon;u(\epsilon))\sqrt{g(\epsilon)}dx$$
$$\le \frac{1}{2}\int_\Omega (\epsilon\psi_\alpha(\epsilon))^2\sqrt{g(\epsilon)}dx + \frac{1}{2}\int_\Omega (\psi_3(\epsilon))^2\sqrt{g(\epsilon)}dx$$
$$+\frac{1}{2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon;\phi(\epsilon))e_{i||j}(\epsilon;\phi(\epsilon))\sqrt{g(\epsilon)}dx$$
$$+\int_0^t\int_\Omega \epsilon f_\alpha(\epsilon)\dot{u}_\alpha(\epsilon)\sqrt{g(\epsilon)}dxdt + \int_0^t\int_\Omega f_3(\epsilon)\dot{u}_3(\epsilon)\sqrt{g(\epsilon)}dxdt \tag{4.5.13}$$

Using the inequality (cf. lemma 2.3.2)

$$\|v\|_{1,\Omega} \le C\left\{\sum_{i,j}\|\tilde{e}_{ij}(v)\|^2\right\}^{\frac{1}{2}} \quad \forall v \in V(\Omega) \tag{4.5.14}$$

the relations (4.4.13) - (4.4.16) and the boundedness of the functions $\phi(\epsilon)$ and $\psi(\epsilon)$, we have

$$||\epsilon\dot{u}_\alpha(\epsilon)||^2_{0,\Omega} + ||\dot{u}_3(\epsilon)||^2_{0,\Omega} + ||u_i(\epsilon)||^2_{1,\Omega}$$
$$\le ||\epsilon\dot{u}_\alpha(\epsilon)||^2_{0,\Omega} + ||\dot{u}_3(\epsilon)||^2_{0,\Omega} + \sum_{i,j}||e_{i||j}(u(\epsilon))||^2_{0,\Omega}$$
$$\le \frac{1}{2}\int_\Omega (\epsilon\dot{u_\alpha}(\epsilon))^2\sqrt{g(\epsilon)}dx + \frac{1}{2}\int_\Omega (\dot{u}_3(\epsilon))^2\sqrt{g(\epsilon)}dx$$
$$+\frac{1}{2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon;u(\epsilon))e_{i||j}(\epsilon;u(\epsilon))\sqrt{g(\epsilon)}dx$$
$$\le C_1\left(1 + \int_0^t\int_\Omega \epsilon f_\alpha(\epsilon)dxdt + \int_0^t\int_\Omega f_3(\epsilon)dxdt + \int_0^t\int_\Omega (\dot{u}(\epsilon))^2dxdt\right) \tag{4.5.15}$$

Therefore by Gronwall's inequality there exists a constant C such that

$$||\epsilon\dot{u}_\alpha(\epsilon)||_{0,\Omega} \leq C, \quad ||\dot{u}_3(\epsilon)||_{0,\Omega} \leq C, \quad ||u_i(\epsilon)||_{1,\Omega} \leq C, \quad ||e_{i||j}(\epsilon; u(\epsilon))||_{0,\Omega} \leq C. \tag{4.5.16}$$

Hence there exists a subsequence and a function $u \in L^\infty(0, T, V(\Omega))$ such that

$$u(\epsilon) \rightarrow \; u \;\; weak^* \;\; in \;\; L^\infty(0, T; V(\Omega)), \tag{4.5.17}$$

$$e_{i||j}(\epsilon; u(\epsilon)) \rightarrow e_{i||j}(u) \;\; weak^* \;\; in \;\; L^\infty(0, T; L^2(\Omega)), \tag{4.5.18}$$

$$\epsilon\dot{u}_\alpha(\epsilon) \rightarrow \; 0 \;\; weak^* \;\; in \;\; L^\infty(0, T; L^2(\Omega)), \tag{4.5.19}$$

$$\dot{u}_3(\epsilon) \rightarrow \dot{u}_3 \;\; weak^* \;\; in \;\; L^\infty(0, T; L^2(\Omega)). \tag{4.5.20}$$

For fixed $t \in [0, T]$, define

$$K_{\alpha\beta}(\epsilon) = \tilde{e}_{\alpha\beta}(u(\epsilon)), \quad K_{\alpha 3}(\epsilon) = \frac{1}{\epsilon}\tilde{e}_{\alpha 3}(u(\epsilon)), \quad K_{33}(\epsilon) = \frac{1}{\epsilon^2}\tilde{e}_{33}(u(\epsilon)) \tag{4.5.21}$$

and

$$K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u), \quad K_{\alpha 3} = 0, \quad K_{33} = -\chi^2\frac{\lambda}{\lambda + 2\mu}\tilde{e}_{\alpha\alpha}(u). \tag{4.5.22}$$

Claim: $K(\epsilon) = (K_{ij}(\epsilon)) \rightharpoonup K = (K_{ij})$ weakly in $L^2(\Omega)$.
From the definition (4.5.21) and relations (4.4.13),(4.4.14), we have

$$\begin{aligned} ||K(\epsilon)||^2_{0,\Omega} \leq\; & 2\Sigma_{i,j}||e_{i||j}(\epsilon; u(\epsilon))||^2_{0,\Omega} + 2\epsilon^4\Sigma_{\alpha\beta}||\tilde{e}^\sharp(\epsilon; u(\epsilon))||^2_{0,\Omega} \\ & + 4\epsilon^2\Sigma_\alpha||\tilde{e}^\sharp(\epsilon; u(\epsilon))||^2_{0,\Omega} \end{aligned} \tag{4.5.23}$$

From the boundedness of $(e_{i||j}(\epsilon, u(\epsilon)))$ and the relation (4.4.15) it follows that $(K(\epsilon))$ is bounded and hence $K(\epsilon) \rightharpoonup K$ in $(L^2(\Omega))^9$ weakly. We next note the following result:

$$\int_\Omega u\partial_3 v dx = 0 \text{ for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0 \Rightarrow u = 0. \tag{4.5.24}$$

Clearly $K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u)$. Multiplying (4.4.11) by ϵ and taking $v_3 = 0$ we get

$$2\int_\Omega A^{\alpha 3\sigma 3}(0)K_{\alpha 3}(\epsilon)\partial_3 v_\alpha \chi dx = \epsilon R(\epsilon, K(\epsilon), u(\epsilon), v), \tag{4.5.25}$$

where $R(\epsilon, K(\epsilon), u(\epsilon), v)$ is bounded independent of ϵ. Letting $\epsilon \to 0$ we get

$$\int_\Omega K_{\alpha 3}\partial_3 v_\alpha \chi dx = 0 \ \ for \ \ all \ \ v_\alpha. \tag{4.5.26}$$

Hence $K_{\alpha 3} = 0$. Multiplying (4.4.11) by ϵ^2 and letting $v_\alpha = 0$ we get

$$\int_\Omega \{A^{33\sigma\tau}(0)K_{\sigma\tau}(\epsilon) + A^{3333}(0)K_{33}(\epsilon)\}\partial_3 v_3 \chi dx \tag{4.5.27}$$
$$= \int_\Omega \left\{\frac{\lambda}{\chi^2}K_{\sigma\sigma}(\epsilon) + \frac{(\lambda+2\mu)}{\chi^4}K_{33}(\epsilon)\right\}\partial_3 v_3 \chi dx$$
$$= \epsilon S(\epsilon, K(\epsilon), u(\epsilon), v) \tag{4.5.28}$$

where $S(\epsilon, K(\epsilon), u(\epsilon), v)$ is independent of ϵ. Letting $\epsilon \to 0$, we get

$$\int_\Omega \left\{\frac{\lambda}{\chi^2}K_{\sigma\sigma} + \frac{(\lambda+2\mu)}{\chi^4}K_{33}\right\}\chi dx = 0. \tag{4.5.29}$$

Hence $K_{33} = -\chi^2 \frac{\lambda}{\lambda+2\mu}\tilde{e}_{\alpha\alpha}(u)$.
Define

$$V_{KL}(\Omega) = \{v \in V(\Omega) : e_{i3}(v) = 0\}. \tag{4.5.30}$$

Using the relation (4.5.22) it follows that for $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$

$$\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon; u(\epsilon))e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}dx$$
$$\to -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3\chi d\omega - \int_\omega (n_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\theta\eta_3 + m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\chi)\eta_3\chi d\omega$$
$$+ \int_\omega n_{\alpha\beta}(\zeta)\partial_\alpha\eta_\beta\chi d\omega \tag{4.5.31}$$

Since $(\epsilon\dot{u}_\alpha(\epsilon), \dot{u}_3(\epsilon)) \to (0, \dot{u}_3)$ weak* in $L^\infty(0,T;L^2(\Omega))$, it follows that for fixed $v = (v_i) = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$,

$$\int_\Omega \epsilon\dot{u}_\alpha(\epsilon)v_\beta\sqrt{g(\epsilon)}dx \to 0 \quad \text{weak* in } L^\infty(0,T)$$

and

$$\int_\Omega \dot{u}_3(\epsilon)v_3\sqrt{g(\epsilon)}dx \to \int_\Omega \dot{u}_3 v_3 dx \quad \text{weak* in } L^\infty(0,T).$$

This gives

$$\int_0^T\int_\Omega \epsilon\ddot{u}_\alpha(\epsilon)v_\beta\zeta\sqrt{g(\epsilon)}dxdt = -\int_0^T\int_\Omega \epsilon\dot{u}_\alpha(\epsilon)v_\beta\dot{\zeta}\sqrt{g(\epsilon)}dxdt \to 0 \quad \forall\zeta\in\mathcal{D}(0,T) \tag{4.5.32}$$

and

$$\int_0^T\int_\Omega \ddot{u}_3(\epsilon)v_3\zeta\sqrt{g(\epsilon)}dxdt = -\int_0^T\int_\Omega \dot{u}_3(\epsilon)v_3\dot{\zeta}\sqrt{g(\epsilon)}dxdt$$
$$\to -\int_\Omega \dot{u}_3v_3\dot{\zeta}\chi dxdt = \int_\Omega \ddot{u}_3v_3\zeta\chi dxdt \quad \forall\zeta\in\mathcal{D}(0,T). \tag{4.5.33}$$

$$i.e., \int_\Omega \epsilon\ddot{u}_\alpha(\epsilon)v_\beta\sqrt{g(\epsilon)}dx\to 0, \text{ and } \int_\Omega \ddot{u}_3(\epsilon)v_3\sqrt{g(\epsilon)}dx \to \int_\Omega \ddot{u}_3v_3\chi dx \quad \text{in } \mathcal{D}'(0,T) \tag{4.5.34}$$

Hence passing to the limit in (4.4.11) by taking $v=(\eta_\alpha - x_3\partial_\alpha\eta_3,\eta_3)\in V_{KL}(\Omega)$, we get

$$\rho\int_\omega \ddot{u}_3\eta_3 d\omega - \int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3\chi d\omega - \int_\omega (n_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\theta\eta_3 + m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\chi)\eta_3\chi d\omega$$
$$+\int_{om} n_{\alpha\beta}(\zeta)\partial_\alpha\eta_\beta\chi d\omega \;=\int_\omega f_3\eta_3\chi d\omega \tag{4.5.35}$$

for all $v=(\eta_\alpha - x_3\partial_\alpha\eta_3,\eta_3)\in V_{KL}(\Omega)$. □

From Lemma (2.5.2) it follows that for a given $\zeta_3\in H_0^2(\omega)$ there exists a unique vector $(\zeta_\alpha)\in(H_0^1(\omega))^2$ such that

$$\int_\omega n_{\alpha\beta}(\zeta)\partial_\beta\eta_\alpha\chi dw = 0 \text{ for all } (\eta_\alpha)\in(H_0^1(\omega))^2 \tag{4.5.36}$$

where $\zeta=(\zeta_\alpha,\zeta_3)$. Given $\zeta_3\in H_0^2(\omega)$, we denote by $T_\chi\zeta_3\in(H_0^1(\omega))^2\times H_0^2(\omega)$ the vector (ζ_α,ζ_3) where $(\zeta_\alpha)\in(H_0^1(\omega))^2$ is the solution of (4.5.36). Hence (4.5.6) can be written as

$$\rho\int_\omega \ddot{u}_3\eta_3\;\chi d\omega + b(\zeta_3,\eta_3) = \int_\omega f_3\eta_3\;\chi d\omega\forall\eta_3\in H_0^2(\omega), \tag{4.5.37}$$

where

$$b(\zeta_3,\eta_3) = -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3\;\chi d\omega - \int_\omega m_{\alpha\beta}(\zeta_3)(\partial_{\alpha\beta}\chi)\eta_3\;\chi d\omega\int_\omega n_{\alpha\beta}(T_\chi\zeta_3)\partial_{\alpha\beta}\theta\eta_3\;\chi d\omega \tag{4.5.38}$$

It has been shown in [73] that $b(\cdot,\cdot)$ is $(H_0^1(\omega))^2 \times H_0^2(\omega)$ elliptic and symmetric. Hence the problem (4.5.37) has a unique solution.

4.6 Conclusion

We have started with dynamic problem for shallow shells with non-uniform thickness and have shown that the solutions of the three dimensional model converges to the soutions of a two dimensional model involving the non-uniform parameter.
The difference between the two dimensional static model and dynamic model is that in the dynamic model we are able to express it involving only the third component of the limit.

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List of Publications

Refereed International Journals

1. Job Mathai and N.Sabu. Asymptotic Analysis of Dynamic Problem for Shallow Shells with Variable Thickness. *Indian Journal of Mathematics*, 61(1),9-25, 2019.

2. Job Mathai and N.Sabu. Lower Dimensional Approximation of Eigenvalue problem for piezoelectric shells with Nonuniform Thickness. *Indian Journal of Mathematics*, 63(1),1-33, 2021.