Analytical and semi-analytical solutions of elastostatic problems in nonlocal beam bending theory

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by

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Abstract

Nanostructures are the fundamental building blocks for micro and nanoscale devices such as Micro Electro Mechanical Systems (MEMS) and Nano Electro Mechanical Systems (NEMS). At these diminutive scales, classical continuum mechanics becomes inadequate due to the emergence of "size effects", attributed to the predominance of interatomic and intermolecular forces, like van der Waals and Langevin interactions. In general, "length scale parameters", which are critical material characteristics, are used to quantify these size-dependent effects.

Several size-dependent continuum theories have been developed to capture the influence of "size effects". Eringen's nonlocal elasticity theory stands out due to its relative simplicity in incorporating size effects through a single scale effect parameter, called the nonlocal parameter, λ. Consequently, it has garnered significant attention from researchers. Initially proposed in the 1970s in its integral form, the strain-driven constitutive model led to complicated Fredholm and Volterra integro-differential equations. In general, closed-form solutions of such equations are not always guaranteed, often leading to ill-posed problems. This prompted researchers to pursue alternative means to arrive at accurate and well-posed solutions. Subsequently, Eringen proposed an equivalent differential form of his integral nonlocal model. Solutions of differential equations are more manageable to derive compared to the integro-differential equations, and thereby, this contribution simplified the static and dynamic analyses of nanostructures.

However, some noticeable deficiencies concerning the solutions derived based on Eringen's differential nonlocal model became apparent, which led to certain "paradoxical" situations for some specific nanobeam boundary and loading conditions. Eventually, researchers reverted to Eringen's integral model and derived analytical solutions for the governing integro-differential equations. However, it was found that these solutions could not satisfy essential and natural boundary conditions. Further investigations indicated that the integro-differential equations arising out of Eringen's integral model is flawed and require the satisfaction of additional "constitutive" boundary conditions, for equivalence between the integral and the differential models. A novel stress-driven constitutive nonlocal model was developed to circumvent this issue.

This research focuses on developing a generalized functions-based approach to analyze the static bending behavior of stress-driven and strain-driven nanobeams under various load-

ing conditions. Special attention is given to deriving the analytical solutions for nanobeams modeled after the Euler-Bernoulli and Timoshenko beam theories by utilizing this approach. The proposed method starts with the governing integro-differential equations and subsequently transforms them into their differential equation form by the Laplace and inverse Laplace transformations. Following this, a technique involving the manipulation of the generalized functions is adopted to split the obtained differential equations into their constituent local and nonlocal parts. Analytical solutions for these individual differential equations are then separately obtained and combined to form a generalized expression. These solutions involved unknown constants of integration found out upon imposing the relevant boundary conditions. A noteworthy contribution in nonlocal elasticity was achieved through a rigorous mathematical proof that established the equivalence between Eringen's integral and differential nonlocal models for the first time. The equivalence was proved without specifying the previously mentioned "constitutive" boundary conditions.

For stress-driven nanobeams, the present study investigates the static bending behavior of stress-driven Euler-Bernoulli and Timoshenko nanobeams subjected to concentrated loads and moments in various end configurations. Governing differential equations were generated from the stress-driven constitutive relations with the input bending moments and shear fields expressed in terms of generalized functions. The resulting analytical solutions were accurate and converged in a single step compared to the existing contemporary solution methods available in recent literature.

The research concludes with a proposed semi-analytical solution methodology based on the Variational Iteration Method (VIM). Imposing the relevant boundary conditions by assumed initial solutions forms the salient feature of this method. The initially assumed solutions are functions of unknown coefficients determined for each iteration by applying relevant boundary conditions depending on the nanobeam configuration. For each iterative step, the solution is updated and compared to its previous solution until a set tolerance condition is met to stop the iterative process. Although in its developmental stages, the method produces solutions comparable to the existing analytical solutions and consistently displays an overall stiffened response.