INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY

A DOCTORAL THESIS

A Study On Subalgebras of Affine Fibrations Using Locally Nilpotent Derivations

Submitted in partial fulfillment for the award of the degree of

Doctor of Philosophy

by

Janaki Raman BABU

Department of Mathematics
Indian Institute of Space Science and Technology
Valiyamala P.O.
Thiruvananthapuram - 695 547

Submitted in September 2022. Revised in April 2023.

Certificate

This is to certify that the doctoral thesis titled ***A study on subalgebras of affine fibrations using locally nilpotent derivations*** submitted by ***Janaki Raman*** **BABU**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of ***Doctor of Philosophy*** in ***Mathematics*** is a bona fide record of the original work carried out by him under my supervision. The contents of this doctoral thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Dr. Prosenjit DAS
Advisor of Janaki Raman BABU
Department of Mathematics
Indian Institute of Space Science and Technology
Valiyamala P.O.
Thiruvananthapuram - 695 547

Dr. Anilkumar CV
The Head
Department of Mathematics
Indian Institute of Space Science and Technology
Valiyamala P.O.
Thiruvananthapuram - 695 547

Place: Thiruvananthapuram
Date:

Declaration

I declare that this thesis titled ***A study on subalgebras of affine fibrations using locally nilpotent derivations*** submitted in partial fulfillment for the award of the degree of ***Doctor of Philosophy*** is a record of the original work carried out by me under the supervision of ***Dr. Prosenjit* DAS**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

Place:	Thiruvananthapuram	*Author:*	Janaki Raman BABU
Date:	22nd September 2022 Revised on 6th April 2023	*Student ID:*	SC16D036

Acknowledgements

First, I would like to thank my supervisor and guide, Dr. Prosenjit Das, for giving me the right ideas and having patience, which gave me adequate time to solve problems. Many active discussions with him taught me to have a sound vision of how to approach a mathematical problem. The freedom he gave helped me to have a glimpse of other areas of study and research works, which are closely related to the field of commutative algebra. The amount of support he gave me throughout my research work is so much that only words cannot describe it, and I am indebted to him throughout my life.

I take this opportunity to thank the entire IIST, including the Directors, Dean R&D, and Dean Academics, for providing a lively and peaceful environment to do research. I extend my thanks to all the members of the mathematics community at IIST. In particular, I want to thank all the Heads of the Mathematics Department during my research tenure at IIST, i.e., Dr. K.S.S Moosath, Dr. Nicholas Sabu, and Dr. C V Anil Kumar, for their kind support and help; and all the faculties for the courses they taught me; and all the office persons for their sincere service and help beyond the academics. I also thank my doctoral committee members, Dr. K.S.S Moosath, Dr. Kaushik Mukherjee, Dr. Priyadarshnam, and Dr. Viji Z. Thomas, for reviewing my work every year and providing general guidance.

Next, I would like to thank all the teachers, professors, and faculties who directly or indirectly impacted me. Though it is not possible to list all of their names, I would like to mention a few of them who have structured my research career. In this regard, I want to convey my special thanks to Dr. Purusottam Rath of CMI for teaching an excellent course on Ring and Module theory, thereby sowing the seed to pursue research in commutative algebra. I would also like to thank Dr. G.P.Youvaraj, Professor Emeritus of RIASM, for enlightening me with two courses on Real and Advanced Analysis and his timely words on how to get the focus back on research during the difficult times of my Ph.D. tenure. Special mention to Dr. Priyavrat C. Deshpande of CMI for teaching the concepts of algebraic topology and helping vastly improve my level of intuition.

Last but not least, I can not forget the support, care and warmth I have received from my friends, parents and family members, in particular, my brother who advised and encouraged me to join CMI to pursue a career in mathematics.

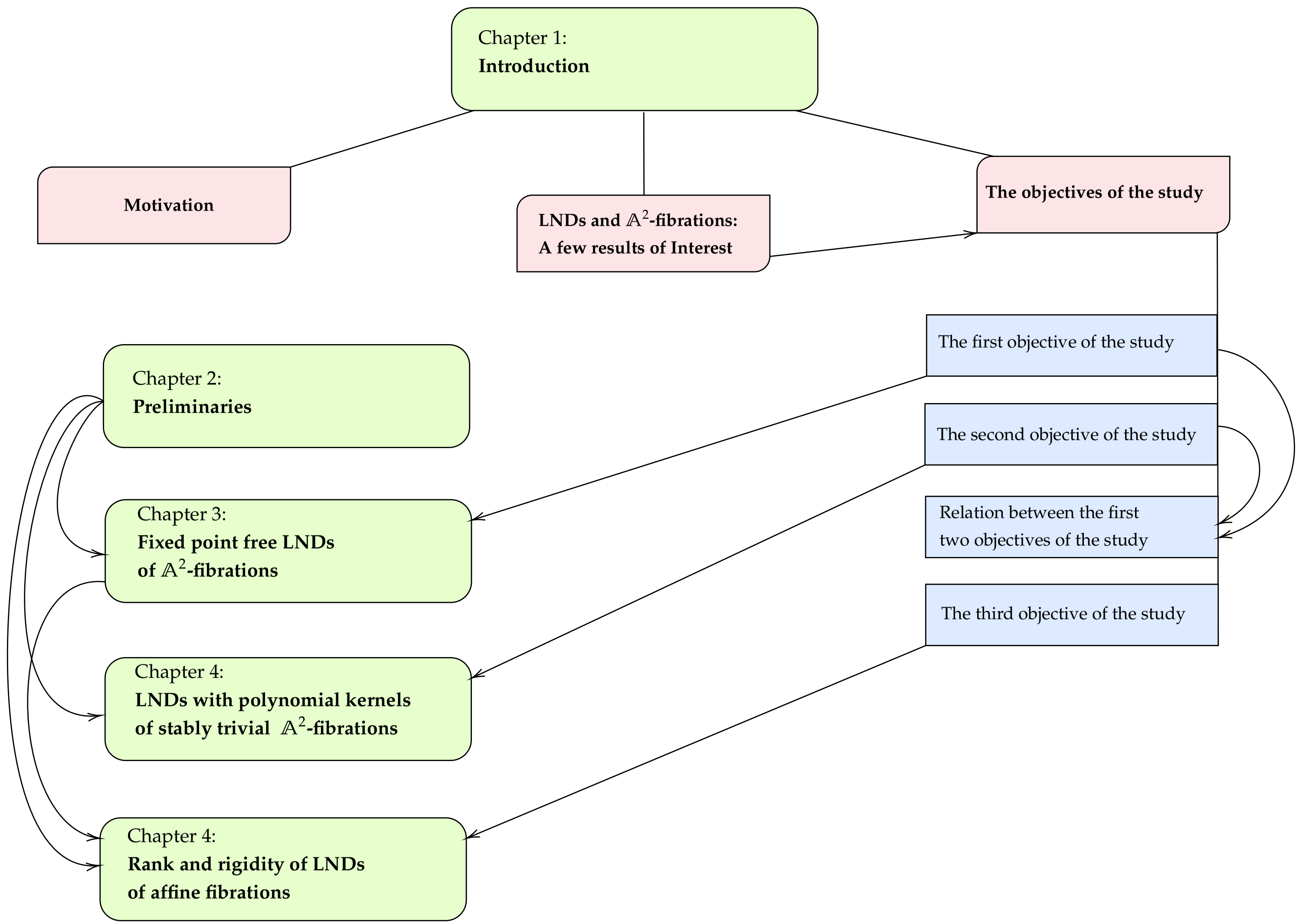
Contents

List of Notation

Throughout the thesis all rings will be assumed to be commutative rings with unity. Let R be a commutative ring, $R[X_1, X_2, \cdots, X_n]$ a polynomial algebra in n indeterminates $X_1, X_2, \cdots, X_n$ over R and A an R-algebra. We fix the following notation.

R^*	:	Group of units of R.
$\mathrm{nil}(R)$	:	Nilradical of R.
$\mathrm{Qt}(R)$	:	Total quotient ring of R.
$\mathrm{Spec}(R)$	:	The set of all prime ideals of R.
$\mathrm{MaxSpec}(R)$	:	The set of all maximal ideals of R.
$k(P)$	:	Residue field R_P/PR_P where $P \in \mathrm{Spec}(R)$.
$ht(I)$	:	The height of an ideal I of R.
$Qt(R)$	:	The field of fractions of R, when R is an integral domain.
$ch(R)$	:	Characteristic of R.
$\mathrm{Pic}(R)$	:	Picard group of R.
$R^{[n]}$	:	Polynomial algebra in n variables over R.
$A = R^{[n]}$	:	A is isomorphic, as an R-algebra, to the polynomial algebra $R^{[n]}$.
$\mathrm{Sym}_R(M)$	:	Symmetric algebra of an R-module M.
$\Omega_R(A)$	:	Universal module of R-differentials of A.
$\mathrm{Der}_R(A)$	:	Module of R-derivations of A.
$\mathrm{tr.deg}_R(A)$	:	Transcendence degree of A over R, where $R \subseteq A$ are domains.
R_P	:	$S^{-1}R$, where $S = R \backslash P$ and $P \in \mathrm{Spec}(R)$.
A_P	:	$A \otimes_R R_P$, for $P \in \mathrm{Spec}(R)$.
$Aut_R(A)$	:	The group of R-algebra automorphisms of A.
DVR	:	Discrete valuation ring.
PID	:	Principal ideal domain.
UFD	:	Unique factorization domain.
LND	:	Locally nilpotent derivation.
$\mathcal{J}(\underline{F})$ or $\mathcal{J}ac_{(\underline{X})}(\underline{F})$	:	Jacobian matrix of $\underline{F} := (F_1, F_2, \cdots, F_m)$ with respect to the indeterminates $\underline{X} := (X_1, X_2, \cdots, X_n)$ in $R[X_1, X_2, \cdots, X_n]$.
$\Delta_{\underline{F}}$ or $\mathcal{JD}_{(\underline{X})}(\underline{F}, -)$	:	Jacobian derivation defined by $\underline{F} := (F_1, F_2, \cdots, F_{n-1})$ with respect to the indeterminates $\underline{X} := (X_1, X_2, \cdots, X_n)$ in $R[X_1, X_2, \cdots, X_n]$.
grade(I)	:	$=$ depth(I, R), which is the length of a maximal R-sequence in I where R is a ring and I is an ideal of R.
D_K	:	The natural extension of an R-derivation $D : A \longrightarrow A$ on $A \otimes_R K$ where R is a domain and $K = \mathrm{Qt}(R)$.

Interdependence of the chapters and their sections



Part I

Introduction and preliminaries

Chapter 1

Introduction

1.1 The motivation

In this title of proposed thesis, the objects of study are locally nilpotent derivations (LNDs) of affine fibrations and subalgebras of affine fibrations. We, mainly, wish to explore the structure of $\mathbb{A}^2$-fibrations using the techniques of locally nilpotent derivations; and also wish to give a useful classification of the locally nilpotent derivations of $\mathbb{A}^2$-fibrations and characterize the classes completely. While locally nilpotent derivations can be realized as a generalization to partial derivatives, they also can be viewed as differential operators on polynomial algebras, and correspondingly the kernel of LNDs represent the solution space of differential equations corresponding to the considered differential operators. For example, one may look the articles [Ess92] and [Ess94] by van den Essen to see how the theory of LNDs can be applied to study certain problems in differential equations. A concise write-up by El Kahoui in [EK04] nicely expresses the applications of the theory of LNDs as follows.

"A classical application of derivations theory is the study of various questions such as first integrals and invariant algebraic sets for ordinary polynomial differential systems over the reals or the complexes. $\cdots$ Very often, the study of practical questions, arising for example from differential equations, leads to dealing with derivations over abstract rings, sometimes even nonreduced, of characteristic zero. One of the fundamental questions in this topic is to describe their rings of constants."

Before we discuss in depth about LNDs, we shall first define some terminologies.

Definition 1.1.1.

- *Let A be an R-algebra and M an A-module. A map $D : A \longrightarrow M$ is called a* ***derivation*** *if it satisfies the following properties.*

  (i) *Additivity:* $D(a+b) = D(a) + D(b)$

  (ii) *The Leibniz rule:* $D(ab) = aD(b) + bD(a)$ *for all* $a, b \in A$.

  The kernel of a derivation D, denoted by $Ker(D)$, forms a subring of A and is called the ***ring of constants*** *of the derivation D.*

- *Let A be an R-algebra and $D : A \longrightarrow A$ a derivation. D is said to be an R-**derivation of** A if $R \subset Ker(D)$. An R-derivation D is called **locally nilpotent** (R-LND or simply LND) if for each $a \in A$ there exists $n \in \mathbb{N}$ such that $D^n(a) = 0$. D is called **irreducible** if $D(A)A \nsubseteq (a)A$ for any non-unit element $a \in A$. D is called **fixed point free** if $D(A)A = A$. A **slice** $s \in A$ is an element satisfying $D(s) = 1$.*

Notation 1.1.2. *For a ring R and a field k, we fix the following notations.*

- *$R^{[n]}$ shall denote the polynomial algebra in n indeterminates over R.*
- *$\mathbb{A}^n_k$ shall denote the affine n-space over k.*
- *For $F_1, F_2, \cdots, F_m \in R[X_1, X_2, \cdots, X_n]$*

  (i) *The notation $\mathcal{J}ac_{(X_1,X_2,\cdots,X_n)}(F_1, F_2, \cdots, F_m)$ will denote the Jacobian matrix $\dfrac{\partial(F_1, F_2, \cdots, F_m)}{\partial(X_1, X_2, \cdots, X_n)}$ of the polynomials $F_1, F_2, \cdots, F_m$ with respect to the indeterminates $X_1, X_2, \cdots, X_n$. If the indeterminates $X_1, X_2, \cdots, X_n$ of the polynomial algebra are fixed, we do not need to mention the indeterminates and in that case the notation $\mathcal{J}(F_1, F_2, \cdots, F_m)$ shall denote $\mathcal{J}ac_{(X_1,X_2,\cdots,X_n)}(F_1, F_2, \cdots, F_m)$.*

  (ii) *The notation $\mathcal{JD}_{(X_1,X_2,\cdots,X_n)}(F_1, F_2, \cdots, F_{n-1}, -)$ will denote the map $g \longmapsto \det \mathcal{J}ac_{(X_1,X_2,\cdots,X_n)}(F_1, F_2, \cdots, F_{n-1}, g)$ [1]. For the cases where mentioning the indeterminates is not necessary, $\mathcal{JD}_{(X_1,X_2,\cdots,X_n)}(F_1, F_2, \cdots, F_{n-1}, -)$ will be denoted by $\Delta_{(F_1,F_2,\cdots,F_{n-1})}$, i.e., $\Delta_{(F_1,F_2,\cdots,F_{n-1})}(g) = \det \mathcal{J}ac_{(X_1,X_2,\cdots,X_n)}(F_1, F_2, \cdots, F_{n-1}, g)$.*

The modern approach to the theory of LNDs was started during 1960s by french mathematicians like Dixmier, Gabriel, Nouaze, and Rentscler while working in the areas of Lie algebras and Lie groups. As the theory of LNDs began to develop it was found that many famous algebraic problems can be translated in terms of LNDs. For example we state the original form of the "Jacobian Conjecture" and the "Zariski Cancellation Problem" and state them in terms of LNDs as follows.

Jacobian Conjecture: Let k be a field of characteristic zero and let $A = k[X_1, X_2, \cdots, X_n]$. Suppose, we have polynomials $G_1, G_2, \cdots, G_n$ in A such that $\det \mathcal{J}(G_1, G_2, \cdots, G_n) \in A^* = k^*$. The conjecture states that $\underline{G} := (G_1, G_2, \cdots, G_n)$ is invertible or equivalently $k[G_1, G_2, \cdots, G_n] = k[X_1, X_2, \cdots, X_n]$.

We now state the Jacobian Conjecture in terms of LNDs ([Fre17, p. 3.2.3]). Let $\underline{F} := F_1, F_2, \cdots, F_{n-1}$ be a sequence of elements in A. One can check that $\Delta_{\underline{F}}$ is a k-derivation of A. The Jacobian Conjecture states that suppose we know that $\Delta_{\underline{F}}$ has a slice, then it follows that $\Delta_{\underline{F}}$ is locally nilpotent and $\mathrm{Ker}(\Delta_{\underline{F}}) = k[F_1, F_2, \cdots, F_{n-1}]$.

[1] which is called the ***Jacobian determinant*** of the elements $F_1, F_2, \cdots, F_{n-1}, g$

Zariski Cancellation Problem: Let k be an algebraically closed field and V be an affine k-variety such that $V \times A^1_k \cong_k A^{n+1}_k$. Does it follow that $V \cong_k A^n_k$? In other words, is the affine n-space A^n_k cancellative?

The above statement can be translated in terms of locally nilpotent derivations as follows ([Ess00, pg. 54]). Let k be a field of characteristic zero and D a locally nilpotent derivation of $k^{[n+1]}$ with a slice. Does it follow that $\mathrm{Ker}(D) = k^{[n]}$?

Though there are many unsolved problems in the study of LNDs, e.g., finding the structure of LNDs of certain algebras, the problem of finite generation of kernels of LNDs e.t.c., solving which will enrich the theory of LNDs, the existing theory of LNDs has vast applications towards the problems related to algebraic geometry and affine algebraic geometry.

In algebraic geometry, for an algebraically closed field k, the $\mathbb{G}_a$-actions (or equivalently the actions of the additive group k^+) on $\mathbb{A}^n_k$ are very important objects of study. When the field k has characteristic zero, the LNDs help translate the study of geometric problems involving $\mathbb{G}_a$-actions on $\mathbb{A}^n_k$ in terms of algebraic problems involving exponential maps on the polynomial algebra $k^{[n]}$.

It can be shown (see [Dai10, Section 4]) that over an algebraically closed field of characteristic zero, there is a bijection between the $\mathbb{G}_a$-actions of k-variety X and the exponential maps of the coordinate ring of X; and further there is a bijection between the exponential maps of the coordinate ring of X and the LNDs of the coordinate ring of X, i.e., if B is the coordinate ring of X, then we have

$$\{\mathbb{G}_a\text{-actions on } X\} \iff \{\text{exponential maps on } B\} \iff \{\text{LNDs of } B\}.$$

So, to study algebraic actions it is enough to study LNDs whenever the base field is algebraically closed of characteristic zero.

For the convenience of the readers, the detailed discussion on the relationship between the $\mathbb{G}_a$-actions, the exponential maps and the LNDs has been done in Appendix A.

In the recent years, LNDs and $\mathbb{G}_a$-actions were used by many authors to give alternative proofs of well established results and to find counterexamples of famous problems. To quote a few:

- Rentschler [Ren68] gave a simple proof of Jung's theorem on automorphisms of polynomial ring $k[X, Y]$ ([Jun42])[2] where $k \hookleftarrow \mathbb{Q}$ using locally nilpotent derivations.

- Miyanishi [Miy73] used $\mathbb{G}_a$ actions (equivalent to exponential maps which in characteristic zero are nothing but LND) to prove the cancellation theorem of Abhyankar, Eakin, and Heinzer.

[2] Jung's Theorem asserts that every plane automorphism is tame in the case when base field is of characteristic zero.

- Hilbert's fourteenth problem: Let K be a field, $K[X] = K[X_1, \cdots, X_m]$ the polynomial ring in m variables over K, and $K(X)$ its field of fractions. Then, *the fourteenth problem of Hilbert* asks whether the K-subalgebra $L \cap K[X]$ of $K[X]$ is finitely generated whenever L is a subfield of $K(X)$ containing K.

  For the case $m \leq 2$ the answer to the Hilbert's Fourteenth problem is affirmative due to Zariski (see [Zar54]). However, it was Nagata, in 1958, who first presented a counterexample to the above problem (see [Nag60]) for the case $m = 2n^2$ where $n \geq 4 \in \mathbb{N}$ (i.e., $m \geq 32$).

  Almost for thirty years since Nagata's construction of the example it was not known whether a counterexample to the problem exists for $m \leq 31$; especially for the case $m = 3$. Roberts, in 1988 ([Rob90]), gave a counterexample to the Hilbert's Fourteenth problem for $m = 7$ using $\mathbb{G}_a$-actions. Subsequently, using techniques of LNDs (or equivalently $\mathbb{G}_a$-actions), counter examples to the problem were given in lower dimensions by Daigle-Freudenburg for $m = 6$ ([DF99]), Freudenburg for $m = 5$ ([Fre00]), and finally by Kuroda in the last two lowest dimensions, i.e., for $m = 4, 3$ ([Kur04] [Kur05]).

- Makar-Limanov in [ML96] (1996) showed the Russel-Koras threefold defined by $f = x + x^2y + z^2 + t^3$ is not isomorphic to $\mathbb{C}^3$. To establish the result, he developed new techniques of locally nilpotent derivations leading to the definition of AK-invariant or the ring of absolute constants (now called as the Makar-Limanov invariant or ML-invariant). To differentiate between algebraic structures ML-invariant is one of the prominent tool which is used extensively.

- Zariski Cancellation Problem:

  1. Crachiola and Makar-Limanov in [CML08], using the techniques of LNDs, gave a simplified algebraic proof to the cancellation theorem of Miyanishi-Sugie and Fujita ([MS80] and [Fuj79]), i.e., the proof that the Zariski Cancellation Problem (see p. 5) has affirmative answer in characteristic zero for $n = 2$.
  2. Gupta in [Gup14a], [Gup14b] and [Gup14c] completely solved the long standing Zariski Cancellation Problem in positive characteristics. Her method mainly depends upon techniques originating from LNDs e.g., exponential maps, Derksen invariant etc. The technique she has developed is now broadly being used to understand different algebraic structures in affine algebraic geometry.

1.2 LNDs and $\mathbb{A}^2$-fibrations: A few results of interest

An important property of LNDs is the "Slice Theorem" which was first discovered by Gabriel-Nouaz ([NG67]). Subsequently, many mathematicians have given proofs to this crucial result, e.g., Dixmier ([Dix77]), Miyanishi ([Miy78]), Wright ([Wri81]). However, we register below a version of "Slice Theorem" by Wright ([Wri81, Proposition 2.1]).

Theorem 1.2.1. *Let R be a ring containing $\mathbb{Q}$, A an R-algebra and $D : A \longrightarrow A$ an R-LND. If D has a slice $s \in A$, then we have $A = Ker(D)[s] = Ker(D)^{[1]}$.*

The converse also holds, provided D is irreducible, i.e., if D is irreducible and $A = Ker(D)^{[1]}$, then D has a slice.

If a locally nilpotent derivation D has a slice, then it can be proven that D is surjective, i.e., $D(A) = A$, provided the base ring R contains $\mathbb{Q}$ (A short proof of it is included here: Let s be a slice then for $y \in A$, by the Slice Theorem (Theorem 1.2.1) write $y = c_0 + c_1 s + \cdots + c_{n-1}s^{n-1}$ with $c_i \in \text{Ker}(D)$ then $x = c + c_0 s + c_1 s^2/2 + \cdots + c_n s^n/n$ for any $c \in \text{Ker}(D)$ will satisfy $D(x) = y$). Since any LND having a slice is fixed point free, naturally the converse asks

Question 1.2.2. *Let R be a ring containing $\mathbb{Q}$, A be an R-algebra and $D : A \longrightarrow A$ an R-LND. Suppose $D(A)A$ is an unit ideal i.e $D(A)A = A$, does it imply D has a slice?*

For the case A is a polynomial algebra, the following results are established.

(a) Answer to Question 1.2.2 is affirmative when A is a polynomial algebra in one indeterminate and is deduced as follows.

Let $D : R[X] \longrightarrow R[X]$ be a fixed point free R-LND. Since D is uniquely determined by its image $D(R[X])$ which is generated by $D(X)$ itself over $R[X]$, and since $D(R[X])R[X] = R[X]$, there exists $a \in R[X]$ such that $aD(X) = 1$. Let $a = a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m$ where $a_i \in R$ for all $i \in \{0, 1, \cdots, m\}$. Then, by the additivity and the Leibniz rule property of D it can be observed that $D(a_0 X + \frac{a_1}{2}X^2 + \frac{a_2}{3}X^3 + \cdots + \frac{a_m}{m+1}X^{m+1}) = 1$. Now since $\mathbb{Q} \subseteq R$, it follows that $\frac{a_i}{i+1} \in R$ for all $i \in \{0, 1, \cdots, m\}$. Hence, $s := a_0 X + \frac{a_1}{2}X^2 + \frac{a_2}{3}X^3 + \cdots + \frac{a_m}{m+1}X^{m+1} \in R[X]$ is a slice of D.

(b) Question 1.2.2 was studied for polynomial algebra in two variables ($A = R^{[2]}$) and positive results were obtained in phases by the following authors.

   (a) Rentschler in [Ren68]: R is a field.
   (b) Daigle and Freudenburg in [DF98]: R is an UFD.
   (c) Bhatwadekar and Dutta in [BD97]: R is a Noetherian domain containing $\mathbb{Q}$. (†)
   (d) Berson-van den Essen-Maubach in [BEM01]: R is any ring containing $\mathbb{Q}$ and D is divergence free.
   (e) van den Essen in [Ess07]: R is any ring containing $\mathbb{Q}$.

The final result can be quoted as

Theorem 1.2.3. *Let R be any ring containing $\mathbb{Q}$ and $D : R[X,Y] \longrightarrow R[X,Y]$ be a fixed point free R-LND, then D has a slice s so that $R[X,Y] = Ker(D)[s]$.*

(c) For the case $A = R^{[3]}$ there exists a counterexample due to the works of Winklemann ([Win90]) and Snow ([Sno89]) .

Winklemann in 1990 constructed a $\mathbb{C}[T]$-LND D on $\mathbb{C}[T][X,Y,Z]$ as follows.

$$D(X) = T \quad D(Y) = X \quad \text{and} \quad D(Z) = X^2 - TY - 1.$$

Now since $-YD(X) + XD(Y) - D(Z) = 1$, it implies that D is fixed point free. Now to prove D does not have a slice, it is enough to show that any one of its conjugate derivations $E = \sigma^{-1}D\sigma$ (where σ denotes an automorphism of $R^{[3]}$) have the same property. Snow, in [Sno89], constructed an LND $E : R^{[3]} \longrightarrow R^{[3]}$ such that $E(X) = T$, $E(Y) = X$ and $E(Z) = 1 + X^2$ which is a conjugate of D. van den Essen ([Ess00, pg. 41]) proved that E does not have a slice, and therefore, it follows that D does not have a slice.

When A is not a polynomial algebra, the first result towards Question 1.2.2 is by Kahoui-Ouali in [EKO12] for the case A is an $\mathbb{A}^2$-fibration over a regular ring. When A is an $\mathbb{A}^1$-fibration, Kahoui-Ouali solved Question 1.2.2 completely in [EKO14, Section 2.1]. Our interest is the case when A is an $\mathbb{A}^2$-fibration. We shall discuss the corresponding results in Section 1.3.1. Before going into the detailed discussion we first define $\mathbb{A}^n$-fibrations (affine n-fibrations). Affine fibrations are important objects of study (see [Miy07]) in affine algebraic geometry, and they occur naturally, for example,

(i) If R is a Noetherian domain containing $\mathbb{Q}$ and $f \in R[X,Y] = R^{[2]}$ is such that $R[X,Y]/(f) = R^{[1]}$, then $R[X,Y]$ is an $\mathbb{A}^1$-fibration over $R[f]$ (see [Bha88, Lemma 3.5]).

(ii) If R is a ring containing $\mathbb{Q}$ and $D : R^{[3]} \longrightarrow R^{[3]}$ is an R-LND with a slice, then one can see that $\operatorname{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R (see [Fre09, Theorem 1.1]).

One can easily notice that for a ring R, the polynomial algebra $R^{[n]}$ necessarily satisfy the following conditions.

1. $R^{[n]}$ is finitely generated over R,
2. $R^{[n]}$ is flat over R, and
3. $R^{[n]} \otimes_R k(P) = k(P)^{[n]}$ for every prime ideal P of R; i.e., the fibers of $R^{[n]}$ at each prime ideal P of R are n-dimensional polynomial algebras.

Definition 1.2.4. *Let R be a ring and A an R-algebra. A is called an* ***$\mathbb{A}^n$-fibration*** *(or* ***affine n-fibration****) if it satisfies the above listed three conditions.*

The above definition of affine fibration was introduced by Sathaye in [Sat83], where he proved the path breaking result that any $\mathbb{A}^2$-fibration over a discrete valuation ring (DVR) containing $\mathbb{Q}$ is a polynomial algebra ([Sat83, Theorem 1]), i.e.,

Theorem 1.2.5. *Let R be a DVR containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Then, $A = R^{[2]}$.*

Definition 1.2.6. *Let A be an $\mathbb{A}^n$-fibration over a ring R. A is said to be* ***trivial*** *if A is a polynomial algebra over R, i.e., $A = R^{[n]}$. A is said to be* ***stably trivial*** *if A is a stably polynomial algebra over R, i.e., there exists $\ell \in \mathbb{N}$ such that $A^{[\ell]} = R^{[\ell+n]}$.*

One should note that an open problem by Dolgačev-Veĭsfeĭler [VD74] itself asks whether an affine fibration A over a ring R is a polynomial algebra. Sathaye's result (Theorem 1.2.5) solved this problem for the case when R is a regular local ring of dimension one containing

$\mathbb{Q}$ and A is an $\mathbb{A}^2$-fibration over R. Answer to this problem is not known even when R is a regular local ring of dimension two containing $\mathbb{Q}$ and A is an $\mathbb{A}^2$-fibration over R.

After Sathaye's result, the most significant work in the theory of affine fibration is the structure theorem by Asanuma ([Asa87, Theorem 3.4]).

Theorem 1.2.7. *Let R be a Noetherian ring and A an $\mathbb{A}^r$-fibration over R. Then, $\Omega_R(A)$ is a projective A-module of rank r and A is an R-subalgebra (up to an isomorphism) of a polynomial ring $R^{[m]}$ for some $m \in \mathbb{N}$ such that $A^{[m]} = Sym_{R^{[m]}}(\Omega_R(A) \otimes_A R^{[m]})$ as R-algebras.*

When the base ring is a one-dimensional Noetherian domain containing $\mathbb{Q}$, Asanuma-Bhatwadekar in [AB97], established that any $\mathbb{A}^2$-fibration has the following nice structure ([AB97, Theorem 3.8]).

Theorem 1.2.8. *Let R be a one-dimensional Noetherian domain containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Then, there exists $W \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[W]$.*

1.3 The objectives of the study

The study in this thesis keeps three aims. The first two aims are related to studying the structure of $\mathbb{A}^2$-fibration using LNDs of it and exploring a certain class of LNDs of $\mathbb{A}^2$-fibrations, which is discussed in Part II, consists of Chapter 3 and Chapter 4. The third aim is to explore a possible concept of rank and rigidity of LNDs of affine fibrations that help to understand the relation between the kernel of the LNDs, the ambient affine fibration and the base space, which is discussed in Part III consisting only Chapter 5.

1.3.1 The first objective of the study

While fixed point free LNDs of $R^{[3]}$ need not have slice, Freudenburg, in [Fre09], observed that the kernels of LNDs of $R^{[3]}$ with slice have nice structures; they are $\mathbb{A}^2$-fibrations over the base ring R ([Fre09, Corollary 2.2]).

Lemma 1.3.1. *Let R be a ring containing $\mathbb{Q}$. If D is an R-LND of $R^{[3]}$ with slice, then $Ker(D)$ is an $\mathbb{A}^2$-fibration over R.*

One may note that Lemma 1.3.1 is due to the fact that the Zariski Cancelation Problem has an affirmative answer in dimension two over fields containing $\mathbb{Q}$ (follows from [Fuj79], [MS80] and [Kam75]); and an alternative version of Lemma 1.3.1 is the following.

Lemma 1.3.2. *If A is a stably polynomial algebra over a Noetherian domain R containing $\mathbb{Q}$ where $tr.deg_R(A) = 2$, then A is an $\mathbb{A}^2$-fibration over R.*

Freudenburg, in [Fre09], also observed that $\mathbb{A}^2$-fibrations over polynomial algebras over fields containing $\mathbb{Q}$ are trivial if and only if they have LNDs with slice ([Fre09, Theorem 3.1]).

Theorem 1.3.3. *Let $\mathbb{Q} \hookrightarrow k$ be a field, $R = k^{[n]}$ and A an $\mathbb{A}^2$-fibration over R. Then, $A = R^{[2]}$ if and only if A has an R-LND with slice.*

And thereby, Freudenburg asked (see [Fre09, Question 2])

Question 1.3.4. *Let $\mathbb{Q} \hookrightarrow k$ be a field, $R = k^{[n]}$ and A an $\mathbb{A}^2$-fibration over R. Suppose, A has a fixed point free LND. Is then $A = R^{[2]}$?*

From Theorem 1.3.3, it follows that if the fixed point free LNDs of $\mathbb{A}^2$-fibrations have slice, then Question 1.3.4 has an affirmative answer. Kahoui-Ouali, in [EKO12], solved Question 1.3.4 completely. To be specific,

Theorem 1.3.5. *Let R be a Noetherian normal domain containing $\mathbb{Q}$ and let A be a locally stably polynomial $\mathbb{A}^2$-fibration over R. Let D be a fixed point free locally nilpotent R-derivation of A. Then D has a slice if and only if $Ker(D)$ is an $\mathbb{A}^1$-fibration over R. Consequently,*

(I) *If R is a regular ring, then $Ker(D)$ is an $\mathbb{A}^1$-fibration; and therefore D has a slice, i.e., $A = Ker(D)^{[1]}$.*

(II) *If R is a regular UFD, then $Ker(D) = R^{[1]}$; and therefore D has a slice, i.e., $A = Ker(D)^{[1]} = R^{[2]}$.*

However, the question whether fixed point free LNDs of $\mathbb{A}^2$-fibrations have slice (i.e., Question 1.2.2 for the case A is an $\mathbb{A}^2$-fibration) remained open. Specifically,

Question 1.3.6. *Let R be a ring containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Suppose, D is a fixed point free locally nilpotent derivation of A. Does D have a slice?*

Here it needs a mention that Kahoui-Ouali, in [EKO14], established that a fixed point free locally nilpotent derivation of an $\mathbb{A}^1$-fibration has a slice, thereby settling the Question 1.2.2 when A is an $\mathbb{A}^1$-fibration over R ([EKO14, Corollary 2.5]).

Theorem 1.3.7. *Let R be a Noetherian domain containing $\mathbb{Q}$, A an $\mathbb{A}^1$-fibration over R and $D : A \longrightarrow A$ a fixed point free R-derivation. Then, D has a slice, i.e., $A = R^{[1]}$ and $Ker(D) = R$.*

In [EKO16], Kahoui-Ouali, in view of Theorem 1.2.3, gave a partial affirmative answer to Question 1.3.6 under the condition that A is a stably polynomial algebra over R (see [EKO16, Theorem 3.1]).

Theorem 1.3.8. *Let R be a ring. If A is a stably polynomial $\mathbb{A}^2$-fibration over R i.e., A also satisfies $A^{[m]} = R^{[m+2]}$ for some $m \in \mathbb{N}$, then $A = R^{[2]}$ if and only if A has a fixed point free R-LND.*

Therefore, Question 1.3.6 remained unsolved for the case either A is a non-trivial $\mathbb{A}^2$-fibration or A is a non-stably polynomial $\mathbb{A}^2$-fibration.

In [BD21], we have completely solved Question 1.3.6 for the case when R is a Noetherian ring (see [BD21, Theorem 4.4 and Theorem 4.7]). Our result proves that if D is a fixed point free R-LND on an $\mathbb{A}^2$-fibration over a Noetherian ring R containing $\mathbb{Q}$, then D has a slice. A consequence of our result proves that when R is a one-dimensional Noetherian ring containing $\mathbb{Q}$, then an $\mathbb{A}^2$-fibration having a fixed point free R-LND has the structure as described in Theorem 1.2.8. The detailed discussion on our results has been done in the Chapter 3 (Part-II) of the thesis.

1.3.2 The second objective of the study

In view of Theorem 1.2.3, one should note that under the case R is a Noetherian ring containing $\mathbb{Q}$, Hamann's cancellation result (see Theorem 2.3.6) establishes that $\text{Ker}(D) = R^{[1]}$. In this regard the contribution of Bhatwadekar-Dutta (†) listed in pg.7 towards Theorem 1.2.3 needs a special mention. In their paper Bhatwadekar-Dutta characterizes all the irreducible locally nilpotent derivations D with polynomial kernels in $R[X,Y]$ when R is a Noetherian ring, to be specific ([BD97, Theorem 4.7 and Corollary 4.9]).

Theorem 1.3.9. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K, $A = R[X,Y]$ and D an R-LND on A. The following are equivalent*
(i) D is irreducible and $\text{Ker}(D) = R^{[1]}$.
(ii) DX and DY form an A-regular sequence or are comaximal in A.
(iii) There exists $F \in R[X,Y]$ such that $D = \mathcal{JD}_{(X,Y)}(F,-)$, $K[X,Y] = K[F]^{[1]}$ and F_X, F_Y either form an A-regular sequence or are comaximal in A.
Moreover if DX and DY are comaximal in A, then $A = \text{Ker}(D)^{[1]}$.

Based on the Theorem 1.3.9 of Bhatwadekar-Dutta, we ask

Question 1.3.10. *Is it possible to characterize the LNDs having polynomial kernels of a stably polynomial algebra A over a Noetherian domain R containing $\mathbb{Q}$ where $\text{tr.deg}_R(A) = 2$?*

In Theorem 1.3.9, since $A = R[X,Y]$, one can naturally talk about the sequential properties of the canonical generators of $D(A)$, i.e., DX and DY. However considering Question 1.3.10, as A is an $\mathbb{A}^2$-fibration over R, there does not exist such concept of canonical generators of D(A) whose sequential properties can be studied.

Remark 1.3.11. *We observe that for the case A = R[X,Y], the sequential properties of D(X) and D(Y) can be translated in terms of grade(D(R[X,Y])R[X,Y]) and further, grade(D(A)A) exists even when A is not a polynomial algebra.*

Before we progress further, we definite a terminology.

Definition 1.3.12. *Let R be a Noetherian ring and I an ideal of R. We shall say $\text{grade}(I) = \infty$ if $I = R$. For $\ell \in \mathbb{N}$, we shall say $\text{grade}(I) \geq \ell$ if either $\text{grade}(I) = \infty$ or $\ell \leq \text{grade}(I) < \infty$.*

As we expect an answer parallel to Theorem 1.3.9 to Question 1.3.10, the above observation in Remark 1.3.11 paves a path to ask a more specific question.

Question 1.3.13. *Let A be an $\mathbb{A}^2$-fibration over Noetherian domain R containing $\mathbb{Q}$ such that $A^{[n]} = A[T_1, T_2, \cdots, T_n] = R[X_1, X_2, \cdots, X_{n+2}] = R^{[n+2]}$ and $D : A \longrightarrow A$ an R-LND. Then, are the following statements equivalent?*

(i) *D is irreducible and* $Ker(D) = R^{[1]}$.

(ii) $grade(D(A)A) \geq 2$.

(iii) $\widetilde{D} = \mathcal{JD}_{(X_1,X_2,\cdots,X_{n+2})}(F, T_1, T_2, \cdots, T_n, -)$ *where* $\widetilde{D}$ *is the trivial extension of* D *on* $A[T_1, T_2, \cdots, T_n]$, $A \otimes_R K = K[F]^{[1]}$ *and* $grade((F_{X_1}, \cdots, F_{X_{n+2}})A[T]) \geq 2$.

It is to be noted that the result of Kahoui-Ouali, i.e., Theorem 1.3.8 gives an affirmative partial answer to Question 1.3.13 when D is fixed point free, i.e., $D(A)A$ is an unit ideal.

Chapter 4 (Part-II) of our thesis discusses on Question 1.3.13 and gives a characterization of LNDs with polynomial kernels of stably polynomial $\mathbb{A}^2$-fibrations.

1.3.3 Relation between the first two objectives of study

It can be observed that the above stated two aims, Question 1.3.6 and Question 1.3.13, of studies have a single origin, i.e., the corresponding problems are related.

Let R be a Noetherian domain containing $\mathbb{Q}$, A an $\mathbb{A}^2$-fibration over R and D a non-zero R-LND of A. Then, D can be of two types:

(I) $\text{grade}(D(A)A) \geq 2$.

(II) $\text{grade}(D(A)A) = 1$.

When $A = R^{[2]}$, from Remark 1.3.11 and Theorem 1.3.9, one can observe that the R-LNDs with $\text{grade}(D(A)A) \geq 2$ have nice structures and those LNDs impart nice structure on the ambient algebra too. Further, such R-LNDs of $R^{[2]}$ have beautiful characterizations; to be specific (see Theorem 1.3.9)

Corollary 1.3.14. *Let R be a Noetherian domain containing $\mathbb{Q}$, $A = R[X, Y]$ and D be an R-LND on A. The following are equivalent*

(i) D is irreducible and $Ker(D) = R^{[1]}$.

(ii) $grade(D(A)A) \geq 2$.

Moreover if $grade(D(A)A) = \infty$, then $A = Ker(D)^{[1]}$.

If $\text{grade}(D(A)A) = 1$, it is clear from Corollary 1.3.14 that either D is reducible or $\text{Ker}(D) \neq R^{[1]}$ and vice versa. However, since R is Noetherian, for any reducible R-LND δ there exists an irreducible R-LND δ^* such that $\delta = \lambda\delta^*$ for some $\lambda \in R \setminus \{0\}$, and in that case we have $\text{Ker}(\delta) = Ker\delta^*$. So, we see that for $A = R^{[2]}$, we have $\text{grade}(D(A)A) = 1$ if and only if $\text{Ker}(D) \neq R^{[1]}$. Bhatwadekar-Dutta, in [BD97, Theorem 3.5], has given the complete description of the kernel of any LND D of a polynomial algebra $R[X, Y]$ over a Noetherian normal domain R containing $\mathbb{Q}$. So, even for the case $\text{grade}(D(A)A) = 1$, the structure of $\text{Ker}(D)$ is known when $A = R^{[2]}$ and R is a Noetherian normal domain. Moreover, a recent result of Khaddah-Kahoui-Ouali ([KEKO22]) gives a description of the structure of $R[X, Y]$, where R is a PID, over the kernel of any of its R-LND D stating that $R[X, Y]$ is a free module over $\text{Ker}(D)$.

So, naturally one asks whether it is possible to give a complete description of the LNDs of $\mathbb{A}^2$-fibrations and their kernels; and the relation of the kernel with the ambient $\mathbb{A}^2$-fibration, similar to the case of polynomial algebra in two indeterminates, i.e.,

Problem 1.3.15. *Let R be a Noetherian domain containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Then, characterize all the R-LNDs $D : A \longrightarrow A$, find the structure of the $Ker(D)$, and find the structure of A over $Ker(D)$ for the following cases*

(a) *$grade(D(A)A) = \infty$, i.e., D is fixed point free.*

(b) *$2 \leq grade(D(A)A) < \infty$.*

(c) *$grade(D(A)A) = 1$.*

Our study focuses on the class of R-LNDs D of A such that grade$(D(A)A) \geq 2$, i.e., either grade$(D(A)A) = \infty$ or $2 \leq$ grade$(D(A)A) < \infty$; and solves Problem 1.3.15(a) completely (see Chapter 3) and partially solve Problem 1.3.15(b) for the case A is a stably polynomial (see Chapter 4). We keep an aim to give a complete solution to Problem 1.3.15(b) and (c) as a post-PhD work.

1.3.4 The third objective of the study

In the theory of LNDs of polynomial algebras the concepts of rank and rigidity are important tools which helped bring in some major results, e.g., [Fre95], [Dai96], [DF98]. While the concept of the rank which is existing only for LNDs of polynomial algebras was introduced by Freudenburg in his article [Fre95], the concept of rigidity of LNDs of polynomial algebras was originated by Daigle in [Dai96]. Rank and rigidity of LNDs of polynomial algebras are defined as follows.

Definition 1.3.16. *Let $A = R^{[n]}$ and $D : A \longrightarrow A$ an R-LND.*

- *The **rank** of D, denoted by $Rk(D)$, is defined to be the least non-negative integer r such that there exists a coordinate system $(X_1, X_2, \cdots, X_n)$ of A satisfying $X_1, X_2, \cdots, X_{n-r} \in Ker(D)$.*
- *A rank-r R-LND D of A is called **rigid** if, for any two coordinate systems $(X_1, X_2, \cdots, X_n)$ and $(Y_1, Y_2, \cdots, Y_n)$ of A satisfying $X_1, X_2, \cdots, X_{n-r}, Y_1, Y_2, \cdots, Y_{n-r} \in Ker(D)$, we have $R[X_1, X_2, \cdots, X_{n-r}] = R[Y_1, Y_2, \cdots, Y_{n-r}]$.*

Since many aspects of affine fibrations can be studied through the LNDs of them, in that case questions arise: whether the kernel B of an LND of an affine fibration A over a base ring R is necessarily an affine fibration over R and whether the given affine fibration

A over R is also an affine fibration over the kernel B. When the affine fibration A is a polynomial algebra, the rank and rigidity of LNDs keep a good amount of information about the kernel of the LNDs and its relation with the the ambient polynomial algebra and the base ring. However, the concept of rank has been defined only for polynomial algebras, and therefore, it is natural to ask whether a suitable notion of rank of LNDs of affine fibrations can be defined that may help study the relations between the kernel of the LNDs, the ambient affine fibration and the base ring, and which is consistent with the existing concept of rank of LNDs of polynomial algebras.

The Part-III of the thesis, i.e., Chapter 5 is devoted towards a notion of rank and rigidity of locally nilpotent derivations of affine fibrations. In Chapter 5, we define residual rank and residual rigidity of LNDs of affine fibrations and show that the concept is analogous to the perception of rank and rigidity of LNDs of polynomial algebras.

1.4 Thesis organization

The thesis has four parts. Part I of the thesis is devoted to the introduction and preliminaries. The main research contributions of our study have been discussed in Part II and Part III. The thesis includes an appendix in Part IV, which discusses the detailed relations between the $\mathbb{G}_a$ actions, exponential maps and LNDs.

Chapter 2

Preliminaries

In this chapter we setup notations, recall definitions and register some known results.

2.1 Notation

Given a ring R and an R-algebra A we fix the following notation.

R^*	:	Group of units of R.
$\text{nil}(R)$	:	Nilradical of R.
$\text{Qt}(R)$	:	Total quotient ring of R.
$\text{Pic}(R)$	:	Picard group of R.
$\text{Sym}_R(M)$	:	Symmetric algebra of an R-module M.
$\Omega_R(A)$	:	Universal module of R-differentials of A.
$\text{Der}_R(A)$	:	Module of R-derivations of A.
$\text{tr.deg}_A(R)$	:	Transcendence degree of A over R, where $R \subseteq A$ are domains.
A_P	:	$A \otimes_R R_P$, for $P \in \text{Spec}(R)$.
grade(I)	:	$= \text{depth}(I, R)$ which is the length of a maximal R-sequence in I where R is Noetherian and I is an ideal of R.

2.2 Definitions

Definition 2.2.1.

- *A reduced ring R is called* ***seminormal*** *if whenever $a^2 = b^3$ for some $a, b \in R$, then there exists $t \in R$ such that $t^3 = a$ and $t^2 = b$.*

- *A subring R of a ring A is called a* ***retract*** *of A, if there exists a ring homomorphism $\phi : A \longrightarrow R$ such that $\phi(r) = r$ for all $r \in R$.*

- *A subring R of a domain A is said to be* ***inert (factorially closed)*** *in A, if $fg \in R$ implies $f, g \in R$ for all $f, g \in A \backslash \{0\}$.*

- *An algebra A over a ring R is said to be an* ***$\mathbb{A}^n$-fibration*** *(or* ***affine n-fibration****) if A is finitely generated and flat over R, and $A \otimes_R k(P) = k(P)^{[n]}$ for all $P \in Spec(R)$, i.e., the fibers of A at each prime ideal P of R are n-dimensional polynomial algebras.*

- *Let A be an $\mathbb{A}^n$-fibration over R.*
  1. *A is said to be **trivial** if A is a polynomial algebra over R, i.e., $A = R^{[n]}$.*
  2. *A is said to be **stably trivial** if A is a stably polynomial algebra over R, i.e., there exists $\ell \in \mathbb{N}$ such that $A^{[\ell]} = R^{[\ell+n]}$.*
  3. *An m-tuple of elements $\underline{W} := (W_1, W_2, \cdots, W_m)$ in A which are algebraically independent over R is called an m-**tuple residual variable** of A if $A \otimes_R k(P) = (R[\underline{W}] \otimes_R k(P))^{[n-m]}$ for all $P \in Spec(R)$.*

- *A domain R is called an **HCF domain** if for any two elements a, b in R, the ideal $(a) \cap (b)$ is principal. HCF domains are often called **GCD domains**.*

2.3 Preliminary results

First we list down some properties of inert subrings and retracts.

Lemma 2.3.1. *Let $B \subseteq A$ be domains.*

(i) *If C is a ring such that $B \subseteq C \subseteq A$ and B is inert in A, then B is inert in C.*

(ii) *If B is inert in A, then B is algebraically closed in A.*

(iii) *Let $B_1 \subseteq B_2 \subseteq A$ be domains such that B_1 is inert in A. If $tr.deg_{B_1}(A) = tr.deg_{B_2}(A) < \infty$, then $B_1 = B_2$.*

(iv) *An inert subring of a HCF domain (UFD) is a HCF domain (UFD); and a polynomial algebra over a HCF domain (UFD) is a HCF domain (UFD).*

(v) *Retract of a UFD is a UFD.*

Proof. Proofs of (i), (ii), (iii) and (iv) are easy. For a proof of (v), one may refer to the arguments by Ed Enochs mentioned in ([EH73], p.69). □

Next we observe a local-global property of the grade of an ideal of an algebra.

Lemma 2.3.2. *Let $R \subseteq A$ be Noetherian domains and I be an ideal of A. Then, the following holds.*

(I) *$grade(I) = \infty$ if and only if $grade(IA_P) = \infty$ for all $P \in Spec(R)$.*

(II) *If $grade(I) < \infty$, then $grade(I) = grade(IA_P)$ for some $P \in Spec(R)$.*

Proof. (I): By a standard local-global principle it quickly follows that $\text{grade}(I) = \infty$ if and only if $IA_P = A_P$ for all $P \in \text{Spec}(R)$.

(II): By [Mat80, pg. 105] there exists $Q \in \text{Spec}(A)$ such that $\text{grade}(I) = \text{grade}(IA_Q)$. Let $P = Q \cap R$. Clearly, $I \subseteq IA_P \subseteq IA_Q$ and hence we have $\text{grade}(I) \leq \text{grade}(IA_P) \leq \text{grade}(IA_Q)$. This shows that $\text{grade}(I) = \text{grade}(IA_P)$. □

We now quote a few results for later use. The first one is by Rentschler ([Ren68])

Theorem 2.3.3. *Let k be a field of characteristic zero, $A = k[X,Y] = k^{[2]}$ and $D : A \longrightarrow A$ a non-trivial k-LND. Then, there exists $F, G \in A$ such that $A = k[F,G]$ and $Ker(D) = k[F]$. Further, there exists $\alpha \in k[F]$ such that $D = \alpha\Delta_F$ where Δ_F is the derivation defined by $\Delta_F(X) = -F_Y$ and $\Delta_F(Y) = F_X$, or in other words $D = \alpha \mathcal{JD}_{(X,Y)}(F, -)$*

The following two Lüroth-type results are by Abhyankar-Eakin-Heinzer ([AEH72, Remark 2.10] and [AEH72, Proposition 4.1 & Proposition 4.8]).

Theorem 2.3.4. *Let k be a field, L a separable algebraic field extension of k and A a normal domain such that $k \subsetneqq A \subseteq L^{[1]}$. If L is algebraically closed in A, then $A = L^{[1]}$.*

Theorem 2.3.5. *Let $R \subseteq B$ be domains such that $tr.deg_R(B) = 1$ and $R \subseteq B \subseteq A = R^{[n]}$. If either R is an HCF domain and B is inert in A or R and B are UFDs, then $B = R^{[1]}$.*

A cancellation result of Hamann ([Ham75, Theorem 2.8]) states

Theorem 2.3.6. *Let R be a Noetherian ring containing $\mathbb{Q}$ and A an R-algebra such that $A^{[m]} = R^{[m+1]}$ for some $m \in \mathbb{N}$. Then, $A = R^{[1]}$.*

The following result of Bass-Connell-Wright ([BCW76, Theorem 4.4]) gives a characterization of locally polynomial algebras.

Theorem 2.3.7. *Let A be a finitely presented R-algebra such that A_P is R_P-isomorphic to the symmetric algebra of some R_P-module for each $P \in Spec(R)$. Then, A is R-isomorphic to the symmetric algebra $Sym_R(M)$ for some finitely presented R-module M.*

We now state a result by Russel-Sathaye, famously known as the Russel-Sathaye criteria for an algebra to be a polynomial algebra ([RS79, Theorem 2.3.1], also see [BD97, Theorem 2.11]).

Theorem 2.3.8. *Let R be a domain and A a finitely generated overdomain of R. Suppose that there exists an element π in R which is prime in A such that $\pi A \cap R = \pi R$, $A_\pi = R_\pi^{[1]}$ and the image of $R/\pi R$ is algebraically closed in $A/\pi A$. Then $A = R^{[1]}$.*

The next result is by Swan ([Swa80, Theorem 6.1]).

Theorem 2.3.9. *Let R be a seminormal ring. Then, $Pic(R) = Pic(R^{[n]})$ for all $n \in \mathbb{N}$.*

As an immediate consequence of Theorem 1.2.7 ([Asa87]), Theorem 2.3.9, Theorem 2.3.6 and Theorem 2.3.7 the following holds.

Corollary 2.3.10. *Let R be a Noetherian ring containing $\mathbb{Q}$ and A an $\mathbb{A}^1$-fibration over R. If $\Omega_R(A)$ is extended from R, specifically when R is seminormal, then $A = Sym_R(N)$ for some finitely generated rank one projective R-module N.*

Next, we state a few properties of affine fibrations.

Lemma 2.3.11. *Let R be a ring and A an $\mathbb{A}^n$-fibration over R. Then A is faithfully flat over R.*

Proof. Since A is flat over R and $A \otimes_R R/\mathfrak{m} = A \otimes_R k(\mathfrak{m}) = k(\mathfrak{m})^{[n]} \neq (0)$ for each maximal ideal $\mathfrak{m}$ of R, it follows that A is faithfully flat over R. □

Lemma 2.3.12. *Let R be a domain and A an $\mathbb{A}^n$-fibration over R. Then, A is a domain and R is inert in A.*

Proof. Since R is a domain, we have $A \hookrightarrow A \otimes_R \mathrm{Qt}(R) = A \otimes_R k(0) = k(0)^{[n]} = \mathrm{Qt}(R)^{[n]}$. This shows that A is a domain and $\mathrm{Qt}(R)$ is inert in $A \otimes_R \mathrm{Qt}(R)$. Let $f, g \in A \backslash \{0\}$ be such that $fg \in R$. We shall show that $f, g \in R$. Due to inertness of $\mathrm{Qt}(R)$ in $A \otimes_R \mathrm{Qt}(R)$, we see that $f, g \in \mathrm{Qt}(R)$, and therefore, $f, g \in A \cap \mathrm{Qt}(R)$. Suppose, $f = r/s$ for some $r \in R$ and $s \in R\backslash\{0\}$, and hence, $sf = r \in R$. Since A is an $\mathbb{A}^n$-fibration over R, by Lemma 2.3.11 it follows that A is faithfully flat over R, and therefore, $sA \cap R = sR$. This shows that $r = sf \in sR$, and hence, $f \in R$. Similarly, we have $g \in R$. □

Asanuma, in his structure theorem of affine fibrations, i.e., Theorem 1.2.7 ([Asa87]), established that for an affine fibration A over a Noetherian ring R, the module of differentials $\Omega_R(A)$ is a projective A-module and A can be viewed as an R-subalgebra of a polynomial algebra B over R in such a way that $A \otimes_R B$ is a symmetric B-algebra of the extended projective B-module $\Omega_R(A) \otimes_A B$. As a consequence of Asanuma's result the following can be observed.

Lemma 2.3.13. *Let R be a Noetherian ring and A an $\mathbb{A}^n$-fibration over R. Then, R is a retract of A and A is a retract of $R^{[t]}$ for some $t \in \mathbb{N}$.*

Proof. Since A is an $\mathbb{A}^n$-fibration over a Noetherian ring R, by Theorem 1.2.7 ([Asa87]), $\Omega_R(A)$ is a projective A-module and there exists $m \in \mathbb{N}$ such that A is a R-subalgebra of $R^{[m]}$ with the property $A^{[m]} = \mathrm{Sym}_{R^{[m]}}(\Omega_R(A) \otimes_A R^{[m]})$. Using Lemma 2.3.11 we see that A is a faithfully flat R-algebra, and therefore, R can be seen as a subring of A. Clearly, R is a retract of $R^{[m]}$, and hence, R is a retract of A. Now, we shall show that A is a retract of $R^{[t]}$ for some $t \in \mathbb{N}$.

Since $\Omega_R(A)$ is a projective A-module, $\Omega_R(A) \otimes_A R^{[m]}$ is a projective $R^{[m]}$-module, and therefore, we have $N \oplus (\Omega_R(A) \otimes_A R^{[m]}) = (R^{[m]})^\ell$ for some projective $R^{[m]}$-module N and $\ell \in \mathbb{N}$. From this we get

$$
\begin{aligned}
R^{[m+\ell]} &= \mathrm{Sym}_{R^{[m]}}((R^{[m]})^{\ell}) \\
&= \mathrm{Sym}_{R^{[m]}}(N \oplus (\Omega_R(A) \otimes_A R^{[m]})) \\
&= \mathrm{Sym}_{R^{[m]}}(N) \otimes_{R^{[m]}} \mathrm{Sym}_{R^{[m]}}(\Omega_R(A) \otimes_A R^{[m]}) \\
&= \mathrm{Sym}_{R^{[m]}}(N) \otimes_{R^{[m]}} A^{[m]}.
\end{aligned}
$$

Since any symmetric algebra has a natural retraction to its base ring, we see that $R^{[m]}$ is a retract of $\mathrm{Sym}_{R^{[m]}}(N)$, and therefore, $R^{[m]} \otimes_{R^{[m]}} A^{[m]} = A^{[m]}$ is a retract of $\mathrm{Sym}_{R^{[m]}}(N) \otimes_{R^{[m]}} A^{[m]} = R^{[m+\ell]}$. Again, since A is a retract of $A^{[m]}$, we see that A is a retract of $R^{[m+\ell]}$. □

A criterion for finite generation of algebras by Onoda ([Ono84, Theorem 2.20]) is as follows.

Theorem 2.3.14. *Let R be a Noetherian domain and A an overdomain of R such that*

(I) *There exists a non-zero $f \in A$ for which $A[1/f]$ is a finitely generated R-algebra.*

(II) *A_m is a finitely generated R_m-algebra for all maximal ideals m of R.*

Then A is a finitely generated R-algebra.

We quote below a result of Daigle on rigidity of LNDs of polynomial algebras over a field containing $\mathbb{Q}$ ([Dai96, Theorem 2.5].

Theorem 2.3.15. *Let k be a field containing $\mathbb{Q}$. Then, all LNDs of $k^{[3]}$ are rigid.*

We again register three results by Daigle ([Dai97, Theorem 3.3], [Dai97, Corollary 3.10] and [Dai97, Corollary 2.4] respectively).

Theorem 2.3.16. *Let A be a UFD containing $\mathbb{Q}$ and let B be inert in A. Assume that A is of finite presentation as a B-algebra. Then A is almost smooth over B.*

Proposition 2.3.17. *Let k be a field of characteristic zero, $B = k[X_1, \cdots, X_n] = k^{[n]}$, $A = k[f_1, \cdots, f_m]$ a k-subalgebra of B of dimension d. Let J denote the ideal of B generated by the $d \times d$ minors of the Jacobian matrix $\mathcal{J}ac_{(X_1,\cdots,X_n)}(f_1, \cdots, f_m)$. Suppose, A is inert in B, then $ht(J) > 1$.*

Corollary 2.3.18. *Let k be a field of characteristic zero and $B = k[X_1, \cdots, X_n] = k^{[n]}$. Suppose, $\underline{f} = (f_1, \cdots, f_{n-1}) \in B^{n-1}$ is such that $f_1, \cdots, f_{n-1}$ are algebraically independent over k and $k[f_1, \cdots, f_{n-1}]$ is inert in B. Then, $\Delta_{\underline{f}} \in Der_k(B)$ is irreducible and $Ker(\Delta_{\underline{f}}) = k[f_1, \cdots, f_{n-1}]$.*

We now register a few results of Das-Dutta from [DD14]). For the first two results one may refer to [DD14, Lemma 2.1] and [DD14, Corollary 3.18] respectively.

Lemma 2.3.19. *Let A be a stably polynomial algebra over a ring R. Then $\Omega_R(A)$ is a stably free A-module*

Corollary 2.3.20. *Let R be a Noetherian domain and A an $\mathbb{A}^{m+1}$-fibration over R. Then an m-tuple $\underline{W}$ of elements from A is an m-tuple residual variable of A over R if and only if A is an $\mathbb{A}^1$-fibration over $R[\underline{W}]$.*

The following is again a result of Das-Dutta ([DD14, Corollary 3.6, Lemma 3.12, Theorem 3.16 & Corollary 3.19]).

Theorem 2.3.21. *Let R be a Noetherian ring and A an $\mathbb{A}^n$-fibration over R. Suppose, $\underline{W} \in A$ is an m-tuple residual variable of A. Then, A is an $\mathbb{A}^{n-m}$-fibration over $R[\underline{W}]$ and $\Omega_R(A) = \Omega_{R[\underline{W}]}(A) \oplus A^m$. Further, if A is stably polynomial over $R \hookleftarrow \mathbb{Q}$ and $n - m = 1$, then $A = R[\underline{W}]^{[1]} = R^{[n]}$.*

It is to be noted that though Das-Dutta, in [DD14], proved Theorem 2.3.21 (see [DD14, Corollary 3.19]) with the hypothesis that the base ring is a Noetherian domain containing $\mathbb{Q}$, from their proof it follows that Theorem 2.3.21 holds over Noetherian rings (not necessarily domains) containing $\mathbb{Q}$.

In [EKO14], Kahoui-Ouali proved the following result ([EKO14, Corollary 2.5]).

Proposition 2.3.22. *Let R be a Noetherian domain containing $\mathbb{Q}$, and let A be an $\mathbb{A}^1$-fibration over R. Then A is trivial over R if and only if there exists $\delta \in Der_R(A)$ which is fixed point free.*

We end this section with one of our observations.

Lemma 2.3.23. *Let R be a one-dimensional Noetherian domain containing $\mathbb{Q}$, $A = R^{[3]}$ and D a fixed point free R-LND of A. Then, the following are equivalent.*

(I) *$Ker(D) = R^{[2]}$ and $A = Ker(D)^{[1]}$.*

(II) *D has a slice.*

Proof. (I) $\implies$ (II): Follows from the converse of the Slice Theorem (Theorem 1.2.1).

(II) $\implies$ (I): Since D has a slice, by the Slice Theorem (Theorem 1.2.1) we have $A = \text{Ker}(D)^{[1]}$, and therefore, by Lemma 1.3.1 it follows that $\text{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R. Since R is a one-dimensional Noetherian domain containing $\mathbb{Q}$, by Theorem 1.2.8 ([AB97, Theorem 3.8]) there exists $W \in \text{Ker}(D)$ such that $\text{Ker}(D)$ is an $\mathbb{A}^1$-fibration over $R[W]$, and hence, by Corollary 2.3.20 W is a residual variable of $\text{Ker}(D)$. Since $\text{Ker}(D)^{[1]} = R^{[3]}$, by Lemma 2.3.19 and Theorem 2.3.21 we get $\text{Ker}(D) = R[W]^{[1]} = R^{[2]}$. This completes the proof. □

Part II

A class of LNDs of $\mathbb{A}^2$-fibrations

Chapter 3

Fixed point free LNDs of $\mathbb{A}^2$-fibrations

In this chapter our main aim is to solve the following problem quoted in Introduction as Question 1.3.6. For our convenience, we rewrite it again.

The main problem to solve in this chapter:
Given an $\mathbb{A}^2$-fibration A over a ring $R \supseteq \mathbb{Q}$ and $D : A \longrightarrow A$ a fixed point free LND of A, our aim is to investigate whether D has a slice.

We show, without any extra hypothesis on A, that Question 1.3.6 has an affirmative answer when the base ring R is Noetherian, specifically (see Theorem 3.3.2)

Theorem 3.A: Let R be a Noetherian ring containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R with a fixed point free R-LND $D : A \longrightarrow A$. Then, $\mathrm{Ker}(D)$ is an $\mathbb{A}^1$-fibration over R and D has a slice, i.e., $A = \mathrm{Ker}(D)^{[1]}$. In particular, if R is a normal domain, then $A = \mathrm{Sym}_R(I)^{[1]}$ for some invertible ideal I of R.

In view of Theorem 3.A, we investigate the hypothesis "A is stably polynomial over R" in Theorem 1.3.8 and find that it helps to generate another fixed point free R-LND of A for which the $\mathbb{A}^2$-fibration A becomes trivial. More precisely, we observe the following (see Theorem 3.3.5).

Theorem 3.B: Let R be a Noetherian domain containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R having a fixed point free R-LND. Then, A has another irreducible R-LND $D : A \longrightarrow A$ such that $\mathrm{Ker}(D) = R^{[1]}$, and A is an $\mathbb{A}^1$-fibration over $\mathrm{Ker}(D)$. Further, the following are equivalent.

(I) D is fixed point free.

(II) A is stably polynomial over R.

(III) $A = R^{[2]}$.

The following is the outline of this chapter. In Section 3.1 we observe some auxilary results; in Section 3.2 we review results on $\mathbb{A}^1$-fibrations having fixed point free derivations; in Section 3.3 we discuss structure of $\mathbb{A}^2$-fibrations having fixed point free LNDs; and in Section 3.4 we discuss a few examples.

3.1 Auxiliary results

In this section, we observe some auxiliary results needed to prove the main results of this chapter.

The following lemma is useful for reduction to the case when R is a reduced ring for proving an R-algebra A is polynomial ring in n variables. To be precise.

Lemma 3.1.1. *Let R be a Noetherian ring, $\eta = nil(R)$ and A an R-algebra. Suppose, $x_i \in A$ where $i = 1, 2, \cdots, n$ are such that $A/\eta A = R/\eta[\bar{x_1}, \bar{x_2}, \cdots, \bar{x_n}]$ where $\bar{x_i}$'s are the images of x_i's in $A/\eta A$. Then, $A = R[x_1, x_2, \cdots, x_n]$.*

Proof. Clearly, $A = R[x_1, x_2, \cdots, x_n] + \eta A$. Since there exists $\ell \in \mathbb{N}$ such that $\eta^\ell = (0)$, we see that $A = R[x_1, x_2, \cdots, x_n]$. □

Next result helps in providing conditions for an R-algebra A to be $R^{[1]}$ given that the induced R/η-algebra is singly generated.

Lemma 3.1.2. *Let R be a Noetherian ring containing $\mathbb{Q}$, $\eta = nil(R)$ and A an R-algebra such that $A/\eta A$ is a singly-generated R/η-algebra and $(A/\eta A)^* = (R/\eta R)^*$. Then, $A = R^{[1]}$ if and only if there exists $D \in Der_R(A)$ such that the induced $R/\eta R$-derivation $D : A/\eta A \longrightarrow A/\eta A$ is fixed point free.*

Proof. If $A = R^{[1]}$, then it is easy to see that there exists a fixed point free R/η-derivation of $A/\eta A$. So, we prove the converse. According to the hypotheses, $A/\eta A$ is generated by a single element over R/η, and therefore, by Lemma 3.1.1, we have $A = R[x]$ for some $x \in A$. Suppose, $D \in \mathrm{Der}_R(A)$ is such that the induced R/η-derivation $\overline{D} : A/\eta A \longrightarrow A/\eta A$ is fixed point free. By letting $\bar{x}$ to be the image of x in $A/\eta A$, it is easy to see that $\overline{D}(\bar{x}) \in (A/\eta A)^* = (R/\eta)^*$, i.e., the image of $D(x)$ in $A/\eta A$ is a unit in R/η, and therefore, $D(x)$ is a unit in R. We claim that x does not satisfy any algebraic relation over R. On the contrary, if there exists $a_i \in R$ for all $i = 0, 1, \cdots, n$, $a_n \neq 0$ such that $a_0 + a_1 x + \cdots + a_n x^n = 0$, then we have $D(x)(a_1 + 2a_2 x + 3a_3 x^2 \cdots + n a_n x^{n-1}) = 0$. Since $D(x) \in R^*$, we get $a_1 + 2a_2 x + 3a_3 x^2 \cdots + n a_n x^{n-1} = 0$, from which again we see that $D(x)(2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots + n(n-1) a_n x^{n-2}) = 0$. Repeating same arguments, we eventually get $D(x)(n!)a_n = 0$, i.e., $a_n = 0$ which is a contradiction to our assumption that $a_n \neq 0$. This proves that $A = R[x] = R^{[1]}$. □

The next lemma shows that a derivation restricts onto a subalgebra provided it is a retraction. i.e,

Lemma 3.1.3. *Let $C \subseteq B$ be algebras over a ring R with a retraction $\phi : B \longrightarrow C$. Suppose, $\widetilde{D} : B \longrightarrow B$ is an R-derivation. Then, $D := (\phi \circ \widetilde{D})|_C : C \longrightarrow C$ is an R-derivation.*

Proof. Easy. □

Suppose an $\mathbb{A}^2$-fibration A over R is a polynomial ring in one variable over a R-subalgebra C. The next lemma describes the structure of such an R-subalgebra C.

Lemma 3.1.4. *Let C, A be algebras over a Noetherian ring R containing $\mathbb{Q}$ such that A is an $\mathbb{A}^2$-fibration over R and $A = C[W] = C^{[1]}$. Then, C is an $\mathbb{A}^1$-fibration over R and A is an $\mathbb{A}^1$-fibration over $R[W]$.*

Proof. Clearly, C is a finitely generated R-subalgebra of A, and further, C, being a direct summand of the flat R-module A, is flat over R. Let $P \in \mathrm{Spec}(R)$. Now, $k(P)^{[2]} = A \otimes_R k(P) = (C \otimes_R k(P))^{[1]}$, and therefore, by Theorem 2.3.6 ([Ham75]), we get $C \otimes_R k(P) = k(P)^{[1]}$. This shows that C is an $\mathbb{A}^1$-fibration over R. Again, as $C \otimes_R k(P) = k(P)^{[1]}$, we see that $A \otimes_R k(P) = (C \otimes_R k(P))[W] = (R[W] \otimes_R k(P))^{[1]}$. This proves that W is a residual variable of A, and therefore, by Theorem 2.3.21 ([DD14]), A is an $\mathbb{A}^1$-fibration over $R[W]$. This completes the proof. □

3.2 Structure of $\mathbb{A}^1$-fibrations having fixed point free LNDs

In [EKO14], Kahoui-Ouali proved Proposition 2.3.22 ([EKO14]) that states that an $\mathbb{A}^1$-fibration over a Noetherian domain containing $\mathbb{Q}$ is trivial if and only if it has a fixed point free derivation. In this section we show that the result of Kahoui-Ouali holds even over Noetherian rings (not necessarily domains) containing $\mathbb{Q}$ (see Proposition 3.2.1), which we shall use in the next section. Though the proof of this observation follows from Kahoui-Ouali's proof of Proposition 2.3.22 ([EKO14]), for the convenience of the readers it is detailed here.

Proposition 3.2.1. *Let R be a Noetherian ring containing $\mathbb{Q}$ and A an $\mathbb{A}^1$-fibration over R. Then the following are equivalent.*

(I) $A = R^{[1]}$.

(II) *$\Omega_R(A)$ is a free A-module.*

(III) *$\Omega_R(A)$ is a stably free A-module.*

(IV) *There exists $D \in Der_R(A)$ such that D is fixed point free.*

Proof. (I) $\implies$ (II), (II) $\implies$ (III) and (I) $\implies$ (IV): Obvious.

(III) $\implies$ (I): Suppose that $\Omega_R(A)$ is a stably free A-module. Then, there exists $n \in \mathbb{N}$ such that $\Omega_R(A) \oplus A^n = A^{n+1}$. Since A is an $\mathbb{A}^1$-fibration over R, by Theorem 1.2.7 ([Asa87]), there exists $\ell \in \mathbb{N}$ such that A is an R-subalgebra of $B = R^{[\ell]}$ and $A^{[\ell]} = \mathrm{Sym}_B(\Omega_R(A) \otimes_A B)$, and therefore, we have $A^{[n+\ell]} = A \otimes_R B^{[n]} = A \otimes_R B \otimes_B B^{[n]} =$

$A^{[\ell]} \otimes_B B^{[n]} = \text{Sym}_B((\Omega_R(A) \otimes_A B) \otimes_B B^{[n]} = \text{Sym}_B((\Omega_R(A) \oplus A^n) \otimes_A B)$. Now, since $(\Omega_R(A) \oplus A^n) \otimes_A B = B^{n+1}$, we essentially have $A^{[n+\ell]} = B^{[n+1]} = R^{[n+\ell+1]}$, and hence, by Theorem 2.3.6 ([Ham75]), we get $A = R^{[1]}$.

(IV) $\implies$ (I): Suppose that $D \in \text{Der}_R(A)$ is fixed point free. Let us assume that R is reduced. Since the total quotient ring K of R is zero-dimensional reduced Noetherian ring, we see that $A \otimes_R K = K^{[1]} = K[U]$, say, for some $U \in A$. Suppose, $D_1 \in \text{Der}_R(A)$. Letting $D(U) = \alpha \in A$ and $D_1(U) = \beta \in A$, we have $\alpha D_1 = \beta D$. Since D is fixed point free there exists $\alpha_1, \alpha_2, \cdots, \alpha_m \in A$ and $u_1, u_2, \cdots, u_m \in A$ such that $\sum_{i=1}^{m} \alpha_i D(u_i) = 1$, and therefore, $\sum_{i=1}^{m} \alpha_i \beta D(u_i) = \beta$. Now, since $\alpha D_1 = \beta D$, we get $\sum_{i=1}^{m} \alpha_i \alpha D_1(u_i) = \beta$, i.e., $\alpha \sum_{i=1}^{m} \alpha_i D_1(u_i) = \beta$. This shows that $\alpha D_1 = \alpha \sum_{i=1}^{m} \alpha_i D_1(u_i) D$.

Let $\widetilde{D} : A \otimes_R K \longrightarrow A \otimes_R K$ be the extension of D. Clearly, $\widetilde{D}$ is fixed point free. Since $A \otimes_R K = K[U]$, we have $\widetilde{D}(U) = D(U) = \alpha \in K^*$, i.e., α is a non-zero divisor in R. Since A is flat over R, α remains a non-zero divisor in A, and therefore, $D_1 = \sum_{i=1}^{m} \alpha_i D_1(u_i) D$. This proves that $\text{Der}_R(A) = \text{Hom}_A(\Omega_R(A), A)$ is a free A-module of rank one. Since $\Omega_R(A)$ is a projective A-module, it is a reflexive A-module, and therefore, $\Omega_R(A)$ is a free A-module. Consequently, by "(II) $\implies$ (I)", we get $A = R^{[1]}$.

Now, we suppose that R is not reduced. Set $\eta := \text{Nil}(R)$. Clearly, the induced R/η-derivation $\overline{D} : A/\eta A \longrightarrow A/\eta A$ is fixed point free. Since $A/\eta A$ is an $\mathbb{A}^1$-fibration over R/η, from the previous discussion we have $A/\eta A = (R/\eta)^{[1]} = (R/\eta)[X]$, say, and therefore, by Lemma 3.1.1, we get $A = R[X]$. Finally, due to Lemma 3.1.2, it follows that $A = R[X] = R^{[1]}$. □

3.3 Structure of $\mathbb{A}^2$-fibrations having fixed point free LNDs

The upcoming proposition illustrates the structure of kernel of a fixed point free derivation on a symmetric algebra over a rank two projective module.

Proposition 3.3.1. *Let R be a Noetherian ring containing $\mathbb{Q}$ and $A = Sym_R(M)$ for some finitely generated rank two projective R-module M. Suppose, $D : A \longrightarrow A$ is a fixed point free R-LND, then $Ker(D) = Sym_R(N)$ for some finitely generated rank one projective R-module N and $A = Ker(D)^{[1]}$.*

Proof. By Theorem 1.2.3 we see that $A_P = \text{Ker}(D)_P^{[1]}$ and $\text{Ker}(D)_P = R_P^{[1]}$ for all $P \in \text{Spec}(R)$. This shows that $A_Q = \text{Ker}(D)_Q^{[1]}$ for all $Q \in \text{Spec}(\text{Ker}(D))$, and therefore, by Theorem 2.3.7 ([BCW76]) we have $A = \text{Sym}_{\text{Ker}(D)}(L)$ for some finitely generated rank one projective $\operatorname{Ker}(D)$-module L, which proves that $\operatorname{Ker}(D)$, being a retract of the finitely generated R-algebra A, is a finitely generated R-subalgebra of A. Since $\operatorname{Ker}(D)_P = R_P{}^{[1]}$ for all $P \in \operatorname{Spec}(R)$, by Theorem 2.3.7 ([BCW76]) $\operatorname{Ker}(D) = \operatorname{Sym}_R(N)$ for some rank one projective R-module N. Moreover, since $A = \operatorname{Sym}_{\operatorname{Ker}(D)}(L)$, by Proposition 3.2.1 we see that $A = \operatorname{Ker}(D)^{[1]}$. □

Now we will state our first main result (Theorem 3.A) which gives answer to Question 1.3.6.

Theorem 3.3.2. *Let R be a Noetherian ring containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Suppose, $D : A \longrightarrow A$ is a fixed point free R-LND. Then, Ker(D) is an $\mathbb{A}^1$-fibration over R and $A = \text{Ker}(D)^{[1]}$. Further, if $\Omega_R(A)$is extended from R, specifically, when R is seminormal, then Ker$(D) = \text{Sym}_R(N)$ for some finitely generated rank one projective R-module N.*

Proof. Since A is an $\mathbb{A}^2$-fibration over R, by Theorem 1.2.7 ([Asa87]) there exists $B = R^{[n]}$ such that $A \subseteq B$ and $A^{[n]} = A \otimes_R B = \operatorname{Sym}_B(\Omega_R(A) \otimes_A B)$ where $\Omega_R(A)$ is a finitely generated projective A-module of rank two. Let $\widetilde{D} := D \otimes 1 : A \otimes_R B \longrightarrow A \otimes_R B$ be the trivial extension of D. Note that $\widetilde{D}$ is fixed point free and $\operatorname{Ker}(\widetilde{D}) = \operatorname{Ker}(D) \otimes_R B$. Since $\Omega_R(A)$ is a projective A-module, $\Omega_R(A) \otimes_A B$ is a projective B-module, and therefore, applying Proposition 3.3.1 we get $\operatorname{Ker}(D) \otimes_R B = \operatorname{Sym}_B(L)$ for some finitely generated rank one projective B-module L and $A \otimes_R B = (\operatorname{Ker}(D) \otimes_R B)^{[1]}$. Since $B = R^{[n]}$, we have $A^{[n]} = A \otimes_R B = (\operatorname{Ker}(D) \otimes_R B)^{[1]} = \operatorname{Ker}(D)^{[n+1]}$, and therefore, by Theorem 2.3.6 ([Ham75]) we have $A = \operatorname{Ker}(D)^{[1]}$. Finally, using Lemma 3.1.4 we see that $\operatorname{Ker}(D)$ is an $\mathbb{A}^1$-fibration over R.

Now, we assume that $\Omega_R(A)$ is extended from R, i.e., $\Omega_R(A) = M \otimes_R A$ for some R-module M. Since $\Omega_R(A)$ is a projective A-module of rank two and A is faithfully flat over R, due to faithful descent property of finite projective module we see that M is a rank two projective R-module. Since A is an $\mathbb{A}^2$-fibration over R, from earlier arguments we have $A \subseteq R^{[n]}$ and

$$
\begin{aligned}
A^{[n]} &= \operatorname{Sym}_{R^{[n]}}(\Omega_R(A) \otimes_A R^{[n]}) \\
&= \operatorname{Sym}_{R^{[n]}}((M \otimes_R A) \otimes_A R^{[n]}) \\
&= \operatorname{Sym}_R(M) \otimes_R R^{[n]} \\
&= \operatorname{Sym}_R(M)^{[n]}
\end{aligned}
$$

Thus, we have $\operatorname{Ker}(D)^{[n+1]} = A^{[n]} = \operatorname{Sym}_R(M)^{[n]}$, and therefore, for each $P \in \operatorname{Spec}(R)$, we get $\operatorname{Ker}(D)_P^{[n+1]} = (\operatorname{Sym}_R(M))_P^{[n]} = R_P^{[n+2]}$, from which, by Theorem 2.3.6 ([Ham75]) we see that $\operatorname{Ker}(D)_P = R_P^{[1]}$. Now, applying Theorem 2.3.7 ([BCW76]) we have $\operatorname{Ker}(D) = \operatorname{Sym}_R(N)$ for some rank one projective R-module N. When R is seminormal, the result follows directly from Corollary 2.3.10, as $\operatorname{Ker}(D)$ is an $\mathbb{A}^1$-fibration over R. □

As a consequence of Theorem 3.3.2 we can summarize the following equivalent conditions for an $\mathbb{A}^2$-fibration over a noetherian ring R containing $\mathbb{Q}$ to have a fixed point free R-LND.

Corollary 3.3.3. *Let R be a Noetherian ring containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Then, the following statements are equivalent.*

(I) *A has a fixed point free R-LND.*

(II) *A has an R-LND with a slice.*

(III) *$A = C[W] = C^{[1]}$ for some R-subalgebra C of A.*

(IV) *$A = C[W] = C^{[1]}$ where $C \subseteq A$ is an $\mathbb{A}^1$-fibration over R.*

(V) *A is an $\mathbb{A}^1$-fibration over $R[W] = R^{[1]}$ where $W \in A$ and there exists $B = R[W][U_1, U_2, \cdots, U_n] = R[W]^{[n]}$, for some $n \in \mathbb{N}$, along with a retraction $\phi : B \longrightarrow A$ such that $\partial_W(\phi(U_i)) = 0$.*

Proof. (I) $\iff$ (II) $\implies$ (III): Follows from Theorem 3.3.2.

(III) $\implies$ (IV): Follows from Lemma 3.1.4

(IV) $\implies$ (II): Since $A = C[W] = C^{[1]}$, A has a C-LND D with a slice, and therefore, D is an R-LND of A with a slice.

(IV) $\implies$ (V): From Lemma 3.1.4 it follows that A is an $\mathbb{A}^1$-fibration over $R[W]$. Now, since C is an $\mathbb{A}^1$-fibration over R, by Theorem 1.2.7 ([Asa87]) there exists $B' = R[U_1, U_2, \cdots, U_n] = R^{[n]}$ for some $n \in \mathbb{N}$ along with a retraction $\phi_1 : B' \longrightarrow C$, which induces a retraction $\phi : B \longrightarrow C[W] = A$ such that $\phi|_{B'} = \phi_1$ where $B = B'[W]$. Clearly, $\partial_W(\phi(U_i)) = 0$ for all $i = 1, 2, \cdots, n$.

(V) $\implies$ (II): Set $D := (\phi \circ \partial_W)|_A : A \longrightarrow A$. By Lemma 3.1.3 it follows that D is an R-derivation of A. We shall show that D is an R-LND with a slice W. Clearly, $D(W) = 1$.

Let $\alpha(\underline{U}) \in A \cap R[U_1, U_2, \cdots, U_n]$. Note that $\phi(\alpha(\underline{U})) = \alpha(\underline{U})$. One may check that

$$D(\alpha(\underline{U})) = 0 \tag{3.1}$$

and

$$D^i(\alpha(\underline{U})W^m) = m(m-1)\cdots(m-i+1)\, \alpha(\underline{U})W^{m-i} \text{ for all } i = 1, \cdots, m \tag{3.2}$$

Let $f \in A$. Then, $f = \alpha_0(\underline{U}) + \alpha_1(\underline{U})W + \alpha_2(\underline{U})W^2 + \cdots + \alpha_m(\underline{U})W^m$ for some $\alpha_i(\underline{U})$'s in $R[U_1, U_2, \cdots, U_n]$, and therefore, $f = \phi(f) = \phi(\alpha_0(\underline{U})) + \phi(\alpha_1(\underline{U}))W + \phi(\alpha_2(\underline{U}))W^2 + \cdots + \phi(\alpha_m(\underline{U}))W^m$. Now, using 3.1 and 3.2 we see that $D^{m+1}(f) = 0$. This shows that D is an R-LND of A with a slice W. $\square$

As a follow up of the above corollary we now state a related problem on the structure of $\mathbb{A}^2$-fibration.

Problem 3.3.4. *Let R be a ring containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Is then A an $\mathbb{A}^1$-fibration over $R[V]$ for some V in A?*

To know the origin of Problem 3.3.4, one may refer to [AB97] (also see [VD74], [Sat83], [Asa87]and [BD94]). While Problem 3.3.4 is open in general, it is known that it has a negative answer even when R is a two-dimensional regular UFD (see Example 3.4.2). However, the following landmark results give partial affirmative answers to Problem 3.3.4. Sathaye ([Sat83]) proved that $A = R^{[2]}$, if R is a DVR. A result of Bass-Connell-Wright ([BCW76]) along with the result of Sathaye show that $A = R^{[2]}$ holds even if R is a PID. Later, Asanuma-Bhatwadekar ([AB97, Theorem 3.8 and Remark 3.13]) showed that A is an $\mathbb{A}^1$-fibration over $R[W]$ for some $W \in A$, if R is an one-dimensional Noetherian ring. For more related results one may look at [DER02], [Ess07], [DF10], [Fre09], [EKO12], [EKO14], [Das15], and [EKO16].

As Kahoui-Ouali's result ([EKO16]) gives an affirmative answer to Question 1.3.6 under the assumption that A is a stably polynomial algebra and in that case the $\mathbb{A}^2$-fibration becomes trivial (i.e., polynomial algebra), we here show that the phenomemon is actually due a "companion" LND of the given fixed point free LND, which becomes fixed point free when the $\mathbb{A}^2$-fibration is stably trivial. For the detailed analysis, one needs to go though Theorem 3.3.5 and Remark 3.3.7(B).

Theorem 3.3.5. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K and A an $\mathbb{A}^2$-fibration over R having a fixed point free R-LND. Then, A has another irreducible R-LND $D : A \longrightarrow A$ such that $\mathrm{Ker}(D) = R^{[1]}$ and A is an $\mathbb{A}^1$-fibration over $\mathrm{Ker}(D)$. Further, the following are equivalent.*

(I) *D is fixed point free.*

(II) *A is stably polynomial over R.*

(III) $A = R^{[2]}$

Proof. Suppose, $\delta : A \longrightarrow A$ is a fixed point free R-LND. Then, by Theorem 3.3.2 $\mathrm{Ker}(\delta)$ is an $\mathbb{A}^1$-fibration over R and $A = \mathrm{Ker}(\delta)[V] = \mathrm{Ker}(\delta)^{[1]}$ for some $V \in A$. Since K is the quotient field of R, we have $\mathrm{Ker}(\delta) \otimes_R K = K[U_0] = K^{[1]}$ for some U_0 in $\mathrm{Ker}(\delta)$, and therefore, $A \otimes_R K = K[V, U_0]$. Since $\mathrm{Ker}(\delta)$ is finitely generated over R, there exists $t \in R \setminus \{0\}$ such that $\mathrm{Ker}(\delta)[1/t] = R[1/t][U_0]$, which enables us to choose $\alpha \in \mathbb{N}$ and a K-LND $\widetilde{D}$ on $A \otimes_R K$ such that $\widetilde{D}(V) = 0$, $\widetilde{D}(U_0) = t^{\alpha}$, and $\widetilde{D}(A) \subseteq A$. So, $D := \widetilde{D}|_A$ is an R-LND of A such that $R[V] \subseteq \mathrm{Ker}(D)$. Since R is Noetherian, through proper reduction, we can ensure irreducibility of D. Now, since $A = \mathrm{Ker}(\delta)[V]$, by Lemma 3.1.4 A is an $\mathbb{A}^1$-fibration over $R[V]$. This shows that $R[V]$ is inert in A, and hence, it is algebraically closed in A. Note that $\mathrm{Ker}(D)$ is also algebraically closed in A. Now, since $R[V] \subseteq \mathrm{Ker}(D)$ and

$\text{tr.deg}_R(R[V]) = \text{tr.deg}_R(\text{Ker}(D))$, we have $\text{Ker}(D)$ is algebraic over $R[V]$, and therefore, $\text{Ker}(D) = R[V]$.

We now prove the equivalence of (I), (II) and (III).

(I) $\Longleftrightarrow$ (III): Follows from Proposition 3.2.1.

(III) $\Longrightarrow$ (II): Obvious.

(II) $\Longrightarrow$ (I): Since A an $\mathbb{A}^1$-fibration over $\text{Ker}(D) = R^{[1]}$, we see that $A \otimes_R k(P)$ is an $\mathbb{A}^1$-fibration over $\text{Ker}(D) \otimes_R k(P) = k(P)^{[1]}$ for all $P \in \text{Spec}(R)$, and therefore, by Corollary 2.3.10 we get $A \otimes_R k(P) = (\text{Ker}(D) \otimes_R k(P))^{[1]}$ for all $P \in \text{Spec}(R)$. Since A is stably polynomial over R, applying Theorem 2.3.21 ([DD14]) we conclude the implication. □

Remark 3.3.6. *As a corollary of Theorem 3.3.2 we get Kahoui-Ouali's result on triviality of stably polynomial $\mathbb{A}^2$-fibration having a fixed point free LND, i.e., Theorem 1.3.8 ([EKO16]).*

Proof. Let $A^{[m]} = R^{[m+2]}$. Using a standard reduction technique (see [EKO16, Lemma 4.3] for the details) we get a finitely generated $\mathbb{Q}$-algebra R_0 which is a subring of R and a finitely presented R_0 subalgebra A_0 of A such that $A_0^{[m]} = R_0^{[m+2]}$, $A_0 \otimes_{R_0} R = A$, $D(A_0) \subseteq A_0$ and $D_0 := D|_{A_0}$ is a fixed point free R_0-LND. Using Theorem 3.3.2 we get $A_0 = \text{Ker}(D_0)^{[1]}$, and therefore, we have ${A_0}^{[m]} = \text{Ker}(D_0)^{[m+1]} = {R_0}^{[m+2]}$, from which, by Theorem 2.3.6 ([Ham75]) it follows that $\text{Ker}(D_0) = {R_0}^{[1]}$. This shows that we have $A_0 = {R_0}^{[2]}$, and therefore, by the properties of A_0 and R_0 it follows that $A = R^{[2]}$. Now, on applying Theorem 1.2.3 we conclude that $\text{Ker}(D) = R^{[1]}$ and $A = \text{Ker}(D)^{[1]}$.

Next, we assume that R is Noetherian and A is a locally stably polynomial algebra over R. Since D is a fixed point free R-LND of A, by Theorem 3.3.2 we have $\text{Ker}(D)$ is an $\mathbb{A}^1$-fibration over R and $A = \text{Ker}(D)^{[1]}$. Since A is locally stably polynomial over R and $A = \text{Ker}(D)^{[1]}$, by Theorem 2.3.6 ([Ham75]) we see that $\text{Ker}(D)$ is a locally polynomial algebra over R, and therefore, by Theorem 2.3.7 ([BCW76]) it follows that $\text{Ker}(D) = \text{Sym}_R(N)$ for some rank one projective R-module N. □

Remark 3.3.7. *From Corollary 3.3.3 and the proof of Theorem 3.3.5 we note the following.*

A. *Let R be a Noetherian domain containing $\mathbb{Q}$, A an $\mathbb{A}^2$-fibration over R and $\delta : A \longrightarrow A$ a fixed point free R-LND. Then, there exists $V \in A$ such that*

   (I) *$\text{Ker}(\delta)$ is an $\mathbb{A}^1$-fibration over R and $A = \text{Ker}(\delta)[V] = \text{Ker}(\delta)^{[1]}$. Further, A is stably polynomial algebra over R if and only if $\text{Ker}(\delta) = R^{[1]}$, i.e.,$A = R^{[2]}$.*

   (II) *A has another irreducible R-LND D, not necessarily fixed point free, such that $\text{Ker}(D) = R[V] = R^{[1]}$ and A is an $\mathbb{A}^1$-fibration over $R[V]$. Further, $A = R[V]^{[1]}$ if and only if D is fixed point free.*

B. *Let R be a Noetherian domain containing $\mathbb{Q}$ and A an $\mathbb{A}^2$-fibration over R. Then $A = R^{[2]}$ if and only if there exist a tuple of R-LNDs (D_1, D_2) of A with slices and an element V in A such that $D_1(V) = 1$ and $D_2(V) = 0$.*

Corollary 3.3.8. *Let R be a Noetherian ring containing $\mathbb{Q}$ and A an $\mathbb{A}^3$-fibration over R. Suppose that there exists a pair of R-LNDs (D_1, D_2) of A with slices and an element V in A such that $D_1(V) = 1$ and $D_2(V) = 0$. Then, $A = C^{[2]}$ for some $\mathbb{A}^1$-fibration C over R. Further, if A has another R-LND D_3 with a slice and an element $W \in A$ such that $D_2(W) = 1$ and $D_3(V) = D_3(W) = 0$, then $A = R^{[3]}$.*

Proof. Since $D_1(V) = 1$, we have $A = \mathrm{Ker}(D_1)[V] = \mathrm{Ker}(D_1)^{[1]}$. As the Zariski's cancellation problem has affirmative answer in dimension three over fields containing $\mathbb{Q}$ (follows from [MS80], [Fuj79], and [Kam75]), from the proof of Lemma 3.1.4 we see that $\mathrm{Ker}(D_1)$ is finitely generated and flat over R and $\mathrm{Ker}(D_1) \otimes_R k(P) = k(P)^{[2]}$ for all $P \in \mathrm{Spec}(R)$, i.e., $\mathrm{Ker}(D_1)$ is an $\mathbb{A}^2$-fibration over R. Now, since D_2 is an R-LND of $A = \mathrm{Ker}(D_1)[V]$ with a slice satisfying $D_2(V) = 0$, it induces an R-LND $\overline{D}_2$ of $A/VA = \mathrm{Ker}(D_1)$ having a slice, and hence, by Corollary 3.3.3 we have $\mathrm{Ker}(D_1) = C^{[1]}$, i.e., $A = C^{[2]}$ where C is an $\mathbb{A}^1$-fibration over R.

We now assume that A has another R-LND D_3 with a slice and an element $W \in A$ such that $D_2(W) = 1$ and $D_3(V) = D_3(W) = 0$. Since $A = \mathrm{Ker}(D_1)[V]$ and $\mathrm{Ker}(D_1)$ is an $\mathbb{A}^2$-fibration over R, we see that V is a residual variable of A, and therefore, by Theorem 2.3.21 ([DD14]), A is an $\mathbb{A}^2$-fibration over $R[V]$. Again, since $R[V] \subset \mathrm{Ker}(D_2) \subset \mathrm{Ker}(D_2)[W] = A$, from Lemma 3.1.4 it follows that $\mathrm{Ker}(D_2)$ is an $\mathbb{A}^1$-fibration over $R[V]$. Now, note that as $V, W \in \mathrm{Ker}(D_3)$ we can see D_3 as an $R[V]$-LND of $A = \mathrm{Ker}(D_2)[W]$ with a slice, and therefore, the corresponding $R[V]$-LND of $A/WA = \mathrm{Ker}(D_2)$ also has a slice. This shows that $\mathrm{Ker}(D_2) = R[V]^{[1]}$, i.e., $A = R[V, W]^{[1]} = R^{[3]}$. □

3.4 Examples

Now let us illustrate with an example that there are non-trivial $\mathbb{A}^2$-fibrations over Noetherian domains containing $\mathbb{Q}$, which have fixed point free LNDs and hence are non-stably polynomial. (refer Theorem 3.3.5)

Example 3.4.1. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K and B a non-trivial $\mathbb{A}^1$-fibration over R (may refer to [Yan81, Example 1]). Set $A := B[X] = B^{[1]}$. It is easy to see that A is a non-trivial $\mathbb{A}^2$-fibration over R. Let $B \otimes_R K = K[Y]$ for some $Y \in B$, and therefore, $A \otimes_R K = K[X, Y]$. Let D be the restriction of the partial derivative $\partial_X : A \otimes_R K \longrightarrow A \otimes_R K$ on A, i.e., $D = \partial_X|_A$. It is easy to see that D is an R-LND of A, $D(X) = 1$ and $\mathrm{Ker}(D) = B$. However, by the Slice Theorem (Theorem 1.2.1) and Theorem 2.3.6 it follows that A is not a stably polynomial algebra over R.*

Now we refer to a collection of examples which shows the existence of non-trivial stably polynomial $\mathbb{A}^2$-fibrations which are not $\mathbb{A}^1$-fibrations over polynomial algebras in one variable (refer Corollary 3.3.3).

Example 3.4.2. *Let A be a non-trivial $\mathbb{A}^2$-fibration over a Noetherian domain R containing $\mathbb{Q}$ such that A is stably polynomial over R, e.g., the examples of Raynaud in [Ray68](also see [Sus82]) and Hochster in [Hoc72] (for both the examples one may also refer to [ER01], [Fre09] and [Fre17], p.272 and p.282). If possible, let $V \in A$ be such that A is an $\mathbb{A}^1$-fibration over $R[V]$. From the Theorem 1.2.7 ([Asa87]) we see that A is an $R[V]$ subalgebra of $R[V]^{[n]}$for some $n \in \mathbb{N}$, and therefore, by the first fundamental exact sequence ([Mat80, Theorem 57, p.186]), we get $\Omega_R(A) = \Omega_{R[V]}(A) \oplus \Omega_R(R[V]) \otimes_{R[V]} A = \Omega_R[V](A) \oplus A$. Since A is stably polynomial over R, by Lemma 2.3.19 ([DD14]) it can be seen that $\Omega_R(A)$ is a stably free A-module, and therefore, $\Omega_{R[V]}(A)$ is a stably free A-module. Now, from Theorem 1.2.7 (see [Asa87]) it directly follows that A is a stably polynomial algebra over $R[V]$, and hence, by Theorem 2.3.6 ([Ham75]) $A = R[V]^{[1]}$, which is a contradiction. This proves that there does not exist $V \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[V]$.*

Asanuma-Bhatwadekar in [AB97, Example 3.12] constructed an example of a non-trivial $\mathbb{A}^2$-fibration which can be written as an $\mathbb{A}^1$-fibration over a polynomial algebra in one variable but not stably polynomial as well as does not possess any fixed point free LND with reference to Corollary 3.3.3.

Example 3.4.3. *Set $T := C[X] = C^{[1]}$ and $R := C[X^2, X^3]$. Let $T[V, W] = T^{[2]}$ and $A = R[V, W + XV^2W^2] + X^2T[V, W]$. One may check that A is a non-stably polynomial $\mathbb{A}^2$-fibration over R and V is a residual variable of A over R; i.e., A is a non-trivial $\mathbb{A}^1$-fibration over $R[V]$.*

The next two examples are due to Winkelmann ([Win90]) which shows the existence of fixed point free LNDs of a polynomial algebra $R^{[3]}$ not having have a slice even when R is a PID containing $\mathbb{C}$.

Example 3.4.4. *Let $R_1 = \mathbb{C}[t] = \mathbb{C}^{[1]}$, $A_1 = R_1[X, Y, Z] = R_1^{[3]}$ and $D_1 : A_1 \longrightarrow A_1$ an R_1-LND given by $D_1(X) = t$, $D_1(Y) = X$ and $D_1(Z) = 1 + 2tY - X^2$.*

Again, let $R_2 = \mathbb{C}[s, t] = \mathbb{C}^{[2]}$, $A_2 = R_2[X, Y, Z] = R_2^{[3]}$ and $D_2 : A_2 \longrightarrow A_2$ an R_2-LND given by $D_2(X) = s$, $D_2(Y) = t$ and $D_2(Z) = 1 + tX - sY$.

Clearly, both LNDs D_1 and D_2 are fixed point free. However, it is well known that none of D_1 and D_2 have slice.

Chapter 4

LNDs with polynomial kernels of stably trivial $\mathbb{A}^2$-fibrations

In this chapter we move on to our next problem stated in the introduction, i.e., Question 1.3.13. Let us restate it here for our convenience.

The main problem to solve in this chapter: *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K, A an $\mathbb{A}^2$-fibration over R such that $A^{[n]} = A[T_1, T_2, \cdots, T_n] = R[X_1, X_2, \cdots, X_{n+2}] = R^{[n+2]}$ and $D : A \longrightarrow A$ an R-LND. Are then the following statements equivalent?*

(i) *D is irreducible and $Ker(D) = R^{[1]}$.*

(ii) *$grade(D(A)A) \geq 2$.*

(iii) *$\widetilde{D} = \mathcal{JD}_{(X_1,X_2,\cdots,X_{n+2})}(F, T_1, T_2, \cdots, T_n, -)$ where $\widetilde{D}$ is the trivial extension of D on $A[T_1, T_2, \cdots, T_n]$, $A \otimes_R K = K[F]^{[1]}$ and $grade((F_{X_1}, \cdots, F_{X_{n+2}})A[T]) \geq 2$.*

In this context we should recall that for an ideal I of a ring R we say grade$(I) = \infty$ if $I = R$; and for $\ell \in \mathbb{N}$, we say grade$(I) \geq \ell$ if either grade$(I) = \infty$ or $\ell \leq$ grade$(I) < \infty$.

A possible approach towards the above stated problem is using the techniques of Bhatwadekar-Dutta applied in their proof of Theorem 1.3.9 ([BD97]). In their proof the following two results ([BD97, Proposition 4.4] and [BD97, Proposition 4.5] respectively) played pivotal roles.

Proposition 4.0.1. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K, and $F \in R[X,Y] \setminus R$. Suppose that $K[X,Y] = K[F]^{[1]}$. Then, $R[X,Y] = R[F]^{[1]}$ if and only if F_X, F_Y are comaximal in $R[X,Y]$*

Proposition 4.0.2. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K, and $F \in R[X,Y] \setminus R$. Then, $R[F]$ is inert in $R[X,Y]$ if and only if $K[F]$ is inert in $K[X,Y]$ and the ideal $(F_X, F_Y)R[X,Y]$ is either the unit ideal, i.e., $grade(F_X, F_Y) = \infty$ or $grade(F_X, F_Y) = 2$. Consequently, $\mathcal{JD}_{(X,Y)}(F, -)$ is an irreducible R-derivation if $R[F]$ is inert in $R[X,Y]$.*

To follow the techniques of Bhatwadekar-Dutta in order to solve Question 1.3.13, the first step would be to check whether the above two results are generalized for multivariate polynomial rings. Since from Theorem 1.3.8 ([EKO16]) it already follows that Proposition 4.0.1 has an extension to stably polynomial algebras (see Proposition 4.3.1), we ask the following questions.

Question 4.0.3. *Let R be a Noetherian domain containing $\mathbb{Q}$.*

(I) *Is it possible to classify the inert subrings of $R^{[n]}$, which are polynomial algebras in one indeterminate? (asks to extend the 1st part of Proposition 4.0.2)*

(II) *If $R[F_1, \cdots, F_{n-1}]$ is inert in $R[X_1, \cdots, X_n]$ where $F_1, \cdots, F_{n-1}$ are algebraically independent over R, is then $\mathcal{JD}_{(X_1,\cdots,X_n)}(F_1, \cdots, F_{n-1}, -)$ an irreducible R-derivation? (asks to generalize the 2nd part of Proposition 4.0.2)*

In Section 4.1 we show that the answer to Question 4.0.3(I) is affirmative. Specifically (see Proposition 4.1.4)

Proposition 4.A. Let R be a Noetherian domain containing $\mathbb{Q}$ and $A = R[X_1, X_2, \cdots, X_n] = R^{[n]}$ where $n \geq 2$. Then, for an element $F \in A \setminus R$, $R[F]$ is an inert subring of A if and only if $K[F]$ is an inert subring of $A \otimes_R K$ and the ideal $(F_{X_1}, F_{X_2}, \cdots, F_{X_n})A$ has grade at least 2.

In section 4.2 we show that from the proof of [Dai97, Corollary 2.4] (see [Dai97, Section 3]) an affirmative answer to Question 4.0.3(II) exists under certain assumptions. To be specific (see Corollary 4.2.2),

Corollary 4.B. Let R be a Noetherian domain containing $\mathbb{Q}$ and $A = R[X_1, X_2 \cdots, X_n] = R^{[n]}$. Suppose that $F_1, F_2, \cdots, F_{n-1} \in A$ are algebraically independent over R; and $R[F_1, F_2, \cdots, F_{n-1}]$ is inert in A. Then, the kernel of $\mathcal{JD}_{(X_1,X_2\cdots,X_n)}(F_1, F_2, \cdots, F_{n-1}, -)$ is $R[F_1, F_2, \cdots, F_{n-1}]$. Further, if R is either a UFD or a regular domain, then $\mathcal{JD}_{(X_1,X_2\cdots,X_n)}(F_1, F_2, \cdots, F_{n-1}, -)$ is irreducible.

In Section 4.4 we investigate Question 1.3.13 and observe that while over Noetherian UFDs containing $\mathbb{Q}$ it has complete affirmative answer, over general Noetherian domains containing $\mathbb{Q}$ the result holds partially (see Theorem 4.5.1 and Corollary 4.5.3).

Theorem 4.C. Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K, A an R-algebra and D an R-LND of A. Suppose, $\underline{X} = (X_1, \cdots, X_{n+2})$ and $\underline{T} = (T_1, \cdots, T_n)$ are two sequence of indeterminates such that $A[\underline{T}] = R[\underline{X}]$ and $\widetilde{D}$ is the trivial extension of D on $A[\underline{T}]$.

(a) Consider the following statements.

  (I) D and D_P are irreducible for each $P \in \operatorname{Spec}(R)$.
  (II) $\operatorname{Ker}(D) = R^{[1]}$.
  (III) $\operatorname{grade}(D(A)A) \geq 2$.
  (IV) There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $\operatorname{grade}((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$.

(V) There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $\operatorname{grade}((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$ where $r \in R$, $h \in \operatorname{Ker}(D)$.

Then, "(II) $\Longleftrightarrow$ (V)", "(IV) $\Longrightarrow$ (V)" and "(III) $\Longrightarrow$ (I)" hold. Further, if R is either a UFD or a regular domain, then (V) implies that $\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ is irreducible.

(b) If R is a UFD, then $\operatorname{Ker}(D) = R^{[1]}$. Further the following statements are equivalent.

(I) D is irreducible.

(II) $\operatorname{grade}(D(A)A) \geq 2$.

(III) There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $\operatorname{grade}((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$.

(c) If R is a regular domain, then $\operatorname{Ker}(D) = \operatorname{Sym}_R(I)$ for some invertible ideal I of R. Further, the following statements are equivalent.

(I) D and D_P are irreducible; and $\operatorname{Ker}(D) = R^{[1]}$.

(II) $\operatorname{grade}(D(A)A) \geq 2$; and $\operatorname{Ker}(D) = R^{[1]}$.

(III) There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $\operatorname{grade}((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$.

4.1 A criterion for one-variable polynomial subrings of $R^{[n]}$ to be inert

In order to explore Question 4.0.3 we first observe a few properties of polynomial rings.

The first lemma shows that for a non-constant element F in the polynomial algebra $K^{[n]}$, the K-algebra generated by $K[F]$ intersected with the ideal generated by partial derivatives of F contains at least one non-zero element. This result is necessary in the proof of Lemma 4.1.2.

Lemma 4.1.1. *Let K be a field containing $\mathbb{Q}$ and $F \in K[X_1, X_2, \cdots, X_n] = K^{[n]}$ be a non-constant element. Then, $(F_{X_1}, F_{X_2}, \cdots, F_{X_n})K[X_1, X_2, \cdots, X_n] \cap K[F] \neq (0)$.*

Proof. Let $S = K[F] \setminus \{0\}$, $L = S^{-1}K[F]$, $C = S^{-1}K[X_1, X_2, \cdots, X_n]$ and $I = (F_{X_1}, F_{X_2}, \cdots, F_{X_n})C$. It is enough to prove that I is not contained in any maximal ideal of C. So, we assume that $I \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{MaxSpec}(C)$. Note that $C_{\mathfrak{m}}$ is a regular local ring. Now, the ring homomorphisms $K \xrightarrow{i} L \xrightarrow{j} C_m$, where i and j are inclusion maps give rise to the following exact sequence.

$$\Omega_K(L) \otimes_L C_m \xrightarrow{\sigma} \Omega_K(C_{\mathfrak{m}}) \longrightarrow \Omega_L(C_{\mathfrak{m}}) \longrightarrow 0$$

Since $C_\mathfrak{m}$ is a regular local domain and $\mathbb{Q} \hookrightarrow L$, one can see that $\Omega_L(C_\mathfrak{m})$ is a free $C_\mathfrak{m}$-module of rank $n-1$ ([AK70, pp. 159-167]), and therefore, the short exact sequence

$$0 \longrightarrow Im(\sigma) \longrightarrow \Omega_K(C_\mathfrak{m}) \longrightarrow \Omega_L(C_\mathfrak{m}) \longrightarrow 0$$

is split exact where $\sigma(d_{L/K}(a) \otimes b) = bd_{C_m/K}(i(a))$. Since $\Omega_K(C_\mathfrak{m})$ is a free $C_\mathfrak{m}$ module of rank n with basis $\{dX_1, dX_2, \cdots, dX_n\}$, $im(\sigma) = (F_{X_1}dX_1 + F_{X_2}dX_2 + \cdots, F_{X_n}dX_n)C_\mathfrak{m}$ and $\Omega_K(C_\mathfrak{m}) = \Omega_L(C_\mathfrak{m}) \oplus im(\sigma)$, we see that there exists a matrix $M := \begin{pmatrix} F_{X_1} & F_{X_2} & \cdots & F_{X_n} \\ * & * & \cdots & * \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ * & * & \cdots & * \end{pmatrix}$ such that MP forms a basis of $\Omega_K(C_\mathfrak{m})$ where $P := (dX_1, dX_2, \cdots, dX_n)^t$. Using the fact that $det(M) = 1$ in $C_\mathfrak{m}$ we get $IC_\mathfrak{m} = C_\mathfrak{m}$ which is a contradiction to the assumption that $I \subseteq \mathfrak{m}$. So, we have $I = C$. This completes the proof. □

The next lemma proves that for a non-constant element F in $K^{[n]}$ the ideal generated by the partial derivatives F_{X_i}s is not contained in any proper principal ideal of $K^{[n]}$, provided $K[F]$ is inert in $K^{[n]}$. This result is used in the Proposition 4.1.4.

Lemma 4.1.2. *Let K be a field containing $\mathbb{Q}$ and $F \in K[X_1, X_2, \cdots, X_n] = K^{[n]}$ be a non-constant element. If $K[F]$ is an inert subring of $K[X_1, X_2, \cdots, X_n]$, then the ideal $(F_{X_1}, F_{X_2}, \cdots, F_{X_n})K[X_1, X_2, \cdots, X_n]$ is not contained in any proper principal ideal of $K[X_1, X_2, \cdots, X_n]$.*

Proof. Set $A := K[X_1, \cdots, X_n]$ and $I := (F_{X_1}, F_{X_2}, \cdots, F_{X_n}) K[X_1, \cdots, X_n]$. If possible let $I \subseteq pA$ for a prime p in A, and therefore, by Lemma 4.1.1 we have $pA \cap K[F] \neq (0)$. Let $0 \neq b \in pA \cap K[F]$. Then $b = \beta p$ where $\beta \in A$. Since $K[F]$ is an inert subring of A, we get $\beta, p \in K[F]$. Write $p = \phi(F) := r_0 + r_1F \cdots + r_mF^m$ where $r_i \in K$. Since $F_{X_1}, F_{X_2}, \cdots, F_{X_n} \in pA$, we get $F_{X_i} = a_i\phi(F)$ where $a_i \in A$ for $i = 1, 2, \cdots, n$. Now, for each $i = 1, 2, \cdots, n$ comparing the degrees of X_is in the equation $F_{X_i} = a_i\phi(F)$ we see that either $a_i = 0$ or $\phi(F) = r_0$ which respectively imply that either $F \in K$ or pA is an unit ideal – a contradiction to our assumptions. So, we conclude that $I \nsubseteq (a)$ for any non zero and non unit $a \in A$. □

The next result shows that an $R^{[n]}$-extended ideal of R will contain a non-constant element of $R^{[n]}$ if and only if all of the partial derivatives of that non-constant elements are in the extended ideal. This result is also needed in the proof of Proposition 4.1.4.

Lemma 4.1.3. *Let R be a domain containing $\mathbb{Q}$, I an ideal of R and F an element of the ideal $(X_1, \cdots, X_n)R[X_1, \cdots, X_n]$. Then, F is in $IR[X_1, \cdots, X_n]$ if and only if all of $F_{X_1}, \cdots, F_{X_n}$ are in $IR[X_1, \cdots, X_n]$*

Proof. Write F as sum of monomials say $F = \sum_{i_1, \cdots, i_n} a_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$, where $a_{i_1 \cdots i_n} \in R$. Then, for each $j = 1, 2, \cdots, n$ we have

$$F_{X_j} = \sum_{i_j} \left(\sum_{i_1} \sum_{i_2} \cdots \sum_{i_{j-1}} \sum_{i_{j+1}} \cdots \sum_{i_n} a_{i_1 \cdots i_n} X_1^{i_1} X_2^{i_2} \cdots X_{j-1}^{i_{j-1}} X_{j+1}^{i_{j+1}} \cdots X_n^{i_n} \right) i_j\, X_j^{i_j - 1}.$$

Since $\mathbb{Q} \hookrightarrow R$, the above expression shows that for each $j = 1, 2, \cdots, n$ we have $F_{X_j} \in IR[X_1, \cdots, X_n]$ if and only if $a_{i_1 \cdots i_n} \in I$ for all $(i_1, i_2, \cdots, i_{j-1}, i_j, i_{j+1}, \cdots, i_n) \in \mathbb{N}^n$ with $i_j > 0$. This proves that $F \in IR[X_1, \cdots, X_n]$ if and only if $F_{X_1}, \cdots, F_{X_n} \in IR[X_1, \cdots, X_n]$. □

We now give an answer to the Question 4.0.3(I) which generalizes the first part of Proposition 4.0.2 ([BD97]).

Proposition 4.1.4. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K and $A = R[X_1, X_2, \cdots, X_n] = R^{[n]}$ where $n \geq 2$. Then for an element $F \in A \setminus R$, the following are equivalent.*

(I) *$R[F]$ is an inert subring of A.*

(II) *$K[F]$ is an inert subring of $A \otimes_R K$ and grade$((F_{X_1}, F_{X_2}, \cdots, F_{X_n})A) \geq 2$.*

Proof. (I) $\Longrightarrow$ (II): Assume that $R[F]$ is inert in A. Let $ab \in K[F]$ with $a, b \in A \otimes_R K$. Write $a = \frac{a_1}{a_2}$ and $b = \frac{b_1}{b_2}$ where $a_1, b_1 \in A$ and $a_2, b_2 \in R$ so that $ab = \frac{a_1 b_1}{a_2 b_2}$ with $a_1 b_1 \in R[F]$. By our hypothesis we have $a_1, b_1 \in R[F]$, and therefore, $\frac{a_1}{a_2}, \frac{b_1}{b_2} \in K[F]$. This proves the first part of (II).

We use induction on the number of indeterminates n to prove the conclusion. By 4.0.2 ([BD97]) the result holds when $n = 2$. We assume that the result holds for $n = r - 1 \geq 2$ and shall show that it holds for $n = r$. We can assume, without loss of generality, that $F \in (X_1, X_2, \cdots, X_r)A$. We further assume that the $(F_{X_1}, F_{X_2}, \cdots, F_{X_r})A \neq A$, i.e., grade$((F_{X_1}, F_{X_2}, \cdots, F_{X_n})A) \neq \infty$, otherwise it directly leads to the conclusion of (II).

Case $F_{X_1} = 0$: In this case, $F \in (X_2, X_3, \cdots, X_r)R[X_2, X_3, \cdots, X_r]$. Since $R[X_2, X_3, \cdots, X_r]$ is inert in A, we see that $R[F]$ is inert in $R[X_2, X_3, \cdots, X_r]$. Now applying the induction hypothesis we get the required result.

Case $F_{X_1} \neq 0$: We will show that one of $F_{X_2}, F_{X_3}, \cdots, F_{X_r}$ is a non-zero divisor in $A/F_{X_1}A$. Let $Q \in \mathrm{Ass}_A(A/F_{X_1}A)$, and hence depth$(A_Q) = 1$. It is enough to prove that one of $F_{X_2}, \cdots, F_{X_r}$ is not in Q. Let $P := Q \cap R$. Then, there are two possibilities: $P = (0)$ and $P \neq (0)$.

Assume that $P = (0)$. Then, $QA \otimes_R K \in \text{Ass}_{A\otimes_R K}(A \otimes_R K/F_{X_1}(A \otimes_R K))$. Since $A \otimes_R K = K^{[r]}$, we have $\text{ht}(QA \otimes_R K) = 1$, and therefore, $QA \otimes_R K$ is a principal ideal. Since $K[F]$ is inert in $K[X_1, X_2, \cdots, X_r]$, by Lemma 4.1.2 we get $(F_{X_1}, F_{X_2}, \cdots, F_{X_r}) \nsubseteq QA \otimes_R K$, i.e., one of $F_{X_2}, F_{X_3}, \cdots, F_{X_r}$ does not belong to $QA \otimes_R K$.

Now assume that $P \neq (0)$. Let $0 \neq a \in P$. Since $\text{depth}(A_Q) = 1$, we have $Q \in \text{Ass}_A(A/aA)$; and since $a \in R$, we see that Q is an extended ideal of R. Thus, $Q = PA$, and hence $P \in \text{Ass}_R(R/aR)$. This shows that there exists $b \in R \setminus aR$ such that $bP \subseteq aR$ along with $bQ \subseteq aA$. We claim that $F \notin Q$. Otherwise, $bF \in bQ \subset aA$, and hence $bF = aG$ for some $G \in A$. Now, since $R[F]$ is inert in A we have $G \in R[F]$, and therefore, $G = cF$ for some $c \in R$. This shows that $bF = aG = acF$, i.e., $b = ac$ which a contradiction to the fact that $b \in R \setminus aR$. So, we have $F \notin Q$. Now, by the Lemma 4.1.3 we get that one of $F_{X_2}, F_{X_3}, \cdots, F_{X_r}$ does not belong to Q.

So, in both the cases we have one of $F_{X_2}, F_{X_3}, \cdots, F_{X_n}$ does not belong to Q.

(II) $\implies$ (I): Assume that (II) holds. Since $K[F]$ is an inert subring of $A \otimes_R K$, it is enough to prove that $A \cap K[F] = R[F]$. We show that $cA \cap R[F] = cR[F]$ for all $c \in R$. Let $G \in A$ and $\phi(F) = \sum_{i=0}^{\ell} a_i F^i$, $a_i \in R$ be such that $cG = \phi(F)$. As $a_0 = cG - (a_1 F + a_2 F^2 + \cdots + a_\ell F^\ell)$, we are done, if we show that $a_i \in cR$, for each $i = \ell, \ell - 1, \cdots, 2, 1$; i.e., enough to show that for each $k = \ell, \ell - 1, \cdots, 2, 1$, $\phi_k^{(k)}(F) \in cA$ where $\phi_k(F) = \sum_{i=0}^{k} a_i F^i$ and where $\phi^{(k)}$ denotes the k-th derivative of ϕ.

First, we shall show that $\phi_\ell^{(\ell)}(F) \in cA$, and for which we start with showing $\phi_\ell^{(1)}(F) \in cA$. Note that $cG_{X_1} = \phi_\ell^{(1)} F_{X_1}$, $cG_{X_2} = \phi_\ell^{(1)} F_{X_2}, \cdots$, $cG_{X_n} = \phi_\ell^{(1)} F_{X_n}$. Let $cA = \bigcap N_j$ be a reduced primary decomposition of cA where $P_j \in \text{Spec}(A)$ be such that $\{P_j\} = \text{Ass}_A(A/N_j)$, and therefore, $\{P_j A_{P_j}\} = \text{Ass}_{A_{P_j}}(A_{P_j}/cA_{P_j})$. This shows that $\text{depth}(A_{P_j}) = 1$. Hence, by the given hypothesis at least one of $F_{X_1}, F_{X_2}, \cdots, F_{X_n}$ becomes a unit in A_{P_j}, i.e., $F_{X_{i_0}} \notin P_j$ for some $i_0 = 1, 2, \cdots, n$, and therefore, since we have $\phi_\ell^{(1)}(F) F_{X_{i_0}} = cG_{X_{i_0}} \in cA \subset N_j$, it follows that $\phi_\ell^{(1)}(F) \in N_j$. So, we have $\phi_\ell^{(1)}(F) \in N_j$ for each j, and therefore, $\phi_\ell^{(1)}(F) \in cA$. Now, for each $m = 1, 2, \cdots, \ell - 1$, by repeating the above argument m-many times on the m-th derivative $\phi_\ell^{(m)}(F)$, we can conclude that $\phi_\ell^{(m+1)}(F) \in cA$. In particular, $\phi^{(\ell)}(F) \in cA$; and as $\mathbb{Q} \hookrightarrow R$, it follows that $a_\ell \in cR$. Note that $\phi(F) = \phi_\ell(F) = \phi_{\ell-1}(F) + a_\ell F^\ell$. Since $a_\ell \in cR$, we get $\phi_{\ell-1}(F) = cG - a_\ell F^\ell \in cA$.

Now, for each $k = \ell - 1, \ell - 2, \cdots, 1$, repeating the whole argument for $\phi_k(F)$ we see that $a_k \in cR$. This completes the proof. □

Remark 4.1.5. *From the proof of Proposition 4.1.4 it follows that under the given hypothesis if* $R[F]$ *is an inert subring of* A*, then either one of* $F_{X_1}, F_{X_2}, \cdots, F_{X_n}$ *is unit or two of* $F_{X_1}, F_{X_2}, \cdots, F_{X_n}$

form a regular sequence in A; and further, if all of $F_{X_1}, F_{X_2}, \cdots, F_{X_n}$ are non-zero and non-unit, then for each F_{X_i} there exists F_{X_j} such that F_{X_i}, F_{X_j} form a regular sequence where $i \neq j$.

4.2 Irreducible Jacobian derivations of polynomial algebras

The following proposition plays a major part in proving the upcoming Corllory 4.2.2. It is an extension of Proposition 2.3.17 ([Dai97]).

Proposition 4.2.1. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K and $R[\underline{X}] = R^{[n]}$. Suppose $\underline{F} = (F_1, F_2, \cdots, F_m) \in R[\underline{X}]^m$ be such that $R[\underline{F}]$ is inert in $R[\underline{X}]$ and $tr.deg_R(R[\underline{F}]) = d$. Let I be the ideal generated by the $d \times d$ minors of $M = \mathcal{J}ac_{(\underline{X})}(\underline{F})$ in $R[\underline{X}]$. Then, grade$(I) \geq 2$ provided any one of the following holds.*

(I) *R is either a UFD or a regular domain.*

(II) *$d = m$ and $(R/P)[\underline{F}]$ remains inert in $(R/P)[\underline{X}]$ for all height one prime ideal P of R.*

Proof. Set $A := R[\underline{X}]$ and $B := R[\underline{F}]$.

(I): Suppose that R is a UFD. Then the conclusion follows from Theorem 2.3.16 ([Dai97, Theorem 3.3]) and the proof of Proposition 2.3.17 (see [Dai97, Corollary 3.10]).

Now, suppose that R is regular. For $P \in \operatorname{Spec}(R)$, consider R_P. Since R_P is a UFD, by the earlier case we have grade$(IA_P) \geq 2$. Since this holds for all $P \in \operatorname{Spec}(R)$, by Lemma 2.3.2 it follows that grade$(I) \geq 2$.

(II): We now assume that $d = m$ and $(R/P)[\underline{F}]$ remains inert in $(R/P)[\underline{X}]$ for all height one prime ideal P of R. Note that since $R[\underline{F}] = R^{[m]}$, $R[F_i]$ is inert in $R[\underline{F}]$ for each $i = 1, 2, \cdots, m$. Note that from Lemma 2.3.2 it follows that to prove grade$(D(A)A) \geq 2$ it is enough to prove that grade$(D(A)A_P) \geq 2$ for each $P \in \operatorname{Spec}(R)$ where $A_P = A \otimes_R R_P$. Therefore, without loss of generality we assume that R is a local domain. We shall show that grade$(D(A)A) \geq 2$. To prove it we use induction on the dimension of the Noetherian local domain R. If $\dim R = 0$, the result follows from Proposition 2.3.17 ([Dai97]). We assume that the assertion holds when $\dim R = \ell$; and now we shall show that it holds when $\dim R = \ell + 1$.

Let $I = (\phi_1, \phi_2, \cdots, \phi_r)A$ where $\phi_1, \phi_2, \cdots, \phi_r$ are the $d \times d$ minors of M and $I \neq A$, i.e., grade$(I) \neq \infty$. We will show that if $\phi_1 \neq 0$, then one of $\phi_2, \cdots, \phi_r$ is a non-zero divisor in $A/\phi_1 A$. So let $\phi_1 \neq 0$ and $\dim(R) = \ell + 1$. Let $Q \in \operatorname{Ass}_A(A/\phi_1 A)$, and hence depth$(A_Q) = 1$. It is enough to prove that one of $\phi_2, \phi_3, \cdots, \phi_r$ is not in Q. Let $P := Q \cap R$. Then, there are two possibilities: $P = (0)$ and $P \neq (0)$.

Assume that $P = (0)$. Then $Q(A \otimes_R K) \in \operatorname{Ass}_{A \otimes_R K}(A \otimes_R K/\phi_1(A \otimes_R K))$. Since $A \otimes_R K = K^{[n]}$, we have ht$(Q(A \otimes_R K)) = 1$, and therefore, $Q(A \otimes_R K)$ is a principal ideal. Since

B is inert in A, it can be seen that $B \otimes_R K$ is inert in $A \otimes_R K$, and therefore, by Proposition 2.3.17 ([Dai97]) we get $(\phi_1, \phi_2, \cdots, \phi_r) \otimes_R K \nsubseteq Q(A \otimes_R K)$, i.e., one of $\phi_2, \phi_3, \cdots, \phi_r$ does not belong to $Q(A \otimes_R K)$.

Now assume that $P \neq (0)$. Let $0 \neq a \in P$. Since $\operatorname{depth}(A_Q) = 1$, we have $Q \in \operatorname{Ass}_A(A/aA)$; and since $a \in R$, we see that Q is an extended ideal of R. Thus, $Q = PA$, and hence $P \in \operatorname{Ass}_R(R/aR)$. This shows that there exists $b \in R \setminus aR$ such that $bP \subseteq aR$ along with $bQ \subseteq aA$. Note that due to inertness of $R[F_i]$ in A we have $F_i \notin Q$ for all $i = 1, 2, \cdots, m$; otherwise it contradicts that $b \in R \setminus aR$. Now, by the Lemma 4.1.3 we get that at least one of $F_{iX_1}, F_{iX_2}, \cdots, F_{iX_r}$ does not belong to Q for each $i = 1, 2, \cdots, m$, i.e., for each $i = 1, 2, \cdots, m$, not all of $F_{iX_1}, F_{iX_2}, \cdots, F_{iX_r}$ are zero in A/Q. Let $R_1 = R/P$ and $A_1 = A/PA = A/Q = R_1[\underline{X}]$. By the hypothesis we have $R_1[\underline{F}]$ is inert in A/Q. Since $\dim(R_1) \leq \ell$, by the induction hypothesis we have $\operatorname{grade}((\phi_1, \phi_2, \cdots, \phi_r)A_1) \geq 2$, and therefore, $(\phi_2, \cdots, \phi_r) \nsubseteq Q$, i.e., one of $\phi_2, \cdots, \phi_r$ does not belong to Q.

So, in both the cases we see that at least one of $\phi_2, \cdots, \phi_r$ does not belong to Q. This completes the proof. □

Our next observation answers the Question 4.0.3(III) and it generalizes the second part of Proposition 4.0.2 ([BD97]).

Corollary 4.2.2. *Let $R \hookleftarrow \mathbb{Q}$ be a Noetherian domain and $R[\underline{X}] = R[X_1, X_2, \cdots, X_n] = R^{[n]}$. Suppose that $\underline{F} = (F_1, F_2, \cdots, F_{n-1}) \in R[\underline{X}]^{n-1}$ is a sequence of algebraically independent elements over R and $R[\underline{F}]$ is inert in $R[\underline{X}]$. Then, $\operatorname{Ker}(\mathcal{JD}_{(\underline{X})}(\underline{F}, -)) = R[\underline{F}]$. Further, $\mathcal{JD}_{(\underline{X})}(\underline{F}, -)$ is irreducible provided either R is a UFD or a regular domain or $(R/P)[\underline{F}]$ remains inert in $(R/P)[\underline{X}]$ for all height one prime ideal P of R.*

Proof. Set $A = R[\underline{X}]$ and $\Delta := \mathcal{JD}_{(\underline{X})}(\underline{F}, -)$. Since $\Delta \neq 0$, it is clear that $\operatorname{Ker}(\Delta) \neq A$ and $\operatorname{tr.deg}_{\operatorname{Ker}(\Delta)}(A) \geq 1$. Again, since $\operatorname{Ker}(\Delta)$ and $R[\underline{F}]$ are algebraically closed in A, $\operatorname{tr.deg}_{R[\underline{F}]}(A) = 1$ and $R[\underline{F}] \subseteq \operatorname{Ker}(\Delta)$, we see that $R[\underline{F}] = \operatorname{Ker}(\Delta)$.

Clearly, $\{\Delta(X_i) \mid i = 1, 2, \cdots, n\}$ are the $(n-1) \times (n-1)$ minors of the Jacobian $\mathcal{J}ac_{(\underline{X})}(\underline{F})$ and $\operatorname{tr.deg}_R(R[\underline{F}]) = n - 1$, and therefore by Proposition 4.2.1 we see that the ideal $(\Delta(X_1), \Delta(X_2), \cdots, \Delta(X_n))A$ is either the unit ideal or has grade at least two. This proves that $(\Delta(X_1), \Delta(X_2), \cdots, \Delta(X_n))A$ is not contained in a principal ideal of A, i.e., Δ is irreducible on A. □

4.3 A criterion for a stably trivial $\mathbb{A}^2$-fibration to be trivial

The next proposition gives a criterion for a stably polynomial $\mathbb{A}^2$-fibration to be a polynomial algebra. The proof is given in the spirit of Bhatwadekar-Dutta's proof of Proposition 4.0.1 ([BD97]). One can see that the result is related to Theorem 1.3.8 ([EKO16]).

Proposition 4.3.1. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K and A a stably polynomial $\mathbb{A}^2$-fibration over R. Suppose, $\underline{X} = (X_1, X_2, \cdots, X_{n+2})$ and $\underline{T} = (T_1, T_2, \cdots, T_n)$ are two sequence of indeterminates such that $A[\underline{T}] = R[\underline{X}]$. Let $F \in A$ be such that $A \otimes_R K = K[F]^{[1]}$. Then $A = R[F]^{[1]}$ if and only if $(F_{X_1}, F_{X_2}, \cdots, F_{X_n})A[\underline{T}] = A[\underline{T}]$.*

Proof. Due to Theorem 2.3.21 ([DD14]) it is enough to prove that F is a residual coordinate of A. Since an element $h \in A$ is a residual coordinate of A over R if and only if h is a residual coordinate of A_P over R_P for all $P \in \mathrm{Spec}(R)$, without loss of generality we assume R to be local with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Since $A \otimes_R K = K[F]^{[1]}$, for each $P \in \mathrm{Spec}(R)\setminus\{0\}$ we only need to show that $A \otimes_R k(P) = k(P)[\overline{F}]^{[1]}$ where $\overline{F}$ denotes the image of F in $A \otimes_R k(P)$. We prove this using induction on the dimension d of the local ring R.

If $d = 0$, there is nothing to prove as F is already a coordinate of $A \otimes_R K = K^{[2]}$. Assume that $d = 1$. Since R is an one-dimensional Noetherian local domain, by the Krull-Akizuki theorem there exists a discrete valuation ring (DVR) (C, π) such that $R \subseteq C \subseteq K$ and the residue field $L = C/(\pi)$ is finite over k. Since $\mathbb{A}^2$-fibrations over a DVR containing $\mathbb{Q}$ are trivial (see Theorem 1.2.5 ([Sat83])), we get $A \otimes_R C = C^{[2]}$. We shall show that $A \otimes_R C = C[F]^{[1]}$. Since F is a generic coordinate of $A \otimes_R C$, by Theorem 2.3.8 ([RS79]) it is enough to show that $L[\overline{F}]$ is algebraically closed in $A \otimes_R L = (A \otimes_R C) \otimes_C L = L^{[2]}$. By Theorem 2.3.4 ([AEH72]) the algebraic closure of $L[\overline{F}]$ in $A \otimes_R L = L^{[2]}$ is of the form $L[U]$, and therefore, we have $L[\overline{F}] \subseteq L[U] \subseteq A \otimes_R L \subseteq A[\underline{T}] \otimes_R L = L[X_1, X_2, \cdots, X_{n+2}]$. Thus, the partial derivatives of U with respect to $X_1, X_2, \cdots, X_{n+2}$ are well defined. Let us write $\overline{F} = a_0 + a_1U + a_2U^2 + \cdots + a_mU^m$ where $a_i \in L$. Applying partial derivatives with respect to $X_1, X_2, \cdots, X_{n+2}$ we get the following equation

$$\frac{\partial F}{\partial X_i} = a_1\frac{\partial U}{\partial X_i} + 2Ua_2\frac{\partial U}{\partial X_i} + \cdots + mU^{m-1}a_m\frac{\partial U}{\partial X_i}$$

where $i = 1, 2, \cdots, n+2$. Since $(F_{X_1}, F_{X_2}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$, there exists $b_1, b_2, \cdots, b_{n+2}$ such that $b_1\frac{\partial F}{\partial X_1} + b_2\frac{\partial F}{\partial X_2} + \cdots + b_{n+2}\frac{\partial F}{\partial X_{n+2}} = 1$, which along with the above equation gives the following

$$(b_1\frac{\partial U}{\partial X_1} + b_2\frac{\partial U}{\partial X_2} + \cdots + b_{n+2}\frac{\partial U}{\partial X_{n+2}})(a_1 + 2a_2U + \cdots + ma_mU^{m-1}) = 1$$

On comparing degrees of U in the above equation, we get $a_i = 0$ for all $i = 2, 3, \cdots, m$. This shows that $\overline{F}$ is linear in U, and therefore, $L[\overline{F}] = L[U]$. So, we have $A \otimes_R C = C[F]^{[1]}$, and hence $A \otimes_R L = L[\overline{F}]^{[1]}$. Since L is a finite separable extension of k, we get $A \otimes_R k = k[\overline{F}]^{[1]}$, i.e., F is a residual coordinate of A.

Now, assume that the result holds for all domains of dimension $d \leq m-1$. Let $\dim(R) = m$. Consider $R_1 := R/P$ where P is a height-one prime ideal of R and set $A_1 := A \otimes_R R_1$. Clearly, A_1 is an $\mathbb{A}^2$-fibration over R_1. Since $dim(R_1) = m-1$, by the induction hypothesis $\overline{F}$ is a residual coordinate of A_1, and therefore, $A_1 \otimes_{R_1} k(\overline{Q}) = k(\overline{Q})[F]^{[1]}$ for all $\overline{Q} \in \operatorname{Spec}(R_1)$ where $\overline{Q}$ denotes the image of $Q \in \operatorname{Spec}(R)$ in R_1 such that $P \subseteq Q$. Since $A_1 = A \otimes_R R_1$ and $k(\overline{Q}) = k(Q)$ we have $A \otimes_R k(Q) = k(Q)[\overline{F}]^{[1]}$ for all $Q \in \operatorname{Spec}(R)$ such that $P \subseteq Q$. This shows we have $A \otimes_R k(P) = k(P)[\overline{F}]^{[1]}$ for all $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) \geq 1$. Since F is already a generic variable of $R[X,Y]$, i.e., $R[X,Y] \otimes_R k(0) = k(0)[F]^{[1]}$, it follows that F is a residual coordinate of A, and hence, by Theorem 2.3.21 ([DD14]) we get $A = R[F]^{[1]}$. □

Remark 4.3.2. *Let R be a regular domain containing $\mathbb{Q}$ with quotient field K and A an $\mathbb{A}^2$-fibration over R. Let $F \in A$ be such that $A \otimes_R K = K[F]^{[1]}$. Fix $P \in Spec(R)$. Then, by [Asa87, Corollary 3.5] there exist sequence of indeterminates $\underline{T} = (T_1, T_2, \cdots, T_n)$ and $\underline{X} = (X_1, X_2, \cdots, X_{n+2})$ such that $A \otimes_R R_P[\underline{T}] = R_P[\underline{X}]$. In that case, we see that $A \otimes_R R_P = R_P[F]^{[1]}$ if and only if $(F_{X_1}, F_{X_2}, \cdots, F_{X_{n+2}})A \otimes_R R_P[\underline{T}] = A \otimes_R R_P[\underline{T}]$*

4.4 LNDs of stably trivial $\mathbb{A}^2$-fibrations with image having grade at least two

The following lemma gives a condition for the ideals of an R-algebra A to be equal.

Lemma 4.4.1. *Let $R \subset A$ be rings, I an ideal of R and $J_1 \subseteq J_2$ ideals of A such that $I \subseteq J_i \cap R$ for $i = 1, 2$. Suppose that the images of J_1 and J_2 are same in $A \otimes_R R/I$, then $J_1 = J_2$.*

Proof. Let for any element $x \in A$, the notation $\bar{x}$ denote the image of x in $A \otimes_R R/I$. Let $x \in J_2$. Then $\bar{x} \in J_2 \otimes_R R/I = J_1 \otimes_R R/I$. This shows that there exists $y \in J_1$ such that $\bar{x} - \bar{y} = \bar{0}$ in $A \otimes_R R/I$, i.e., $x - y \in IA$, i.e., $x \in J_1 + IA$. Since $IA \subseteq J_1$, it is clear that $x \in J_1$, and therefore, we have $J_1 = J_2$. □

The next result states that for a Noetherian domain R containing $\mathbb{Q}$, a stably polynomial R- algebra A of transcendence degree two is also an $\mathbb{A}^2$-fibration.

Lemma 4.4.2. *Let R be a Noetherian domain containing $\mathbb{Q}$ and A an R-algebra such that $A^{[n]} = R^{[n+2]}$. Then, A is an $\mathbb{A}^2$-fibration over R.*

Proof. Since $A^{[n]} = R^{[n+2]}$, A is finitely generated and flat over R and further we have $(A \otimes_R k(P))^{[n]} = k(P)^{[n+2]}$ for all $P \in \operatorname{Spec}(R)$, and therefore, the result follows from the cancellation result of Fujita, Miyanishi and Sugie ([Fuj79], [MS80]. □

The next lemma is a part of proof of Bhatwadekar-Dutta's ([BD97]) result Theorem 1.3.9 which we state here as it is used in the proof of our main result Theorem 4.C (Theorem 4.5.1 and Corollary 4.5.3).

Lemma 4.4.3. *Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K, $A = R[X,Y] = R^{[2]}$ and $D : A \longrightarrow A$ an R-LND. Then, there exists $F \in Ker(D)$ such that $K[X,Y] = K[F]^{[1]}$. If $D(A)A = A$ or $D(X), D(Y)$ form a regular sequence in A, then $\alpha D(X) = F_Y$ and $-\alpha D(Y) = F_X$ for some $\alpha \in A$. Further, F can be chosen to be an irreducible element from the ideal $(X,Y)A$, and in that case $\alpha \in R^*$, i.e., $D(X) = F_Y$ and $D(Y) = -F_X$, and consequently, $(D(X), D(Y))A = (F_X, F_Y)A$.*

Proof. By Theorem 2.3.3 ([Ren68]) there exists $F \in \text{Ker}(D)$ such that $K[X,Y] = K[F]^{[1]}$. Since $F \in \text{Ker}(D)$, we have $F_X D(X) + F_Y D(Y) = 0$ and therefore, due to the hypothesis $D(A)A = A$ or $D(X), D(Y)$ form a regular sequence in A, we get $\alpha D(X) = F_Y$ and $-\alpha D(Y) = F_X$ for some $\alpha \in A$.

Let $F(0,0) = b \in R$. Clearly, $F - b \in (X,Y)R[X,Y]$, $D(F-b) = 0$ and $K[X,Y] = K[F-b]^{[1]}$. This shows that F can be chosen to be from the ideal $(X,Y)R[X,Y]$. Further, since R is Noetherian, F can be chosen to be an irreducible in A. Now, since $K[X,Y] = K[F]^{[1]}$ and $\alpha D(X) = F_Y$ and $-\alpha D(Y) = F_X$ for some $\alpha \in A$, by Jacobian criterion we have $\alpha \in K^* \cap A = R$. Thus, $\alpha \in R$ is such that $\alpha \mid F_X$, and $\alpha \mid F_Y$ and therefore, it follows that $\alpha \mid F$ in A. Since F is an irreducible in A, we have $\alpha \in R^*$, and hence $(D(X), D(Y))A = (F_X, F_Y)A$. □

The next result describes the condition for two elements b_1, b_2 in R to be a regular sequence which are R-multiples of regular sequence a_1, a_2 in R.

Lemma 4.4.4. *Let R be a domain and let $r, s, a_i, b_i \in R$, $i = 1,2$ be such that $ra_i = sb_i$. Suppose that a_1, a_2 form a regular sequence in R. Then the following holds.*

(I) *If $s = 0$, then $r = 0$. If $s \neq 0$, then $s \mid r$.*

(II) *Suppose $r, s \in R \setminus \{0\}$. Then, b_1, b_2 also form a regular sequence in R if and only if either $s = r$ upto units or $r, s \in R^*$.*

Proof. <u>(I)</u>: Since a_1, a_2 is a regular sequence, if $s = 0$ then clearly $r = 0$. Now assume that $s \neq 0$.

$ra_1 = sb_1$ and $ra_2 = sb_2$ together gives $ra_1 b_2 = sb_1 b_2 = rb_1 a_2$ so that we have $r(a_1 b_2 - b_1 a_2) = 0$. This implies either $r = 0$ or $a_1 b_2 - b_1 a_2 = 0$. If $r = 0$, then clearly $s \mid r$. Assume $r \neq 0$. Then as $a_1 b_2 - b_1 a_2 = 0$ and since $\{a_1, a_2\}$ is a sequence, we have $b_1 \in (a_1)$ and $b_2 \in (a_2)$. Thus $b_2 = \alpha a_2$ and substituting this back in $ra_2 = sb_2$ we have $ra_2 = s\alpha a_2$ and hence $(r - \alpha s)a_2 = 0$. Since by hypothesis $a_2 \neq 0$, we get $s \mid r$.

(II): If either $s = r$ upto units or $r, s \in R^*$, then clearly b_1, b_2 form a regular sequence in R. Now, suppose that b_1, b_2 form a regular sequence in R. If $r, s \in R^*$ we are done. Clearly, under the given hypotheses only one of r and s cannot be a unit of R. So assume that both r and s are not in R^*. Now, since $r, s \in R \setminus \{0\}$, by repeating the arguments in (I) we have $r \mid s$ and also from (I) we have $s \mid r$; these two together gives us $r = s$ upto units. □

Now we will establish the relationship between irreducibility of a derivation over an R-algebra A and the grade of its image ideal.

Lemma 4.4.5. *Let $R \subseteq A$ be Noetherian domains and $D : A \longrightarrow A$ an R-derivation. Then, the following holds.*

(I) *If grade$(D(A)A) \geq 2$, then D and D_P are irreducible for all $P \in$ Spec(R).*

(II) (a) *If D is irreducible and A is a UFD, then grade$(D(A)A) \geq 2$.*

(b) *For each $P \in$ Spec(R), if D_P is irreducible and A_P is a UFD, then grade$(D(A)A) \geq 2$.*

Proof. (I): Assume that grade$(D(A)A) \geq 2$. If possible let D be reducible. Then, there exists $a \in A$ such that $D(A)A \subset aA$, which clearly contradicts the assumption that grade$(D(A)A) \geq 2$. Therefore, D is irreducible.

Let $P \in$ Spec(R) and D_P denote the extension of D to A_P. Since grade$(D(A)A) \leq$ grade$((D_P(A_P)A_P))$, we have grade$(D_P(A_P)A_P) \geq 2$. This tells us that D_P is irreducible.

(II): Suppose that D is irreducible. Let $I = D(A)A$. If $I = A$, i.e., grade$(I) = \infty$, we are done. So assume that $I \neq A$.

(a): Since A is Noetherian, there exists $P = (f_1, f_2, \cdots, f_r) \in$ Spec(A) such that $I \subseteq P$ and grade$(I) =$ grade(PA_P). Now suppose that A is a UFD. Since A_P is a UFD and PA_P is a prime ideal of A_P, there exists a prime $p_1 \in A$ such that p_1 remains a prime in A_P and $p_1 \mid f_1$ in A_P; and from which it follows that $p_1 \mid f_1$ in A. In that case we have $P = (p_1, f_2, \cdots, f_r)$. Note that there exists $j \neq 1$ such that $p_1 \nmid f_j$ in A; otherwise we shall have $D(A)A = I \subset P \subset (p_1)$, which is a contradiction to the assumption that D is irreducible. So, without loss of generality we assume that $p_1 \nmid f_2$. In that case, on repeating previous arguments we get a prime $p_2 \in A$ such that p_2 remains prime in A_P, $p_2 \mid f_2$ in A and $p_1 \neq p_2$, and hence $P = (p_1, p_2, f_3, \cdots, f_r)$. Since p_1 and p_2 are distinct primes in A_P, they form a sequence in A_P, i.e., grade$(PA_P) \geq 2$, and hence we have grade$(I) \geq 2$.

(b): Assume that for each $P \in$ Spec(R), D_P is irreducible and A_P is a UFD. Then by (a) we have grade$(D_P(A_P)A_P) \geq 2$ for each $P \in$ Spec(R). Now, since grade$(D(A)A) =$ grade$(D_{P_0}(A_{P_0})A_{P_0}) =$ grade$(D(A)A_{P_0})$ for some $P_0 \in$ Spec(R), by Lemma 2.3.2 it follows that grade$(D(A)A) \geq 2$.

□

As a consequence of the above Lemma 4.4.5 we get the following results.

Corollary 4.4.6. *Let R be a Noetherian domain, A a stably polynomial algebra over R and $D : A \longrightarrow A$ an R-derivation.*

(I) *Suppose that R is a UFD. Then, D is irreducible if and only if $grade(D(A)A) \geq 2$.*

(II) *Suppose that R is a regular domain. Then, D and D_P are irreducible for each $P \in Spec(R)$ if and only if $grade(D(A)A) \geq 2$.*

Corollary 4.4.7. *Let $R \subseteq A$ be Noetherian domains containing $\mathbb{Q}$ and $D, E : A \longrightarrow A$ two R-LNDs. Assume that $rD = sE$ for some $r, s \in A$ and $grade(D(A)A) \geq 2$. Then D is irreducible and $s|r^e$ for some $e \in \mathbb{N}$.*

Proof. Clearly, by Lemma 4.4.5(I) D is irreducible. For rest of the proof, let us fix the following notation.

$\mathcal{P}$:= *"ideal generated by the image of the derivation has grade at least two"*

Given that D satisfies $\mathcal{P}$ on A. Let $R_r = R[1/r]$, $A_r = A \otimes_R R_r$, and D_r and E_r denote the natural extension of D and E respectively on A_r. Since A_r is flat over A, we see that D_r satisfies $\mathcal{P}$ on A_r. Now, since $rD_r = sE_r$ on A_r where r is an unit in R_r and since D_r satisfies $\mathcal{P}$ on $A_r[T]$, it follows that s is a unit in R_r, and therefore, $s \mid r^e$ in R for some $e \in \mathbb{N}$. □

For the next results of this section we use the following hypothesis.

Let $\mathcal{H}$:= *"Let R be a Noetherian domain containing $\mathbb{Q}$ with quotient field K and A an R-algebra. Suppose, $\underline{X} = (X_1, X_2, \cdots, X_{n+2})$ and $\underline{T} = (T_1, T_2, \cdots, T_n)$ are two sequences of indeterminates such that $A^{[n]} = A[\underline{T}] = R[\underline{X}] = R^{[n+2]}$. Let $D : A \longrightarrow A$ be a non-trivial R-LND, $\widetilde{D}$ the trivial extension of D on $A[\underline{T}]$ and D_K the natural extension of D on $A \otimes_R K$."*

We now observe some properties of LNDs of stably polynomial $\mathbb{A}^2$-fibrations.

Proposition 4.4.8. *Under the hypothesis $\mathcal{H}$ the following holds.*

(I) *$D_K = h\frac{\partial}{\partial G}$, $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(F, \underline{T}, -)$ for some $r \in R \setminus \{0\}$, $h \in Ker(D) \setminus \{0\}$ and irreducibles $F, G \in A$ such that $Ker(D) \otimes_R K = K[F]$, $A \otimes_R K = K[F, G] = K^{[2]}$ and $Ker(\widetilde{D}) = Ker(D)[\underline{T}]$. Further, $\mathcal{JD}_{(\underline{X})}(F, \underline{T}, -)$ is irreducible in $A[\underline{T}] \otimes_R K = K[\underline{X}]$.*

(II) (a) *Suppose that $grade(D(A)A) \geq 2$. Then, $h \in R$ and $h|r^e$ for some $e \in \mathbb{N}$ and therefore, $rD = \mathcal{JD}_{(\underline{X})}(hF, \underline{T}, -)$, i.e., rD is a Jacobian derivation.*

(b) *Suppose that $grade(D(A)A) = \infty$. Then, $grade((F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}]) = \infty$ and $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(\mu F, \underline{T}, -)$ for some $\mu \in R^*$.*

(c) *Suppose that* $2 \leq grade(D(A)A) < \infty$*. Then, there exist* $f_i \in (F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}]$ *and* $g_i \in \widetilde{D}(A[\underline{T}])A[\underline{T}]$ *for* $i = 1, 2$ *such that* g_1, g_2 *form a regular sequence where* $hf_i = rg_i$ *and* $h \mid r$*. Further,* f_1, f_2 *form a regular sequence in* $A[\underline{T}]$ *if and only if either* $r = h$ *upto units or* $r, h \in R^*$*, and therefore,* $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(\mu F, \underline{T}, -)$ *for some* $\mu \in R^*$*.*

Proof. **(I):** By Lemma 4.4.2 A is an $\mathbb{A}^2$-fibration over R and therefore, by Theorem 2.3.3 ([Ren68]), one can find $F, G \in A$ such that F, G are irreducible in A, $A \otimes_R K = K[F, G]$, $\text{Ker}(D_K) = K[F]$ and $D_K = h\frac{\partial}{\partial G}$ where $h = D_K(G) = D(G) \in \text{Ker}(D)$. Since D is non-trivial, $h \neq 0$, i.e., $h \in \text{Ker}(D) \setminus \{0\}$. Consider the Jacobian derivation $\Delta := \mathcal{JD}_{(\underline{X})}(F, \underline{T}, -)$ defined on $A[\underline{T}]$ and its extension Δ_K on $A[\underline{T}] \otimes_R K$. Note that $r\widetilde{D}_K = h\Delta_K$ where $r = \Delta_K(G) = \Delta(G)$. Since $A[\underline{T}] \otimes_R K = K[F, G, \underline{T}] = K[\underline{X}]$, it follows that $r \in K^* \cap A$. Further, since A is faithfully flat over R, we have $K \cap A = R$, and therefore, $r \in R \setminus \{0\}$. Now, observe that the equation $r\widetilde{D}_K = h\Delta_K$ also holds on $A[\underline{T}]$, and therefore we have $r\widetilde{D} = h\Delta$ on $A[\underline{T}]$. Clearly, $\text{Ker}(\widetilde{D}) = \text{Ker}(D)[\underline{T}]$. Further, since $A[\underline{T}] \otimes_R K = K[F, G][\underline{T}] = K[\underline{X}]$, we see that $K[F, \underline{T}]$ is inert in $K[\underline{X}]$, and therefore, by Corollary 2.3.18 ([Dai97]) Δ is irreducible in $K[\underline{X}]$.

(II)(a): Assume that $\text{grade}(D(A)A) \geq 2$. Since $A[\underline{T}]$ is faithfully flat over A, we see that $\text{grade}(\widetilde{D}(A[\underline{T}])A[\underline{T}]) \geq 2$. By Proposition 4.4.8(I) we have $r\widetilde{D} = h\Delta$ for some $r \in R$ and $h \in \text{Ker}(D)$, and hence by Corollary 4.4.7 we get $h \mid r^e$ for some $e \in \mathbb{N}$. Since $h \in \text{Ker}(D)$, and $r \in R \subset \text{Ker}(D)$, it follows that $h \in R$. Thus, we have $r\widetilde{D} = \mathcal{JD}_{(\underline{X})}(hF, \underline{T}, -) = h\Delta$ where $r, h \in R$.

(II)(b): Though the proof follows from Theorem 1.3.8 ([EKO16]) and Theorem 1.3.9 (iii) ([BD97]), we give an independent proof. We suppose that $\text{grade}(D(A)A) = \infty$, i.e., $D(A)A = A$. Note that $A[\underline{T}] \otimes_R K = K[F, G][\underline{T}] = K[\underline{X}]$, and therefore, $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] \otimes_R K = A[\underline{T}] \otimes_R K$, i.e., $a_1F_{X_1} + a_2F_{X_2} + \cdots + a_{n+2}F_{X_{n+2}} = t$ where $t \in R$ and $a_i \in A[\underline{T}]$ for all $i = 1, 2, \cdots, n+2$. Set $I := R \cap (F_{X_1}, F_{X_2}, \cdots, F_{X_{n+2}})A[\underline{T}]$. Note that I is an ideal of R, $t \in I$ and $I = R$ if and only if $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$. We claim that $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$. Since by a standard local-global principle we have $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$ if and only if $(F_{X_1}, \cdots, F_{X_{n+2}})A_P[\underline{T}] = A_P[\underline{T}]$ for all $P \in \text{Spec}(R)$ where $A_P = A \otimes_R R_P$, without loss of generality we may assume that R is local. We shall use induction on $\dim(R)$ to prove our claim. If $\dim(R) = 0$, the claim holds true obviously. So, assume that $\dim(R) \neq 0$.

If possible let $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] \neq A[\underline{T}]$. Then, there exists $\mathfrak{P} \in \text{Spec}(A[\underline{T}])$ such that $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] \subseteq \mathfrak{P}$. Clearly, $I \subset \mathfrak{P}$ and hence there exists $\mathfrak{p} \in \text{Spec}(R)$ such that $I \subset \mathfrak{p}$ and $\mathfrak{P} \cap R = \mathfrak{p}$. Consider the domain $R_1 = R_{\mathfrak{p}}$. Let $(\widehat{R_1}, \widehat{\mathfrak{p}})$ be the completion of $(R_1, \mathfrak{p})$. Since the non-zero divisors of R remains non-zero divisor in $\widehat{R}$, going modulo a minimum prime ideal of $\widehat{R_1}$, we may assume that $\widehat{R_1}$ is a domain such that $I = R$ if and only if $\widehat{I} := I \otimes_R \widehat{R_1} = \widehat{R_1}$. Let $(\widetilde{\widehat{R_1}}, \widetilde{\widehat{\mathfrak{p}}})$ denote the normalization of $(\widehat{R_1}, \widehat{\mathfrak{p}})$.

(†): Note that $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$ if and only if $(F_{X_1}, \cdots, F_{X_{n+2}})\widetilde{\widehat{A_1}}[\underline{T}] = \widetilde{\widehat{A_1}}[\underline{T}]$. One implication is obvious. We let $(F_{X_1}, \cdots, F_{X_{n+2}})\widetilde{\widehat{A_1}}[\underline{T}] = \widetilde{\widehat{A_1}}[\underline{T}]$. Since $\widetilde{\widehat{A_1}}[\underline{T}]$ is integral over $\widehat{A_1}[T]$, it follows that $(F_{X_1}, \cdots, F_{X_{n+2}})\widehat{A_1}[\underline{T}] = \widehat{A_1}[\underline{T}]$, and therefore, due to faithful flatness of $\widehat{A_1}[\underline{T}]$ over $A_1[\underline{T}]$, we have $(F_{X_1}, \cdots, F_{X_{n+2}})A_1[\underline{T}] = A_1[\underline{T}]$. This shows that $(F_{X_1}, \cdots, F_{X_{n+2}})\widehat{A}[\underline{T}] = \widehat{A}[\underline{T}]$.

Let $\dim(R) = 1$. Then, $(\widetilde{\widehat{R_1}}, \widetilde{\widehat{\mathfrak{p}}})$ is a Noetherian normal local domain of dimension one, i.e., $\widetilde{\widehat{R_1}}$ is a DVR. Set $\widetilde{\widehat{A_1}} := A \otimes_R \widetilde{\widehat{R_1}}$. Clearly, $\widetilde{\widehat{A_1}}[\underline{T}] = \widetilde{\widehat{R_1}}[\underline{X}] = \widetilde{\widehat{R_1}}^{[n+2]}$. One can observe that $\widetilde{\widehat{A_1}}$ is an $\mathbb{A}^2$-fibration over $\widetilde{\widehat{R_1}} \hookleftarrow \mathbb{Q}$, and therefore, by Theorem 1.2.5 ([Sat83]) $\widetilde{\widehat{A_1}} = \widetilde{\widehat{R_1}}[U,V] = \widetilde{\widehat{R_1}}^{[2]}$. So, we have $\widetilde{\widehat{A_1}}[\underline{T}] = \widetilde{\widehat{R_1}}[\underline{X}] = \widetilde{\widehat{R_1}}[U,V,\underline{T}]$. Now observe that

$$\begin{aligned}(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] \otimes_{A[\underline{T}]} \widetilde{\widehat{A_1}}[\underline{T}] &= (F_{X_1}, \cdots, F_{X_{n+2}})\widetilde{\widehat{A_1}}[\underline{T}] \\ &= (F_U, F_V, F_{T_1}, F_{T_2} \cdots, F_{T_n})\widetilde{\widehat{A_1}}[\underline{T}] \\ &= (F_U, F_V)\widetilde{\widehat{A_1}}[\underline{T}].\end{aligned}$$

Since the natural extension of D on $\widetilde{\widehat{A_1}}$ is fixed point free and since $\widetilde{\widehat{A_1}} = \widetilde{\widehat{R_1}}[U,V]$, by Lemma 4.4.3 it follows that $(F_U, F_V)\widetilde{\widehat{A_1}}[\underline{T}] = \widetilde{\widehat{A_1}}[\underline{T}] = (F_{X_1}, \cdots, F_{X_{n+2}})\widetilde{\widehat{A_1}}[\underline{T}]$. Now, using the argument in (†) we have $(F_{X_1}, \cdots, F_{X_{n+2}})A_1[\underline{T}] = A_1[\underline{T}]$, which gives a contradiction.

We now assume that our claim holds for R where $\dim(R) \leq \ell$. Let $\dim(R) = \ell + 1$.

Let $\{P_1, P_2, \cdots, P_m\} = \mathrm{Ass}_{\widetilde{\widehat{R_1}}}(\widetilde{\widehat{R_1}}/t\widetilde{\widehat{R_1}})$. Note that $\mathrm{ht}(P_i) = \mathrm{grade}(P_i) = 1$ for all $i = 1, 2, \cdots, m$ (see [Mat80, Theorem 38]).

Case $\bigcup_{i=1}^m P_i \supseteq \widetilde{\widehat{I}}$: By prime avoidance lemma we have $\widetilde{\widehat{I}} \subseteq P_j$ for some $j = 1, 2, \cdots, m$. Note that $P_j\widetilde{\widehat{A_1}}[\underline{T}] \in \mathrm{Ass}_{\widetilde{\widehat{A_1}}[\underline{T}]}(\widetilde{\widehat{A_1}}[\underline{T}]/t\widetilde{\widehat{A_1}}[\underline{T}])$ and $P_j\widetilde{\widehat{A_1}}[\underline{T}] \bigcap \widetilde{\widehat{R_1}} = P_j$. Now, by the induction hypothesis, considering the LND induced by D on $\widetilde{\widehat{A_1}}[\underline{T}] \otimes_{\widetilde{\widehat{R_1}}} \widetilde{\widehat{R_1}}/P_j$ we have $(F_{X_1}, \cdots, F_{X_{n+2}})\widetilde{\widehat{A_1}}[\underline{T}] \otimes_{\widetilde{\widehat{R_1}}} \widetilde{\widehat{R_1}}/P_j = \widetilde{\widehat{A_1}}[\underline{T}] \otimes_{\widetilde{\widehat{R_1}}} \widetilde{\widehat{R_1}}/P_j$. Since we assumed that $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] \neq A[\underline{T}]$, by the arguments in (†) there exists $\widetilde{\widehat{\mathfrak{P}}} \in \mathrm{Spec}(\widetilde{\widehat{A_1}}[\underline{T}])$ such that $(F_{X_1}, \cdots, F_{X_{n+2}})\widetilde{\widehat{A_1}}[\underline{T}] \subseteq \widetilde{\widehat{\mathfrak{P}}}$ and $\widetilde{\widehat{\mathfrak{P}}} \otimes_{\widetilde{\widehat{R_1}}} \widetilde{\widehat{R_1}}/P_j = \widetilde{\widehat{A_1}} \otimes_{\widetilde{\widehat{R_1}}} \widetilde{\widehat{R_1}}/P_j$. Since $P_j \subseteq \widetilde{\widehat{\mathfrak{P}}} \bigcap \widetilde{\widehat{R_1}}$, by Lemma 4.4.1 we see that $\widetilde{\widehat{\mathfrak{P}}} = \widetilde{\widehat{A_1}}[\underline{T}]$, which is a contradiction.

Case $\bigcup_{i=1}^m P_i \not\supseteq \widetilde{\widehat{I}}$: Clearly $P_1 \not\supseteq \widetilde{\widehat{I}}$. Consider the domain $\widetilde{\widehat{R_1}}/P_1$. Then $\dim(\widetilde{\widehat{R_1}}/P_1) \leq \ell$, and therefore, by the induction hypothesis we have $\widetilde{\widehat{R_1}}/P_1 = I \otimes_R \widetilde{\widehat{R_1}}/P_1$ which is a contradiction to the fact that $\widetilde{\widehat{R_1}}$ is a local domain with maximal ideal $\widetilde{\widehat{\mathfrak{p}}} \supseteq I \otimes_R \widetilde{\widehat{R_1}}$.

So, we get $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$ for $\dim(R) = \ell + 1$. This establishes our claim, i.e., $\text{grade}((F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}]) = \infty$.

We now show that $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(\mu F, \underline{T}, -)$. From the equation $r\widetilde{D} = h\Delta$ we get the following system of equations

$$r\widetilde{D}(X_i) = h(a_{i1}F_{X_1} + a_{i2}F_{X_2} + \cdots + a_{in+2}F_{X_{n+2}}) = h\Delta(X_i)$$

where, for $i, j = 1, 2 \cdots, n+2$, $a_{ij} \in A[\underline{T}]$ and $a_{ii} = 0$. Since $D(A) = A$, we have $\widetilde{D}(A[T])A[T] = A[T]$, and therefore, there exists $b_i \in A[\underline{T}]$, $i = 1, 2, \cdots, n+2$ such that $\sum_{i=1}^{n+2} b_i\widetilde{D}(X_i) = 1$, i.e., $\sum_{i=1}^{n+2} b_i r\widetilde{D}(X_i) = r$, which, along with the above set of equations, gives us $h\sum_{i=1}^{n+2} b_i\Delta(X_i) = r$. Thus, we have $c := \sum_{i=1}^{n+2} b_i\Delta(X_i) \in R$, i.e., $(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}]$ contains a constant $c \in R$ such that $hc = r$ and therefore $c\widetilde{D} = \Delta$. Note that by Proposition 4.3.1 we have $A = R[F, G]$ for some $G \in A$, and therefore, $R[F, G][\underline{T}] = R[\underline{X}]$. From this we see that $\mu := \det(\mathcal{J}ac_{(F,G,\underline{T})}(\underline{X})) \in R^*$. Further, by Theorem 1.3.9 ([BD97]) we have $\text{Ker}(D) = R[F]$ and $D = \mathcal{JD}_{(F,G)}(F, -)$. This shows that

$$\begin{aligned}
\widetilde{D} &= \mathcal{JD}_{(F,G,\underline{T})}(F, \underline{T}, -) \\
&= \det(\mathcal{J}ac_{(F,G,\underline{T})}(\underline{X}))\mathcal{JD}_{(\underline{X})}(F, \underline{T}, -) \\
&= \mu \,.\, \mathcal{JD}_{(\underline{X})}(F, \underline{T}, -) \\
&= \mathcal{JD}_{(\underline{X})}(\mu F, \underline{T}, -) = \mu\Delta \text{ where } \mu \in R^*.
\end{aligned}$$

(II)(c): Since $A[\underline{T}]$ is faithfully flat over A and $2 \leq \text{grade}(D(A)A) < \infty$, we have $2 \leq \text{grade}(\widetilde{D}(A[\underline{T}])A[\underline{T}]) < \infty$. So, there exists $\alpha_i, \beta_i \in A[\underline{T}]$, $i = 1, 2, \cdots, n+2$ such that $g_1 := \alpha_1\widetilde{D}(X_1) + \alpha_2\widetilde{D}(X_2) + \cdots + \alpha_{n+2}\widetilde{D}(X_{n+2})$ and $g_2 := \beta_1\widetilde{D}(X_1) + \beta_2\widetilde{D}(X_2) + \cdots + \beta_{n+2}\widetilde{D}(X_{n+2})$ form a regular sequence in $A[\underline{T}]$. Set $f_1 := \alpha_1\Delta(X_1) + \alpha_2\Delta(X_2) + \cdots + \alpha_{n+2}\Delta(X_{n+2})$ and $f_2 := \beta_1\Delta(X_1) + \beta_2\Delta(X_2) + \cdots + \beta_{n+2}\Delta(X_{n+2})$. Note that by (II)(a) we have $r\widetilde{D} = h\Delta$ where $r, h \in R$ and $h \mid r^e$ for some $e \in \mathbb{N}$ and also $hf_i = rg_i$ for $i = 1, 2$. Now, since g_1, g_2 forms a sequence, by Lemma 4.4.4 we have $h \mid r$.

We now assume that f_1, f_2 form a regular sequence in R, then by Lemma 4.4.4 it follows that either $r = h$ upto units or $r, h \in R^*$, and therefore, $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(\mu F, \underline{T}, -)$ for some $\mu \in R^*$. Conversely, we assume that $r, h \in R^*$. Then, it is easy to see that f_1, f_2 form a regular sequence in R as g_1, g_2 form the same. □

Remark 4.4.9. (I) *In Proposition 4.4.8 under the hypothesis "*$\text{grade}(D(A)A) \geq 2$*" if we further have* $r \in R^*$*, then in view of Lemma 4.4.4 it is clear that* $h \in R^*$*, and in that case* *grade*$((F_{X_1}; \cdots, F_{X_{n+2}})A[T]) \geq 2$, *and therefore, by Proposition 4.1.4 we have Ker*$(D) = R[F]$.

(II) *One can see that Proposition 4.4.8(I) also follows from [ML98, Lemma 8] (also see [Fre17, Theorem 3.20]).*

4.5 LNDs with polynomial kernels of stably trivial $\mathbb{A}^2$-fibrations

We are now ready to prove our main results. First we shall prove Theorem 4.C(a).

Theorem 4.5.1. *Let the hypothesis $\mathcal{H}$ holds and consider the following statements.*

(I) *D is irreducible.*

(I′) *D and D_P are irreducible for each $P \in Spec(R)$.*

(II) *$Ker(D) = R^{[1]}$.*

(III) *$grade(D(A)A) \geq 2$.*

(IV) *There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $grade((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$.*

(V) *There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $grade((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$ where $r \in R$, $h \in Ker(D)$.*

Then, we have the following

(a) *"(I′) $\Longrightarrow$ (I)", "(II) $\Longleftrightarrow$ (V)", "(IV) $\Longrightarrow$ (V)" and "(III) $\Longrightarrow$ (I′)" hold.*

(b) *Suppose that (V) holds and assume that either R is a UFD or a regular domain or $R[W] \otimes_R R/P$ is inert in $A \otimes_R R/P$ for all height-one prime ideal P of R. Then, $\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ is irreducible in $R[\underline{X}]$ and $R_P[\underline{X}]$ for all $P \in Spec(R)$.*

(c) *If R is a UFD, then (II) holds and further (I) and (I′) are equivalent.*

(d) *Suppose that R is either a UFD or a regular domain. Then, "(I′) $\Longleftrightarrow$ (III)", "(IV) $\Longrightarrow$ (III)" and "(I′) & (II) $\Longrightarrow$ (IV)" hold true.*

Proof. (a): "(I′) $\Longrightarrow$ (I)" and "(IV) $\Longrightarrow$ (V)" are trivial. "(III) $\Longrightarrow$ (I′)" follows from Lemma 4.4.5(I).

Now, we shall prove "(II) $\Longleftrightarrow$ (V)". Assume that (II) holds, i.e., Ker$(D) = R^{[1]} = R[W]$ for some $W \in A$. We shall show that (V) holds. By Proposition 4.4.8(I) we have an irreducible element F in A, $r \in R$ and $h_1 \in$ Ker(D) such that $A \otimes_R K = K[F]^{[1]}$, $r\widetilde{D} =$

$h_1\mathcal{JD}_{(\underline{X})}(F,\underline{T},-)$. Since $R[F] \subset R[W]$ and $K[F] = K[W]$, we see that $F = r_0 + r_1W$ for some $r_0 \in R$ and $r_1 \in R\setminus\{0\}$. In that case we have $r\widetilde{D} = h_1r_1\mathcal{JD}_{(\underline{X})}(W,\underline{T},-) = h\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ where $h = h_1r_1 \in \mathrm{Ker}(D)$. Since $R[W] = \mathrm{Ker}(D)$ is inert in A, it is inert in $A[\underline{T}] = R[\underline{X}]$, and hence by Proposition 4.1.4 we have $\mathrm{grade}((W_{X_1},\cdots,W_{X_{n+2}})A[\underline{T}]) \geq 2$.

We now assume that (V) holds. Since $A\otimes_R K = K[W]^{[1]}$, $K[W]$ is inert in $A\otimes_R K$ and hence it is inert in $A[T]\otimes_R K = K[\underline{X}]$. Now, since $\mathrm{grade}((W_{X_1},\cdots,W_{X_{n+2}})A[\underline{T}]) \geq 2$ whereas $A[\underline{T}] = R[\underline{X}]$, by Proposition 4.1.4 we have $R[W]$ is inert in $A[T]$. Further, since $R[W] \subset A$, we have $R[W]$ is inert in A and hence $R[W,\underline{T}]$ is inert in $A[\underline{T}]$, and therefore, by Corollary 4.2.2 we have $\mathrm{Ker}(\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)) = R[W,\underline{T}]$. Since $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ for some $r \in R$, $h \in \mathrm{Ker}(D)$ and since $\widetilde{D}$ is a natural extension of D, $R[W] \subset \mathrm{Ker}(D)$. Since both $R[W]$ and $\mathrm{Ker}(D)$ are inert in A and $\mathrm{tr.deg}_{\mathrm{Ker}(D)}(A) = \mathrm{tr.deg}_{R[W]}(A)$, we have $\mathrm{Ker}(D) = R[W]$.

<u>(b):</u> We assume the hypothesis. Since (II) $\Longleftrightarrow$ (V) hold, we have from (a) that $\mathrm{Ker}(D) = R[W] = R^{[1]}$, and therefore, $R[W,\underline{T}]$ is inert in $A[\underline{T}] = R[\underline{X}]$. This also shows that $R_P[W,\underline{T}]$ is inert in $A_P[\underline{T}] = R_P[\underline{X}]$ for each $P \in \mathrm{Spec}(R)$. Now, due to the given hypothesis, from Corollary 4.2.2 it follows that $\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ is irreducible in $R[\underline{X}]$ and $R_P[\underline{X}]$ for all $P \in \mathrm{Spec}(R)$.

<u>(c):</u> Assume that R is a UFD. By Theorem 2.3.5 ([AEH72]) and Theorem 2.3.6 ([Ham75]), one can easily see that $\mathrm{Ker}(D) = R^{[1]}$ (also see Corollary 5.1.8). We now show that (I) and (I′) are equivalent. It is enough to show that "(I) $\Longrightarrow$ (I′)". So, assume that (I) holds, i.e., D is irreducible. Since R is a UFD, we already have $\mathrm{Ker}(D) = R[W]$ for some $W \in A$. Now, by Corollary 4.4.6 we see that $\mathrm{grade}(D(A)A) \geq 2$ and hence by Proposition 4.4.8 (I) & (II)(a) it follows that $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ where $r,h \in R$. Since $\mathrm{Ker}(D) = R[W]$ is inert in A we see that $R[W,\underline{T}]$ is also inert in $A[\underline{T}] = R[\underline{X}]$, and therefore, by Corollary 4.2.2 we have $\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ is irreducible. Since R is a UFD, both $\widetilde{D}$ and $\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ are irreducible and $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ where $r,h \in R$, it follows that $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$. Rest follows from (b).

<u>(d):</u> We assume that R is either a UFD or a regular domain. "(I′) $\Longleftrightarrow$ (III)" follows directly from Corollary 4.4.6.

We now prove "(IV) $\Longrightarrow$ (III)". So, assume that (IV) holds. Note that since "(IV) $\Longrightarrow$ (V)" holds true, from (b) it follows that $\mathcal{JD}_{(\underline{X})}(W,\underline{T},-)$ is irreducible in $R[\underline{X}]$ and $R_P[\underline{X}]$ for all $P \in \mathrm{Spec}(R)$. From this it follows that $\widetilde{D}$, and therefore D is irreducible in $R[\underline{X}]$ and $R_P[\underline{X}]$ for all $P \in \mathrm{Spec}(R)$. Now, by Corollary 4.4.6 we see that $\mathrm{grade}(D(A)A) \geq 2$; and therefore (III) holds.

Next we prove "(I′) & (II) $\implies$ (IV)". Assume that (I′) and (II) holds. Now, since (II) $\iff$ (V) holds true, we assume that both (II) and (V) holds, and therefore by (b) we see that $\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ is irreducible where $\mathrm{Ker}(D) = R[W] = R^{[1]}$. Again, since D is irreducible, $\widetilde{D}$ is also irreducible. Since R is either a UFD or a regular domain, in view of Corollary 4.4.6, we see that an iterative use of Corollary 4.4.7, for both $\widetilde{D}$ and $\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$, on the given equation $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$, where $r \in R$ and $h \in \mathrm{Ker}(D)$, gives us $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$. Now since we already have (II) $\iff$ (V), it follows that $\mathrm{grade}((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$. This establishes (IV). □

Remark 4.5.2. *(a) From the proof of Theorem 4.5.1 the following is observed easily. If we assume (II) and (III) hold, then in (V) we have $r\widetilde{D} = h\mathcal{JD}_{(\underline{X})}(F, \underline{T}, -)$ where $r, h \in R$; as due to Corollary 4.4.7 it would follow that $h|r^e$ for some $e \in \mathbb{N}$, implying $h \in R$.*

(b) It is to be noted that under the hypothesis of Theorem 1.3.9 ([BD97]) the following statements can be seen to be equivalent.

- (I) *D is irreducible and $\mathrm{Ker}(D) = R^{[1]}$.*
- (I′) *D and D_P are irreducible for each $P \in \mathrm{Spec}(R)$ and $\mathrm{Ker}(D) = R^{[1]}$.*
- (II) *Either D is fixed point free or $D(X)$ and $D(Y)$ form an $R[X,Y]$-regular sequence.*
- (III) *$D = \mathcal{JD}_{(X,Y)}(F, -)$, $K[X,Y] = K[F]^{[1]}$ and F_X, F_Y either form an $R[X,Y]$-regular sequence or are comaximal in $R[X,Y]$.*

As a particular application of the Theorem 4.5.1 we have the following corollary.

Corollary 4.5.3. *Let the hypothesis $\mathcal{H}$ holds.*

(A) *Assume that R is a UFD. Then, $\mathrm{Ker}(D) = R[W] = R^{[1]}$ for some $W \in R$. Further, the following are equivalent.*

- (I) *D is irreducible.*
- (I′) *D and D_P are irreducible for all $P \in \mathrm{Spec}(R)$.*
- (II) *$\mathrm{grade}(D(A)A) \geq 2$.*
- (III) *$\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$.*

(B) *Assume that R is a regular domain. Then, $\mathrm{Ker}(D) = \mathrm{Sym}_R(I)$ for some invertible ideal I of R. Further the following are equivalent.*

- (I) *D and D_P are irreducible and $\mathrm{Ker}(D) = R^{[1]}$ for all $P \in \mathrm{Spec}(R)$.*
- (II) *$\mathrm{grade}(D(A)A) \geq 2$ and $\mathrm{Ker}(D) = R^{[1]}$.*
- (III) *There exists $W \in A$ such that $A \otimes_R K = K[W]^{[1]}$, $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(W, \underline{T}, -)$ and $\mathrm{grade}((W_{X_1}, \cdots, W_{X_{n+2}})A[\underline{T}]) \geq 2$.*

Proof. (A): By Theorem 4.5.1(c) we have $\text{Ker}(D) = R[W] = R^{[1]}$ for some $W \in A$; and therefore, it follows that $R[W]$ is inert in $A[\underline{T}] = R[\underline{X}]$. Again directly by Theorem 4.5.1(c) we have (I) $\iff$ (I'). For the rest of the proof, first notice that by Proposition 4.1.4 we have $\text{grade}((F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}]) \geq 2$. Now, "(I) $\iff$ (II)" follows from Theorem 4.5.1(c) and (d); and "(II) $\iff$ (III)" follows from Theorem 4.5.1 (c), (d) and (a).

(B): Since R_P is regular local, it is a UFD and therefore, by Corollary 4.5.3(A) we have $\text{Ker}(D)_P = R_P^{[1]}$ for all $P \in \text{Spec}(R)$. Now, since D is an R-LND of A, D has a pre-slice, i.e, there exists $z \in A$ such that $a := D(z) \in \text{Ker}(D)$, and therefore, D_a, the natural extension of D on $A[1/a]$, has a slice. So, by the Slice Theorem (Theorem 1.2.1) we have $A[1/a] = \text{Ker}(D)[1/a]^{[1]}$. This shows that $\text{Ker}(D)[1/a]$ is a finitely generated algebra over R, and therefore, by Theorem 2.3.14 ([Ono84]), $\text{Ker}(D)$ is finitely generated over R. Since R is Noetherian, by Theorem 2.3.7 ([BCW76]) we have $\text{Ker}(D) = \text{Sym}_R(I)$ for some invertible ideal I of R.

Now, "(I) $\iff$ (III)" follows from Theorem 4.5.1(a) and (d); and "(I) $\iff$ (II)" directly follows from Theorem 4.5.1(c).

□

Though the implication "(I) $\implies$ (III)" or even "(I) & (II) $\implies$ (III)" in Theorem 4.5.1 may not hold if R is not a UFD or not a regular domain (see Corollary 4.5.3), the following result specify an extra condition under which the implication holds when R is a general Noetherian domain containing $\mathbb{Q}$.

Proposition 4.5.4. *Let the hypothesis $\mathcal{H}$ holds. Assume that D is irreducible and $\text{Ker}(D) = R[F]$. Then, there exist $d_i \in A[\underline{T}]$ such that $aF_{X_i} = sd_i$ and $d_i = \frac{\partial U}{\partial X_i}DV - \frac{\partial V}{\partial X_i}DU$ for all $i = 1, 2, \cdots, n+2$ where $U, V \in A$ satisfying $A \otimes_R K = K[U, V]$. Further, if there exist i, j such that d_i, d_j form a regular sequence in $A[T]$ whenever F_{X_i}, F_{X_j} form the same, then $\text{grade}(D(A)A) \geq 2$.*

Proof. By Theorem 2.3.3 ([Ren68]) there exists $G \in A$ and $h \in R[F]$ such that $A \otimes_R K = K[F, G]$ and $D_K = h\dfrac{\partial}{\partial G}$. Let $U, V \in A$ be such that $A \otimes_R K = K[U, V]$ and $D(U) \neq 0 \neq D(V)$. Since $K[U, V][\underline{T}] = K[F, G][\underline{T}]$, clearly we have

$$\begin{aligned}
(F_{X_1}, \cdots, F_{X_{n+2}})A[\underline{T}] \otimes_R K &= (F_{X_1}, \cdots, F_{X_{n+2}})K[U, V][\underline{T}] \\
&= (F_U, F_V, F_{T_1}, \cdots, F_{T_n})K[U, V][\underline{T}] \\
&= (F_U, F_V)K[U, V][\underline{T}] \\
&= (F_F, F_G)K[F, G][\underline{T}] \\
&= A[\underline{T}] \otimes_R K
\end{aligned}$$

Therefore, for $0 = D(F) = F_U D(U) + F_V D(V)$ we have $D(U) = \alpha F_V$ and $D(V) = -\alpha F_U$ where $\alpha \in A \otimes_R K$.

Again since $K[U, V, \underline{T}] = K[X_1, X_2, \cdots, X_{n+2}]$, we have

$$\begin{pmatrix} F_{X_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_{X_{n+2}} \end{pmatrix} = J^t \begin{pmatrix} F_U \\ F_V \\ F_{T_1} \\ \cdot \\ \cdot \\ F_{T_n} \end{pmatrix} \quad \text{where } J = \begin{pmatrix} \frac{\partial U}{\partial X_1} & \frac{\partial U}{\partial X_2} & \cdot & \cdot & \cdot & \frac{\partial U}{\partial X_{n+2}} \\ \frac{\partial V}{\partial X_1} & \frac{\partial V}{\partial X_2} & \cdot & \cdot & \cdot & \frac{\partial V}{\partial X_{n+2}} \\ \frac{\partial T_1}{\partial X_1} & \frac{\partial T_1}{\partial X_2} & \cdot & \cdot & \cdot & \frac{\partial T_1}{\partial X_{n+2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial T_n}{\partial X_1} & \frac{\partial T_n}{\partial X_2} & \cdot & \cdot & \cdot & \frac{\partial T_n}{\partial X_{n+2}} \end{pmatrix} \tag{4.1}$$

Clearly, $\det(J) \in K^* \cap A[\underline{T}] = R \setminus \{0\}$. Note that $F_{T_i} = 0$ for each i and $\alpha F_V = D(U)$ and $\alpha F_U = -D(V)$, and therefore,

$$\alpha F_{X_i} = \left(\frac{\partial U}{\partial X_i} DV - \frac{\partial V}{\partial X_i} DU\right) \quad i.e., \quad aF_{X_i} = sd_i$$

where $d_i = \left(\frac{\partial U}{\partial X_i} DV - \frac{\partial V}{\partial X_i} DU\right)$ for all $i = 1, 2, \cdots, n+2$, $s\alpha = a \in A$ and $s \in R$.

We now prove the next part.

Case $(F_{X_1}, F_{X_2}, \cdots F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$: By Proposition 4.3.1 we have $A = R[F, G] = R^{[2]}$ for some $G \in A$, and therefore, by Theorem 1.3.9 ([BD97]) $\text{grade}(D(A)A) \geq 2$.

Case $(F_{X_1}, F_{X_2}, \cdots F_{X_{n+2}})A[\underline{T}] \neq A[\underline{T}]$: In view of Remark 4.1.5, without loss of generality, we assume that F_{X_1}, F_{X_2} form a regular sequence. By hypothesis d_1, d_2 form a regular sequence, and therefore $\text{grade}(D(A[\underline{T}])A[\underline{T}]) \geq 2$. Since $A[\underline{T}]$ is faithfully flat over A, it follows that $\text{grade}(D(A[\underline{T}])A[\underline{T}]) = \text{grade}(D(A)A) \geq 2$. □

4.6 Examples

We shall now workout on two examples of stably polynomial $\mathbb{A}^2$-fibrations in complete details; one by Hochster ([Hoc72]) and another by Raynaud ([Ray68]). These examples of $\mathbb{A}^2$-fibrations belong to the class of one-stably polynomial algebras and can not be written as $\mathbb{A}^1$-fibrations over polynomial algebras (see Example 3.4.2 in Chapter 3 for details), and therefore, by Theorem 1.3.8 ([EKO16]) they do not possess fixed point free LNDs.

First we compute the example by Hochster.

Example 4.6.1. *Let $R = \mathbb{R}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2 - 1)$, $B = R[Y_0, Y_1, Y_2]$ and K denote the quotient field of R. Let x_0, x_1 and x_2 denote the image of indeterminates X_0, X_1 and X_2 respectively in R. Since the polynomial $X_0^2 + X_1^2 + X_2^2 - 1$ have the Jacobian matrix as $(2X_0, 2X_1, 2X_2)$ $(\neq 0)$ which is a row matrix of rank-1, using Jacobian criterion for regular rings we notice that R is a regular ring. Further, by Nagata's criterion it can be seen that R is a UFD.*

Define an R-LND $D_0 : B \longrightarrow B$ *by* $D_0(Y_0) = x_0$, $D_0(Y_1) = x_1$, $D_0(Y_2) = x_2$. *Let* $c := x_0Y_0 + x_1Y_1 + x_2Y_2 \in B$. *See that* c *is a slice of* D_0*, i.e.,* $D_0(c) = 1$. *Let* $A := Ker(D_0) = R[Y_0 - x_0c,\ Y_1 - x_1c,\ Y_2 - x_2c] = B/(c)$. *Clearly, by the Slice Theorem (Theorem 1.2.1)* $B = A[c]$*, and therefore by Lemma 1.3.1* A *is a* 1*-stably trivial* $\mathbb{A}^2$*-fibration over* R.

Set $\widetilde{Y_0} := Y_0 - x_0c$, $\widetilde{Y_1} := Y_1 - x_1c$ *and* $\widetilde{Y_2} := Y_2 - x_2c$. *Observe that* $B \otimes_R K = K[Y_0, Y_1, Y_2] = K^{[3]}$, $D_0(x_1Y_0 - x_0Y_1) = x_1x_0 - x_0x_1 = 0$ *and* $D_0(x_2Y_1 - x_1Y_2) = x_2x_1 - x_1x_2 = 0$. *Now* $x_1Y_0 - x_0Y_1 = x_1\widetilde{Y_0} - x_0\widetilde{Y_1}$ *and* $x_2\widetilde{Y_1} - x_1\widetilde{Y_2}$ *are K-linear transformations of variables in* $K^{[3]}$ *and hence are again variables in* $K^{[3]}$. *Thus,*

$$B \otimes_R K = K^{[3]} = K[Y_0, Y_1, Y_2] = K[x_1Y_0 - x_0Y_1,\ x_2Y_1 - x_1Y_2,\ c] = K[x_1\widetilde{Y_0} - x_0\widetilde{Y_1},\ x_2\widetilde{Y_1} - x_1\widetilde{Y_2},\ c].$$

We use the relation $\widetilde{Y_0} = -(x_1\widetilde{Y_1} + x_2\widetilde{Y_2})/x_0$ *to eliminate* $\widetilde{Y_0}$ *in* $B \otimes_R K$ *and* $A \otimes_R K$ *and get the following.*

$$B \otimes_R K = K\left[\frac{(-x_0^2 - x_1^2)}{x_0}\widetilde{Y_1} - \frac{x_1x_2}{x_0}\widetilde{Y_2},\ x_2\widetilde{Y_1} - x_1\widetilde{Y_2}, c\right]$$

$$A \otimes_R K = B/(c) \otimes_R K = K\left[\frac{(-x_0^2 - x_1^2)}{x_0}\widetilde{Y_1} - \frac{x_1x_2}{x_0}\widetilde{Y_2},\ x_2\widetilde{Y_1} - x_1\widetilde{Y_2}\right]$$

Set $S := x_2\widetilde{Y_1} - x_1\widetilde{Y_2} \in A$ *and* $T := \dfrac{(-x_0^2 - x_1^2)}{x_0}\widetilde{Y_1} - \dfrac{x_1x_2}{x_0}\widetilde{Y_2}$. *One can check that* $T = -(x_1Y_0 - x_0Y_1) = -(x_1\widetilde{Y_0} - x_0\widetilde{Y_1}) \in A$. *Due to finite generation of* A *over* R, $\exists\, r \in R$ *such that* $A[1/r] = R[1/r][S, T] = R[1/r]^{[2]}$. *To obtain* r*, first note that*

$$\begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} x_2 & -x_1 \\ \frac{-x_0^2 - x_1^2}{x_0} & \frac{x_1x_2}{x_0} \end{pmatrix} \begin{pmatrix} \widetilde{Y_1} \\ \widetilde{Y_2} \end{pmatrix}$$

We now calculate the generators $Y_0 - x_0c$, $Y_1 - x_1c$ *and* $Y_2 - x_2c$ *of* A *in terms of* S *and* T *in* $A \otimes_R K$.

$$\begin{aligned} Y_0 - x_0c &= \frac{-x_0^2x_2 - 2x_1x_2^2}{x_0x_1(2x_2^2 - 1)}S + \frac{-x_1^2 - x_2^2}{x_1(2x_2^2 - 1)}T \\ Y_1 - x_1c &= \frac{x_2}{2x_2^2 - 1}S + \frac{x_0}{2x_2^2 - 1}T \\ Y_2 - x_2c &= \frac{x_0^2 + x_1^2}{x_1(2x_2^2 - 1)}S + \frac{x_0x_2}{x_1(2x_2^2 - 1)}T \end{aligned}$$

Hence it is clear that we can choose r *to be* $x_1(2x_2^2 - 1)$. *Define a new R-LND* D_K *on* $A \otimes_R K$ *as follows*

$$D_K : K[S, T] \longrightarrow K[S, T] \quad by \quad S \mapsto 0,\ T \mapsto x_1(2x_2^2 - 1)$$

Notice that $D_K|_A(A) \subseteq A = R[Y_0 - x_0c,\ Y_1 - x_1c,\ Y_2 - x_2c]$ *and therefore* $D := D_K|_A$ *is an* R*-LND of* A*. Let* $\widetilde{D}$ *be the trivial extension of* D *on* $A[c] = B = R[Y_0, Y_1, Y_2]$*, i.e.,* $\widetilde{D}|_A = D$ *and* $\widetilde{D}(c) = 0$*. Using the equations above we have*

$$\widetilde{D}(Y_0) = -x_1^2 - x_2^2 = D(\widetilde{Y_0}), \quad \widetilde{D}(Y_1) = x_0x_1 = D(\widetilde{Y_1}), \quad \widetilde{D}(Y_2) = x_0x_2 = D(\widetilde{Y_2})$$

It is easy to see that $\widetilde{D}$ *is irreducible, and therefore, so is* D*. Also,* $Ker(D) \otimes_R K = K[x_2\widetilde{Y_1} - x_1\widetilde{Y_2}] = K[x_2Y_1 - x_1Y_2]$*. Now, as* $x_2 = \dfrac{\partial(x_2\widetilde{Y_1} - x_1\widetilde{Y_2})}{\partial Y_1}$*,* $-x_1 = \dfrac{\partial(x_2\widetilde{Y_1} - x_1\widetilde{Y_2})}{\partial Y_2}$ *form a sequence in* $R[Y_0, Y_1, Y_2]$*, by Proposition 4.1.4 it follows that* $Ker(D) = R[x_2\widetilde{Y_1} - x_1\widetilde{Y_2}] = R^{[1]}$*.*

One can check that $\widetilde{D} = -\mathcal{JD}_{(Y_0,Y_1,Y_2)}(x_2\widetilde{Y_1} - x_1\widetilde{Y_2},\ c, -) = -\mathcal{JD}_{(Y_0,Y_1,Y_2)}(x_2Y_1 - x_1Y_2,\ c, -)$ *and* $grade(D(A)A) = 2$*.*

The next example is by Raynaud ([Ray68]).

Example 4.6.2. *Let* $R = \mathbb{R}[X, Y, Z, F, G, H]/(XF + YG + ZH - 1)$*,* $K := Qt(R)$ *and* $B = R[U, V, W] = R^{[3]}$*. Let* x, y, z, f, g *and* h *denote the images of indeterminates* X, Y, Z, F, G *and* H *respectively in* R*. Since the polynomial* $XF + YG + ZH - 1$ *has the Jacobian matrix* (F, G, H, X, Y, Z) $(\neq 0)$ *and hence using Jacobian criterion for regular rings we notice that* R *is a regular ring.*

Define an R*-LND* D_0 *on* B *by*

$$D_0(U) = f, \quad D_0(V) = g, \quad D_0(W) = h.$$

Let $c := xU + yV + zW \in B$*. Observe that* $D_0(c) = 1$*, i.e.,* c *is a slice of* D_0*. Let* $A := \ker(D_0) = R[U - fc, V - gc, W - hc] = B/(c)$*. By the Slice Theorem (Theorem 1.2.1) we have* $B = A[c]$*, and therefore by Lemma 1.3.1* A*, is a* 1*-stably trivial* $\mathbb{A}^2$*-fibration over* R*.*

Note that $B \otimes_R K = K[U, V, W] = K[S, T, c] = K^{[3]}$ *for some* $S, T \in B$*. We now calculate the elements* S *and* T*. Set* $\widetilde{U} = U - fc$*,* $\widetilde{V} = V - gc$*,* $\widetilde{W} = W - hc$*. Observe that* $D_0(gU - fV) = D_0(g\widetilde{U} - f\widetilde{V}) = gf - fg = 0$ *and* $D_0(hV - gW) = D_0(h\widetilde{V} - g\widetilde{W}) = hg - gh = 0$*. Now* $gU - fV$ *and* $hV - gW$ *are* K*-linear transformations of variables in* $B \otimes_R K$ *and hence are again variables in* $B \otimes_R K$*. See that*

$$B \otimes_R K = K^{[3]} = K[U, V, W] = K[gU - fV,\ hV - gW,\ c] = K[g\widetilde{U} - f\widetilde{V},\ h\widetilde{V} - g\widetilde{W},\ c].$$

Note that $g\widetilde{U} - f\widetilde{V}$, $h\widetilde{V} - g\widetilde{W} \in A$. *Using the relation* $\widetilde{U} = -(y\widetilde{V} + z\widetilde{W})/x$ *we eliminate* U *in* $B \otimes_R K$ *and* $A \otimes_R K$ *and get the following.*

$$B \otimes_R K = K\left[\frac{(xf+gy)}{x}\widetilde{V} + \frac{gz}{x}\widetilde{W},\ h\widetilde{V} - g\widetilde{W},\ c\right]$$

$$A \otimes_R K = K\left[\frac{(xf+gy)}{x}\widetilde{V} + \frac{gz}{x}\widetilde{W},\ h\widetilde{V} - g\widetilde{W}\right]$$

Set $S := h\widetilde{V} - g\widetilde{W} \in A$ *and* $T := \dfrac{(xf+gy)}{x}\widetilde{V} + \dfrac{gz}{x}\widetilde{W}$. *Note that*

$$\begin{aligned} xT &= (xf+yg)\widetilde{V} + zg\widetilde{W} \\ &= (xf+yg)V + zgW - gc(xf+yg+zh) \\ &= (xf+yg)V + zgW - gc \\ &= -x(gU - fV) \\ &= -x(g\widetilde{U} - f\widetilde{V}) \in A, \end{aligned}$$

i.e., $T = -(gU - fV) = -(g\widetilde{U} - f\widetilde{V}) \in A$. *As* A *is finitely generated,* $\exists\, r \in R$ *such that* $A[1/r] = R[1/r][S,T] = R[1/r]^{[2]}$. *We shall now find* r. *First, note that*

$$\begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} h & -g \\ \frac{xf+gy}{x} & \frac{gz}{x} \end{pmatrix} \begin{pmatrix} \widetilde{V} \\ \widetilde{W} \end{pmatrix}$$

The generators $\widetilde{U} = U - fc$, $\widetilde{V} = V - gc$ *and* $\widetilde{W} = W - hc$ *of* A *are calculated as follows:*

$$\begin{aligned} \widetilde{U} &= \frac{zf}{g}S + \frac{-gy-hz}{g}T \\ \widetilde{V} &= zS + xT \\ \widetilde{W} &= \frac{-xf-gy}{g}S + \frac{hx}{g}T \end{aligned}$$

Clearly, we can set $r := g \in R$. *Now, define a new LND* D_K *on* $A \otimes_R K$ *as follows*

$$D : K[S,T] \longrightarrow K[S,T] \quad by \quad S \mapsto 0,\ T \mapsto g$$

Notice that $Ker(D_K) = K[S] = K[h\widetilde{V} - g\widetilde{W}]$, $D_K|_A(A) \subseteq A = R[U - fc,\ V - gc,\ W - hc]$. *This shows that* $D := D_K|_A$ *is an* R*-LND of* A. *Using the above equations we see that*

$$D(U) = -gy - hz = D(\widetilde{U}), \quad D(V) = gx = D(\widetilde{V}), \quad D(W) = hx = D(\widetilde{W})$$

Let $\widetilde{D}$ *be the trivial extension of* D *on* $A[c] = B = R[U, V, W]$*, i.e.,* $\widetilde{D}|_A = D$ *and* $\widetilde{D}(c) = 0$*. Note that* $\widetilde{D}$ *is irreducible, and therefore, so is* D*. Also,* $Ker(D) \otimes_R K = K[h\widetilde{V} - g\widetilde{W}] = K[hV - gW]$*. Now, since* $h = \dfrac{\partial(h\widetilde{V} - g\widetilde{W})}{\partial V}$, $-g = \dfrac{\partial(h\widetilde{V} - g\widetilde{W})}{\partial W}$ *form a sequence in* $R[U, V, W]$*, by Proposition 4.1.4 it follows that* $Ker(D) = R[h\widetilde{V} - g\widetilde{W}]$*.*

One can easily check that $\widetilde{D} = -\mathcal{JD}_{(U,V,W)}(h\widetilde{V} - g\widetilde{W},\, c, -) = -\mathcal{JD}_{(U,V,W)}(hV - gW,\, c, -)$ *and* $grade(D(A)A) = 2$*.*

Part III

Rank and rigidity

Chapter 5

Rank and rigidity of LNDs of affine fibrations

The main aim of this chapter: To define a notion of rank and rigidity of LNDs of affine fibrations which is analogous to the perception of rank and rigidity of LNDs of polynomial algebras.

Before we go to the definition of rank and rigidity of LNDs of affine fibrations we note some important results related to rank and rigidity of LNDs of polynomial algebras.

The following two results show that when the rank of an LND of a polynomial algebra is at most two, then the LND satisfies some nice properties.

Theorem 5.0.1. *Let R be a domain containing $\mathbb{Q}$, $A = R^{[n]}$ and $D : A \longrightarrow A$ an R-LND. Then, the following hold.*

(I) *Suppose that the rank of D is one. Then, $Ker(D) = R^{[n-1]}$ and $A = Ker(D)^{[1]}$.*

(II) *Suppose that the rank of D is two.*

  (a) *If R is an HCF domain or a UFD, then $Ker(D) = R^{[n-1]}$.*

  (b) *If D is fixed point free, then $Ker(D) = R^{[n-1]}$ and $A = Ker(D)^{[1]}$.*

Theorem 5.0.1(I) follows from the property that kernel of an LND of a domain B is an inert subring of B and the trancendence degree of B over the kernel is one. Theorem 5.0.1(II)(a) holds due to Theorem 2.3.5 ([AEH72]); and Theorem 5.0.1(II)(b) appears as a corollary of [Ess07, Remark 3.2].

As a consequence of Theorem 5.0.1 we have the following characterization of the R-LNDs of $R^{[3]}$ having a slice when R is a PID.

Corollary 5.0.2. *Let R be a PID containing $\mathbb{Q}$, $A = R^{[3]}$ and D a fixed point free R-LND of A. Then, the following are equivalent.*

(I) *The rank of D is at most two.*

(II) *$Ker(D) = R^{[2]}$ and $A = Ker(D)^{[1]}$.*

(III) *D has a slice.*

The equivalence of (I) and (II) of Corollary 5.0.2 follows from Theorem 5.0.1. (II) implies (III) follows from the converse of the Slice Theorem (Theorem 1.2.1); and (III) implies (II) follows from Lemma 1.3.1, Theorem 1.2.5 ([Sat83]) and Theorem 2.3.7 ([BCW76]).

Remark 5.0.3. (I) It is to be noted that in Theorem 5.0.1, if rank of D is three, then $\text{Ker}(D)$ need not be a polynomial ring even when R is a PID and D is fixed point free (see [Win90] or [Fre17, pp.104 – 105]).

(II) In Corollary 5.0.2, if the rank of D is three, then D can not have a slice. See [Win90] or [Fre17, pp.104 – 105] for example of such LNDs. Note that Corollary 5.0.2 holds even over one-dimensional Noetherian domains containing $\mathbb{Q}$; see Lemma 2.3.23 for details.

In section 5.1, we define residual rank of LNDs of affine fibrations. We observe that if an affine fibration is a polynomial algebra, then the rank of an LND equals to its residual rank under certain conditions (see Remark 5.1.4(3)); otherwise, in general, residual rank is dominated by rank. Further, we get results analogous to the existing results on the rank of LNDs of polynomial rings, specifically (see Corollary 5.1.7 and Corollary 5.1.9).

Theorem 5.A: Let A be an $\mathbb{A}^n$-fibration over a Noetherian domain R containing $\mathbb{Q}$ and $D : A \longrightarrow A$ an R-LND. Then, the following hold.

(I) If the residual rank of D is one, then $\text{Ker}(D)$ is an $\mathbb{A}^{n-1}$-fibration over R and A is an $\mathbb{A}^1$-fibration over $\text{Ker}(D)$. Further, if R is a UFD, then $A = \text{Ker}(D)^{[1]}$.

(II) If the residual rank of D is two and R is a UFD, then $\text{Ker}(D) = B^{[1]}$ for some $\mathbb{A}^{n-2}$-fibration B over R.

(III) Suppose that D is fixed point free and the residual rank of D is at most two, then D has a slice.

Clearly, the above result is an analogue to Theorem 5.0.1. As an immediate application of Theorem 5.A, we get a characterization of the LNDs of $\mathbb{A}^3$-fibrations with slice as follows (see Corollary 5.1.11). The result is analogous to Corollary 5.0.2.

Corollary 5.B: Let R be a Noetherian domain containing $\mathbb{Q}$, A an $\mathbb{A}^3$-fibration over R and $D : A \longrightarrow A$ a fixed point free R-LND. Then, the following are equivalent.

(I) The residual rank of D is at most two.

(II) $\text{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R and A is an $\mathbb{A}^1$-fibration over $\text{Ker}(D)$.

(III) D has a slice.

Further, if the residual rank of D is three, then $\text{Ker}(D)$ need not be an $\mathbb{A}^2$-fibration over R (see Example 5.3.3).

Notation: Let R be a domain with quotient field K, A an R-algebra and $D : A \longrightarrow A$ an R-LND. The natural extension $D \otimes_R K : A \otimes_R K \longrightarrow A \otimes_R K$ of D on $A \otimes_R K$ will be denoted by D_K.

Keshari and Lokhande proved the following result on rigidity of LNDs ([KL14, Theorem 3.1 & Corollary 3.2]) as an extension of a result by Daigle (Theorem 2.3.15).

Theorem 5.0.4. *Let R be a domain containing $\mathbb{Q}$ with $Qt(R) = K$ and D an R-LND of $R^{[n]}$ such that the rank of D equals to the rank of D_K. If D_K is rigid, then D is also rigid. Consequently, an R-LND D of $R^{[3]}$ is rigid if the rank of D equals to the rank of D_K.*

In section 5.2, we define residual-rigidity using residual rank and show that our notion of rigidity of LNDs of affine fibrations also enjoys similar property as in Theorem 5.0.4, specifically (see Theorem 5.2.4 and Corollary 5.2.5).

Theorem 5.C: Let A be an $\mathbb{A}^n$-fibration over a Noetherian domain R containing $\mathbb{Q}$ with $\mathrm{Qt}(R) = K$ and $D : A \longrightarrow A$ an R-LND. If the residual rank of D equals to the rank of D_K and D_K is rigid, then D is residually rigid. Consequently, if $n = 3$ and the residual rank of D equals to the rank of D_K, then D is residually rigid.

In section 5.3, we discuss a few examples of LNDs of affine fibrations and calculate their residual ranks.

5.1 Rank of LNDs of affine fibrations

We first define rank of an LND of an affine fibration.

Definition 5.1.1. *Let R be a ring and A an $\mathbb{A}^n$-fibration over R.*

1. *For an R-subalgebra B of A, the sequence (R, B, A) is called an (n,r)-residual system if B is an $\mathbb{A}^{n-r}$-fibration over R and $A \otimes_R k(P) = (B \otimes_R k(P))^{[r]}$ for all prime ideals P of R.*
2. *Let $D : A \longrightarrow A$ be an R-LND; D is said to have residual rank r if r is the least non-negative integer for which there exists an (n,r)-residual system (R, B, A) such that $B \subseteq \mathrm{Ker}(D)$.*

The residual rank of D is denoted by Res-Rk(D). Note that Res-Rk(D) belongs to $\{0, 1, \cdots, n\}$.

Remark 5.1.2. Given a non-trivial $\mathbb{A}^n$-fibration A over a ring R, there may not exist an (n,r)-residual system (R, B, A) where $1 \leq r < n$ even for the case $n = 2$ (see Example 5.3.2). However, Asanuma and Bhatwadekar proved that (see Theorem 1.2.8 or [AB97, Theorem 3.8]) when R is a one-dimensional Noetherian domain containing $\mathbb{Q}$ and A is an $\mathbb{A}^2$-fibration over R, then there exists $W \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[W]$, and therefore, by Lemma 5.1.5 it follows that $(R, R[W], A)$ is a $(2,1)$-residual system. From their result it also follows that A has an R-LND D such that $\mathrm{Ker}(D) = R[W]$, and therefore, the residual rank of D is one.

In view of Definition 5.1.1 a result on residual variables by Das and Dutta ([DD14, Corollary 3.6, Theorem 3.13, Corollary 3.19 & Appendix A]) can be stated as follows.

Remark 5.1.3. *Let R be a Noetherian ring, A an $\mathbb{A}^n$-fibration over R and (R, B, A) an $(n, n-r)$-residual system. Then, A is an $\mathbb{A}^{n-r}$-fibration over B and $\Omega_R(A) = \Omega_B(A) \oplus (\Omega_R(B) \otimes_B A)$. Further, suppose $B = R[\underline{W}] = R^{[r]}$, i.e., $\underline{W}$ is an r-tuple residual variable of A, and $\Omega_R(A)$ is a stably free A-module. Then,*

(I) *$A^{[\ell]} = B^{[n-r+\ell]}$ for some $\ell \in \mathbb{N}$.*

(II) *$A = B^{[1]}$, provided $n - r = 1$ and $\mathbb{Q} \hookrightarrow R$.*

It is to be noted that though Das and Dutta, in [DD14], proved Remark 5.1.3(II) (see [DD14, Corollary 3.19]) over Noetherian domains containing $\mathbb{Q}$, from their proof it follows that Remark 5.1.3(II) holds over Noetherian rings (not necessarily domains) containing $\mathbb{Q}$.

Remark 5.1.4. Let R be a ring, A an $\mathbb{A}^n$-fibration over R and $D : A \longrightarrow A$ an R-LND. Then, the following can be observed from Definition 5.1.1.

1. **Residual system implies tower of affine fibrations:** If R is Noetherian and (R, B, A) an (n, r)-residual system, then by Remark 5.1.3 we see that A is an $\mathbb{A}^r$-fibration over B. If we further assume that R is a domain, then by Lemma 2.3.12 it follows that R is inert in both B and A, and B is inert in A.

2. **Condition for residual rank of an LND to be zero:** If R is a domain, then it is easy to see from Lemma 2.3.12 and Lemma 2.3.1 that $\text{Res-Rk}(D) = 0$ if and only if $D = 0$.

3. **Relation between residual rank and rank:** If R is a domain with $K = \text{Qt}(R)$ and D is non-trivial, then the following hold.

   (a) Since $A \otimes_R K = K^{[n]}$, it directly follows from the definition that $\text{Rk}(D_K) = \text{Res-Rk}(D_K) \leq \text{Res-Rk}(D)$.

   (b) Suppose $A = R^{[n]}$. Clearly, $\text{Rk}(D_K) \leq \text{Res-Rk}(D) \leq \text{Rk}(D)$. Hence, if we suppose that $\text{Rk}(D) = \text{Rk}(D_K)$, then we have $\text{Res-Rk}(D) = \text{Rk}(D)$.

Now, we discuss the case residual rank is at most two. Collectively, the discussion proves Theorem 5.A. At first we observe a few results on residual systems.

Lemma 5.1.5. *Let R be a domain, A an R-algebra and B an R-subalgebra of A such that B is an $\mathbb{A}^{n-r}$-fibration over R and A is an $\mathbb{A}^r$-fibration over B. Then, the following hold.*

(I) *If $r = 1$, then (R, B, A) is an $(n, 1)$-residual system.*

(II) *If R contains $\mathbb{Q}$, $n = 3$ and $r = 2$, then (R, B, A) is a $(3, 2)$-residual system.*

Proof. Note that A is finitely generated and flat over R. Since A is an $\mathbb{A}^r$-fibration over B, we have $A \otimes_R k(P)$ is an $\mathbb{A}^r$-fibration over $B \otimes_R k(P) = k(P)^{[n-r]}$ for all $P \in \operatorname{Spec}(R)$. This implies that $\operatorname{tr.deg}_{B\otimes_R k(P)}(A \otimes_R k(P)) = r$ for all $P \in \operatorname{Spec}(R)$.

(I): Let $r = 1$. Fix $P \in \operatorname{Spec}(R)$. By Lemma 2.3.13 and Lemma 2.3.1 we see that $B \otimes_R k(P) \subseteq A \otimes_R k(P) \subseteq B \otimes_R k(P)^{[t]}$ is a chain of UFDs for some $t \in \mathbb{N}$. Since $\operatorname{tr.deg}_{B\otimes_R k(P)}(A \otimes_R k(P)) = 1$, Theorem 2.3.5 ([AEH72]) implies $A \otimes_R k(P) = (B \otimes_R k(P))^{[1]} = k(P)^{[n]}$. This shows that A is an $\mathbb{A}^n$-fibration over R, and hence, (R, B, A) is an $(n, 1)$-residual system.

(II): Assume that $R \hookleftarrow \mathbb{Q}$, $n = 3$ and $r = 2$. Fix $P \in \operatorname{Spec}(R)$. Since $B \otimes_R k(P) = k(P)^{[1]}$ is a PID, by Theorem 1.2.5 ([Sat83]) and Theorem 2.3.7 ([BCW76]) it follows that $A \otimes_R k(P) = (B \otimes_R k(P))^{[2]} = k(P)^{[3]}$. This shows that A is an $\mathbb{A}^3$-fibration over R, and hence, (R, B, A) is a $(3, 2)$-residual system. □

Theorem 5.1.6. *Let A be an $\mathbb{A}^n$-fibration over a domain R containing $\mathbb{Q}$, $D : A \longrightarrow A$ a non-trivial R-LND and (R, B, A) be an (n, r)-residual system such that $B \subseteq \operatorname{Ker}(D)$. Then, the following hold.*

(I) *If $r = 1$ and R is Noetherian, then A is an $\mathbb{A}^1$-fibration over B and $\operatorname{Ker}(D) = B$, i.e., A is an $\mathbb{A}^1$-fibration over $\operatorname{Ker}(D)$, and $\operatorname{Ker}(D)$ is an $\mathbb{A}^{n-1}$-fibration over R. If we further assume that either A is stably polynomial over R and $B = R^{[n-1]}$; or D is fixed point free, then $A = \operatorname{Ker}(D)^{[1]}$.*

(II) *If $r = 2$, R is Noetherian and D is fixed point free, then $A = \operatorname{Ker}(D)^{[1]}$ and $\operatorname{Ker}(D)$ is an $\mathbb{A}^1$-fibration over B as well as an $\mathbb{A}^{n-1}$-fibration over R. If we further assume that A is stably polynomial over B, then $\operatorname{Ker}(D) = B^{[1]}$.*

(III) *Suppose that R is a Noetherian UFD.*

  (a) If $r = 1$, then $A = \operatorname{Ker}(D)^{[1]}$ and $\operatorname{Ker}(D) = B$.

  (b) If $r = 2$, then A is an $\mathbb{A}^2$-fibration over B and $\operatorname{Ker}(D) = B^{[1]}$.

(IV) *Suppose that R is an HCF domain and A is stably polynomial over R as well as over B.*

  (a) If $r = 1$, then $A = \operatorname{Ker}(D)^{[1]}$ and $\operatorname{Ker}(D) = B$.

  (b) If $r = 2$, then $\operatorname{Ker}(D) = B^{[1]}$.

Proof. Since $D \neq 0$ and A is a domain, by ([Fre17, Principle 1 & Principle 11]), $\operatorname{tr.deg}_{\operatorname{Ker}(D)}(A) = 1$ and $\operatorname{Ker}(D)$ is inert in A.

(I): Let R be Noetherian and $r = 1$. By Remark 5.1.4(1) A is an $\mathbb{A}^1$-fibration over B and B is inert in A. Since $B \subseteq \operatorname{Ker}(D) \subseteq A$ and $\operatorname{tr.deg}_B(A) = \operatorname{tr.deg}_{\operatorname{Ker}(D)}(A) = 1$, using Lemma 2.3.1 we get $\operatorname{Ker}(D) = B$.

Now we further assume that A is stably polynomial over R. By Lemma 2.3.19 ([DD14]) $\Omega_R(A)$ is stably free over A. Since (R, B, A) is a residual system and $\text{Ker}(D) = B = R^{[n-1]}$, applying Remark 5.1.3 we get $A = \text{Ker}(D)^{[1]}$.

Again, along with the hypotheses $r = 1$ and R is Noetherian, if we further suppose that D is fixed point free, then by Proposition 2.3.22 ([EKO14]) we get that $A = \text{Ker}(D)^{[1]}$.

(II): Let R be Noetherian, $r = 2$, and D a fixed point free R-LND. Since (R, B, A) is an $(n, 2)$-residual system, from Remark 5.1.4(1) we have A is an $\mathbb{A}^2$-fibration over B. Now, since $B \subseteq \text{Ker}(D)$, we see that D is a B-LND of A, and hence by Remark 3.3.7 ([BD21]) it follows that $A = \text{Ker}(D)^{[1]}$ and $\text{Ker}(D)$ is an $\mathbb{A}^1$-fibration over B. Since B is an $\mathbb{A}^{n-2}$-fibration over R, by Lemma 5.1.5(I) we get $(R, B, \text{Ker}(D))$ is an $(n-1, 1)$-residual system and which implies that $\text{Ker}(D)$ is an $\mathbb{A}^{n-1}$-fibration over R. If we further assume that A is stably polynomial over B, then by Remark 3.3.7 ([BD21]) $\text{Ker}(D) = B^{[1]}$.

(III): Let us assume that the hypothesis holds. Since both A and B are affine fibrations over R, by Lemma 2.3.13 and Lemma 2.3.1 we see that both A and B are UFDs. Again, since $\text{Ker}(D)$ is inert in A, by Lemma 2.3.1 it follows that $\text{Ker}(D)$ is also a UFD.

(a) Let $r = 1$. By (I) we have $\text{Ker}(D) = B$. Since A is an $\mathbb{A}^1$-fibration over $\text{Ker}(D)$, by Theorem 1.2.7 ([Asa87]) we find a $t \in \mathbb{N}$ such that A is a $\text{Ker}(D)$-subalgebra of $\text{Ker}(D)^{[t]}$. This shows that $\text{Ker}(D) \subseteq A \subseteq \text{Ker}(D)^{[t]}$ is a chain of UFDs where $\text{tr.deg}_{\text{Ker}(D)}(A) = 1$. Therefore, by Theorem 2.3.5 ([AEH72]) we conclude that $A = B^{[1]}$.

(b) Let $r = 2$. By Remark 5.1.4(1), A is an $\mathbb{A}^2$-fibration over B and therefore, using Theorem 1.2.7 ([Asa87]) we get an $\ell \in \mathbb{N}$ such that A is an B-subalgebra of $B^{[\ell]}$. Notice that $B \subseteq \text{Ker}(D) \subseteq A \subseteq B^{[\ell]}$ is a chain of UFDs where $\text{tr.deg}_B(\text{Ker}(D)) = 1$. Therefore, by Theorem 2.3.5 ([AEH72]), we get $\text{Ker}(D) = B^{[1]}$.

(IV): We assume the hypothesis. Since A is stably polynomial over both R and B, there exist $s, t \in \mathbb{N}$ such that $A^{[s]} = R^{[n+s]}$ and $A^{[t]} = B^{[r+t]}$. Since $R \subseteq A \subseteq R^{[n+s]}$ and $B \subseteq A \subseteq B^{[r+t]}$, by Lemma 2.3.1 both R and B are inert in A, and therefore, repeated application of Lemma 2.3.1 implies that A, B and $B^{[m]}$ are HCF domains for all $m \in \mathbb{N}$.

(a) Let $r = 1$. By (I) we have $\text{Ker}(D) = B$, and hence $A^{[t]} = B^{[t+1]} = \text{Ker}(D)^{[t+1]}$. Clearly, A is inert in $\text{Ker}(D)^{[t+1]}$ and $\text{Ker}(D) \subseteq A \subseteq \text{Ker}(D)^{[t+1]}$ is a chain of HCF domains. By Theorem 2.3.5 ([AEH72]) we get $A = \text{Ker}(D)^{[1]}$.

(b) Let $r = 2$. Since A is stably polynomial over B, we see that $B \subseteq \text{Ker}(D) \subseteq A^{[t]} = B^{[t+2]}$ is a chain of HCF domains for some there $t \in \mathbb{N}$. Note that since $\text{Ker}(D)$ is inert in A, it is also inert in $A^{[t]}$. Now, by Theorem 2.3.5 ([AEH72]) we conclude that $\text{Ker}(D) = B^{[1]}$.

□

As a consequence of Theorem 5.1.6, we get the following analogue of Theorem 5.0.1(I) and (II)(a).

Corollary 5.1.7. *Let A be an $\mathbb{A}^n$-fibration over a Noetherian domain R containing $\mathbb{Q}$ and $D : A \longrightarrow A$ a non-trivial R-LND. Then, the following hold.*

(I) *Suppose that Res-Rk$(D) = 1$. Then, Ker(D) is an $\mathbb{A}^{n-1}$-fibration over R and A is an $\mathbb{A}^1$-fibration over Ker(D). Furthermore, if either A is assumed to be stably polynomial over R or R is assumed to be a UFD, then $A =$ Ker$(D)^{[1]}$.*

(II) *Suppose that Res-Rk$(D) = 2$ and R is a UFD. Then, Ker$(D) = B^{[1]}$ for some $\mathbb{A}^{n-2}$-fibration B over R.*

Proof. (I) : Since Res-Rk$(D) = 1$, there exists an $(n, 1)$-residual system (R, B, A) such that $B \subseteq$ Ker(D), and therefore, the result follows due to Theorem 5.1.6[(I) & (III)].

(II) : Since Res-Rk$(D) = 2$, there exists an $(n, 2)$-residual system (R, B, A) such that $B \subseteq$ Ker(D), and therefore, by Theorem 5.1.6(III) we get the result.

□

As an immediate corollary of Theorem 5.1.6(III) we observe the following result.

Corollary 5.1.8. *Let R be a Noetherian UFD containing $\mathbb{Q}$, A an $\mathbb{A}^2$-fibration over R and $D : A \longrightarrow A$ a non-trivial R-LND. Then, Ker$(D) = R^{[1]}$.*

Proof. Clearly, (R, R, A) is a $(2, 2)$-residual system such that $R \subseteq$ Ker(D), and therefore, by Theorem 5.1.6(III) we directly get Ker$(D) = R^{[1]}$. □

Next, we get an analogue of Theorem 5.0.1(II)(b).

Corollary 5.1.9. *Let A be an $\mathbb{A}^n$-fibration over a Noetherian domain R containing $\mathbb{Q}$ and $D : A \longrightarrow A$ a fixed point free R-LND. Suppose Res-Rk$(D) \leq 2$, then $A =$ Ker$(D)^{[1]}$ and Ker(D) is an $\mathbb{A}^{n-1}$-fibration over R, i.e., Res-Rk$(D) = 1$.*

Proof. Let Res-Rk$(D) \leq 2$. Then, there exists an (n, r)-residual system (R, B, A) such that $B \subseteq$ Ker(D) where either $r = 1$ or $r = 2$ (note that $r \neq 0$ because D is fixed point free). Now, from Theorem 5.1.6[(I) & (II)] it follows that $A =$ Ker$(D)^{[1]}$ and Ker(D) is an $\mathbb{A}^{n-1}$-fibration over R, which, by Lemma 5.1.5, is equivalent to say that Res-Rk$(D) = 1$.

□

Remark 5.1.10. The phenomenon in Corollary 5.1.9(I) is very specific for fixed point free LNDs, i.e., if the LND D is not fixed point free then the condition Res-Rk$(D) \leq 2$ need not imply Res-Rk$(D) = 1$. One may look at Example 5.3.4 for details.

We now prove Corollary 5.B.

Corollary 5.1.11. *Let R be a Noetherian domain containing $\mathbb{Q}$, A an $\mathbb{A}^3$-fibration over R and $D : A \longrightarrow A$ a fixed point free R-LND. Then, the following are equivalent.*

(I) *D has a slice.*

(II) *$\text{Res-Rk}(D) = 1$.*

(III) *$\text{Res-Rk}(D) \leq 2$.*

(IV) *$\text{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R and $A = \text{Ker}(D)^{[1]}$.*

(V) *$\text{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R and A an $\mathbb{A}^1$-fibration over $\text{Ker}(D)$.*

(VI) *$\text{Ker}(D)$ is Noetherian and A is an $\mathbb{A}^1$-fibration over $\text{Ker}(D)$.*

Further, if $\text{Res-Rk}(D) = 3$, then $\text{Ker}(D)$ need not be an $\mathbb{A}^2$-fibration over R.

Proof. (I) $\implies$ (II): Since D has a slice, $A = \text{Ker}(D)^{[1]}$, and therefore, finite generation and flatness of A over R will imply the finite generation and flatness of $\text{Ker}(D)$ over R. Further, we see that $k(P)^{[3]} = A \otimes_R k(P) = (\text{Ker}(D) \otimes_R k(P))^{[1]}$ for all $P \in \text{Spec}(R)$. Now, since the Zariski cancellation problem has affirmative answer in dimension two over fields containing $\mathbb{Q}$ (follows from [MS80], [Fuj79], and [Kam75]), we conclude that $\text{Ker}(D) \otimes_R k(P) = k(P)^{[2]}$ for all $P \in \text{Spec}(R)$ and therefore, it follows that $\text{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R. Since $A = \text{Ker}(D)^{[1]}$, we see that $\text{Res-Rk}(D) \leq 1$. Since D is non-trivial, we have $\text{Res-Rk}(D) = 1$.

(II) $\implies$ (III), (IV) $\implies$ (V) and (V) $\implies$ (VI): Obvious.

(III) $\implies$ (IV): Directly follows from Corollary 5.1.9(I).

(VI) $\implies$ (I): Directly follows from Proposition 2.3.22 ([EKO14]) and the converse of the Slice Theorem (Theorem 1.2.1).

Example 5.3.3 exhibits an R-LND D such that $\text{Res-Rk}(D) = 3$, but $\text{Ker}(D)$ is not an $\mathbb{A}^2$-fibration over R. This shows that if $\text{Res-Rk}(D) = 3$, then $\text{Ker}(D)$ need not be an $\mathbb{A}^2$-fibration over R. □

5.2 Rigidity of LNDs of affine fibrations

First, we define rigidity of LNDs of affine fibrations.

Definition 5.2.1. *Let A be an $\mathbb{A}^n$-fibration over a ring R and $D : A \longrightarrow A$ an R-LND with residual rank r. We define D to be residually rigid if, for any two (n, r)-residual systems (R, B_1, A) and (R, B_2, A) with $B_1, B_2 \subseteq \text{Ker}(D)$ we have $B_1 = B_2$.*

Remark 5.2.2. Let R be a ring, A an R-algebra and $D : A \longrightarrow A$ an R-LND.

1. If $A = R^{[n]}$ and $\text{Rk}(D) = \text{Res-Rk}(D)$, then one can see that the residual-rigidity of D implies rigidity of D.
2. If R is a domain, $A = R^{[n]}$ and $\text{Rk}(D) = 1$, then it can be seen, due to inertness of $\text{Ker}(D)$, that D is rigid. In the context of affine fibrations one may observe a similar phenomenon, also caused by the inertness of the kernel of the LNDs: If R is a Noetherian domain, A is an $\mathbb{A}^n$-fibration over R and $\text{Res-Rk}(D) = 1$, then D is residually rigid.

Before proving Theorem 5.C, we note the following lemma which can be seen as an extension of an observation by Abhyankar-Eakin-Heinzer ([AEH72, p. 1.7]).

Lemma 5.2.3. *Let A be a domain and B_1, B_2 subdomains of A. Suppose B_2 is inert in A. If $b \in B_1$ is such that $bA \cap B_2 \neq \{0\}$, then $b \in B_2$.*

Proof. Let $d \in bA \cap B_2$. Then, $d = bc \in B_2$ for some $c \in A$. Since $b, c \in A$ and B_2 is inert in A, we have $b, c \in B_2$. □

We now present Theorem 5.C.

Theorem 5.2.4. *Let A be an $\mathbb{A}^n$-fibration over a Noetherian domain R and $D : A \longrightarrow A$ an R-LND. Suppose that $\text{Res-Rk}(D) = \text{Rk}(D_K)$. If D_K is rigid, then D is residually rigid.*

Proof. Let $\text{Res-Rk}(D) = \text{Rk}(D_K) = r$. Let us assume that D_K is rigid. Since K is a field, we have $\text{Res-Rk}(D_K) = \text{Rk}(D_K) = r$. Let (R, B_1, A) and (R, B_2, A) be two (n, r)-residual systems such that $B_1, B_2 \subseteq \text{Ker}(D)$. By Remark 5.1.4(1), we get A is an $\mathbb{A}^r$-fibration over both B_1 and B_2; and both B_1 and B_2 are inert in A. We shall show that $B_1 = B_2$.

Let $\underline{U} \in {B_1}^{n-r}$ and $\underline{V} \in {B_2}^{n-r}$ be such that $B_1 \otimes_R K = K[\underline{U}]$ and $B_2 \otimes_R K = K[\underline{V}]$. Since A is an $\mathbb{A}^r$-fibration over both B_1 and B_2, we have $A \otimes_R K = K[\underline{U}]^{[r]} = K[\underline{V}]^{[r]} = K^{[n]}$, and therefore, since D_K is rigid and $\text{Rk}(D_K) = r$, we have $K[\underline{U}] = K[\underline{V}]$, i.e., $B_1 \otimes_R K = K[\underline{V}]$. Suppose that $x \in B_1$. Since $B_1 \otimes_R K = K[\underline{U}] = K[\underline{V}] = B_2 \otimes_R K$, there exists $r \in R$ such that $rx \in B_2$. This shows that $rx \in xA \cap B_2$, and therefore, by Lemma 5.2.3 we have $x \in B_2$. So, we get $B_1 \subseteq B_2$. Now, interchanging the roles of B_1 and B_2 we get $B_2 \subseteq B_1$. Hence, $B_1 = B_2$. This completes the proof. □

As a direct consequence of Theorem 5.2.4 and Theorem 2.3.15 ([Dai96]) we get the following.

Corollary 5.2.5. *Let A be an $\mathbb{A}^3$-fibration over a Noetherian domain R and $D : A \longrightarrow A$ an R-LND such that $\text{Res-Rk}(D) = \text{Rk}(D_K)$, then D is residually rigid.*

5.3 Examples

We now discuss a few examples. The first example involves a non-trivial $\mathbb{A}^2$-fibration along with a fixed point free LND.

Example 5.3.1. Let R be a Noetherian domain containing $\mathbb{Q}$, A a non-trivial $\mathbb{A}^2$-fibration over R and $D : A \longrightarrow A$ a fixed point free R-LND. We shall show Res-Rk$(D) = 1$.

By Remark 3.3.7 ([BD21]) we have $A = \mathrm{Ker}(D)^{[1]}$ and $\mathrm{Ker}(D)$ is an $\mathbb{A}^1$-fibration over R, and therefore, by Lemma 5.1.5 we see that $(R, \mathrm{Ker}(D), A)$ is a $(2,1)$-residual system. This shows that Res-Rk$(D) \leq 1$. Since D is non-trivial, Res-Rk$(D) \neq 0$, and hence Res-Rk$(D) = 1$.

The next example is by Hochster (see [Hoc72] or [Fre17, p. 10.1.5]).

Example 5.3.2. Let $R = \mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1) = \mathbb{R}[x,y,z]$ where x,y,z denote the images of X,Y,Z in R. Let $A = R[U,V,W]/(xU+yV+zW)$. One can see that R is a Noetherian UFD, A is a non-trivial $\mathbb{A}^2$-fibration over R and $A^{[1]} = R^{[3]}$. We claim that there does not exist $B \subseteq A$ such that (R,B,A) is a $(2,1)$-residual system. On the contrary, let (R,B,A) be a $(2,1)$-residual system. Since R is a Noetherian domain and B is an $\mathbb{A}^1$-fibration over R, by Lemma 2.3.13 and Lemma 2.3.1 we see that $R \subseteq B \subseteq R^{[m]}$ is a sequence of UFDs for some $m \in \mathbb{N}$ where $\mathrm{tr.deg}_R(B) = 1$, and therefore, by Proposition 2.3.5 ([AEH72]) we have $B = R^{[1]}$. Now, since A is stably polynomial over R, by Lemma 2.3.19 ([DD14]) and Remark 5.1.3 it follows that $A = R^{[2]}$ which is a contradiction to the fact that A is a non-trivial $\mathbb{A}^2$-fibration over R. This shows that for any non-trivial R-LND D of A, the residual rank of D is always two, and therefore, in view of Example 5.3.1, A does not have any fixed point free R-LND. In this context, one should note that by Corollary 5.1.8 we have $\mathrm{Ker}(D) = R^{[1]}$; however, A can not be an $\mathbb{A}^1$-fibration over $\mathrm{Ker}(D)$.

The following example is by Winkelmann ([Win90], also see [Fre17, pp.104 – 105]).

Example 5.3.3. Let $R = \mathbb{C}[X] = \mathbb{C}^{[1]}$, $A = R[U,V,W] = R^{[3]}$ and $D : A \longrightarrow A$ be an R-LND defined by $D(U) = X$, $D(V) = U$ and $D(W) = U^2 - 2XV - 1$. One can easily see that D is fixed point free. It is known that $\mathrm{Ker}(D) = R[f,g,h] \neq R^{[2]}$ where

$$\begin{aligned} &f = U^2 - 2XV, \\ &g = XW + (1-f)U, \\ &Xh = g^2 - f(1-f)^2, \ \textit{i.e.},\ h = XW^2 + 2(1-f)(UW + (1-f)V). \end{aligned}$$

By Theorem 5.0.1 it follows that Rk$(D) = 3$. We shall calculate Res-Rk(D). Note that $\mathrm{Ker}(D)$ is not an $\mathbb{A}^2$-fibration over R, otherwise by Theorem 1.2.5 ([Sat83]) and Theorem 2.3.7 ([BCW76]) we get $\mathrm{Ker}(D) = R^{[2]}$ which is a contradiction. Further, D has no slice; otherwise by Lemma 1.3.1 ([Fre09]) it would follow that $\mathrm{Ker}(D)$ is an $\mathbb{A}^2$-fibration over R, which is a contradiction. Thus, D is fixed point free without a slice, and therefore, by Corollary 5.1.9, we have $\text{Res-Rk}(D) = 3$. So, by Remark 5.1.4(3) we see that $3 = \text{Res-Rk}(D) \leq \text{Rk}(D) = 3$, i.e., $\text{Res-Rk}(D) = \text{Rk}(D) = 3$.

We now consider an example by Bhatwadekar and Dutta ([BD94, Example 4.13]) (also see [VÓ1]).

Example 5.3.4. Let $\mathbb{F}$ be a field containing $\mathbb{Q}$, $R = \mathbb{F}[\pi]_{(\pi)}$ and $A = R[X, Y, Z]$. Set $F := \pi^2 X + \pi Y(YZ + X + X^2) + Y$. One can check that $A \otimes_R k(P) = (R[F] \otimes_R k(P))^{[2]}$ for all $P \in \text{Spec}(R)$, i.e., F is a residual variable of A, and therefore, by Remark 5.1.3, A is an $\mathbb{A}^2$-fibration over $R[F]$, and hence, by Theorem 1.2.7 ([Asa87]), A is stably polynomial over $R[F]$. It is not known whether $A = R[F]^{[2]}$.

Define an R-LND D of A by $D(X) = Y^2$, $D(Y) = 0$ and $D(Z) = -(\pi + Y + 2XY)$. Then, $R[F] \subseteq \text{Ker}(D)$. We shall find $\text{Rk}(D)$ and $\text{Res-Rk}(D)$. Clearly, D is irreducible and triangular. By Corollary 5.1.8 we get $\text{Ker}(D) = R[F]^{[1]} = R^{[2]}$. We now show that D is not fixed point free. On the contrary, assume that D is fixed point free, and therefore, there exists $f_1, f_2, f_3 \in R[X, Y, Z]$ such that $D(X)f_1 + D(Y)f_2 + D(Z)f_3 = 1$. Since $D(Y) = 0$, we have $D(X)f_1 + D(Z)f_3 = 1$, i.e., $Y^2 f_1 - (\pi + Y + 2XY)f_3 = 1$. Hence, in $A/YA = R[X, Z]$ we get $-\pi f_3 = 1$, i.e., π is a unit in $R[X, Z]$, giving a contradiction to the fact that π is a prime in R.

As $A = R[X, Y, Z]$ and $Y \in \text{Ker}(D)$, we see that $\text{Rk}(D) \leq 2$. Since R is a UFD and D is a non-trivial irreducible R-LND without having a slice, by Corollary 5.1.7 we have $\text{Res-Rk}(D) = 2$; and also by Theorem 5.0.1 it follows that $\text{Rk}(D) = 2$. So, we have $\text{Res-Rk}(D) = \text{Rk}(D) = 2$. Note that since A is an $\mathbb{A}^2$-fibration over $R[F]$, by Lemma 5.1.5 we have $(R, R[F], A)$ is a $(3, 2)$-residual system such that $R[F] \subseteq \text{Ker}(D)$.

Part IV

Appendix

Appendix A

$\mathbb{G}_a$-actions, exponential maps and LNDs

Here we get into the details of the relationship between the $\mathbb{G}_a$-actions, exponential maps and LNDs. Throughout this section k will denote an algebraically closed field.

A.1 Relationship between the $\mathbb{G}_a$-actions and the exponential maps

Definition A.1.1. *Let $B = k^{[n]}$ and $X := Spec(B)$. Then the $\mathbb{G}_a$**-action on** X is a morphism of varieties $\theta : \mathbb{G}_a \times X \longrightarrow X$ satisfying*

(i) $\theta(0, x) = x \quad \forall\, x \in X$, *and*

(ii) $\theta(\alpha, \theta(\beta, x)) = \theta(\alpha + \beta, x) \quad \forall x \in X,\ \alpha, \beta \in \mathbb{G}_a$

In fact for the above definition, B need not necessarily be $k^{[n]}$ but it can also be any affine k-domain.

Definition A.1.2. *Let B be a ring, h an element of $B[X,Y]$ and $\mu_h : B[T] \longrightarrow B[X,Y]$ the B-algebra evaluation map $T \mapsto h$. Given a ring homomorphism $\phi : B \longrightarrow B[T]$, we define $\phi^h : B[X,Y] \longrightarrow B[X,Y]$ to be the unique lift of the composite ring homomorphism $\mu_h \circ \phi : B \longrightarrow B[X,Y]$ satisfying $\phi^h(X) = X$ and $\phi^h(Y) = Y$. One can see that the ring homomorphism ϕ^h makes the following diagram commutative*

$$\begin{array}{ccc} B[X,Y] & \xrightarrow{\phi^h} & B[X,Y] \\ \uparrow{\scriptstyle \nu} & & {\scriptstyle \mu_h}\uparrow \\ B & \xrightarrow{\phi} & B[T] \end{array}$$

where ν is the inclusion map.
For the field k we shall denote the additive group $(k, +)$ by $\mathbb{G}_a(k)$.

Proposition A.1.3. *Let B be an affine k-algebra and $\phi : B \longrightarrow B[T]$ be a k-homomorphism. Then the following are equivalent.*

1. *$Spec(\phi) : \mathbb{G}_a(k) \times Spec(B) \longrightarrow Spec(B)$ is a $\mathbb{G}_a$-action*

2. *$\phi^0 = id_{B[X,Y]}$ and $\phi^{X+Y} = \phi^Y \circ \phi^X$.*

Proof. The condition that $\phi^0 = \mathrm{id}_{B[X,Y]}$ is equivalent to say that the composition map $B \xrightarrow{\phi} B[T] \xrightarrow{\mu_0} B$ is the identity map on B (see Definition A.1.2). Now applying the Spec functor on the composite map $B \xrightarrow{\phi} B[T] \xrightarrow{\mu_0} B$ gives the composite map $\mathrm{Spec}(B) \xrightarrow{\mathrm{Spec}(\mu_0)} \mathbb{G}_a(k) \times \mathrm{Spec}(B) \xrightarrow{\mathrm{Spec}(\phi)} \mathrm{Spec}(B)$ to be the identity map on $\mathrm{Spec}(B)$; which is equivalent to say that $\mathrm{Spec}(\phi)(0, x) = x$ for all $x \in \mathrm{Spec}(B)$ (the first condition of the Definition A.1.1 of $\mathbb{G}_a$ action). Note that $\mathrm{Spec}(B[T]) = G_a(k) \times \mathrm{Spec}(B)$

Now one can see that the second condition of the G_a-action's Definition A.1.1 is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{G}_a(k) \times \mathbb{G}_a(k) \times \mathrm{Spec}(B) & \xrightarrow{1_{\mathbb{G}_a} \times \mathrm{Spec}(\phi)} & \mathbb{G}_a(k) \times \mathrm{Spec}(B) \\ \downarrow{\scriptstyle \psi \times 1_{\mathrm{Spec}(B)}} & & \downarrow{\scriptstyle \mathrm{Spec}(\phi)} \\ \mathbb{G}_a(k) \times \mathrm{Spec}(B) & \xrightarrow{\mathrm{Spec}(\phi)} & \mathrm{Spec}(B) \end{array} \tag{A.1}$$

Where $\psi : \mathbb{G}_a(k) \times \mathbb{G}_a(k) \longrightarrow \mathbb{G}_a(k)$ is the morphism corresponding to the k-homomorphism $k[T] \longrightarrow k[X, Y]$ which maps T to $X + Y$. Now the commutativity of diagram A.1 is equivalent to the commutativity of the below diagram (via contraviant equivalence of the Spec functor)

$$\begin{array}{ccc} B[X, Y] & \xleftarrow[\eta]{} & B[T] \\ \uparrow{\scriptstyle \mu_{X+Y}} & & \uparrow{\scriptstyle \phi} \\ B[T] & \xleftarrow[\phi]{} & B \end{array} \tag{A.2}$$

Where μ_{X+Y} is the B-homomorphism evaluation map $T \mapsto X + Y$ and η is defined as $\eta(T) = X$ along with $\eta(b) = \phi^Y(b)$.

Note that under the composite of maps $B[T] \xrightarrow{\mu_X} B[X, Y] \xrightarrow{\phi^Y} B[X, Y]$ one can see that $T \mapsto X$ and $b \mapsto \phi^Y(b)$ $\forall b \in B$ and therefore $\phi^Y \circ \mu_X = \eta$. Now from the Definition A.1.2 it follows that $\mu_X \circ \phi = \phi^X \circ \nu$ as well as $\mu_{X+Y} \circ \phi = \phi^{X+Y} \circ \nu$. Combining these relations along with the commutativity of the diagram A.2 it follows that $\phi^{X+Y} \circ \nu = \phi^Y \circ \phi^X \circ \nu$. Now since ν is injective as well as $\phi^h(X) = X$, $\phi^h(Y) = Y$ for all $h \in B[X, Y]$, it follows that $\phi^{X+Y} = \phi^Y \circ \phi^X$. This proves the equivalence of the two statements. □

Any k homomorphism ϕ satisfying the second property of the statement of Proposition A.1.3 is called as an exponential map. The proposition A.1.3 establishes bijection between the $\mathbb{G}_a$ -actions and the exponential maps.

A.2 Relationship between the exponential maps and the LNDs

We now discuss the bijection between the exponential maps and the locally nilpotent k-derivations of B. Throughout this section we assume that the field k has characteristic zero.

Set $\Gamma := \{\phi \in \mathrm{Hom}_k(B, B[T]) \mid \phi^0 = \mathrm{id}_{B[X,Y]} \text{ and } \phi^{X+Y} = \phi^Y \circ \phi^X\}$.

Given $\phi \in \Gamma$, define $D_\phi : B \longrightarrow B$ to be the composition of the maps

$$B \xrightarrow{\phi} B[T] \xrightarrow{d/dT} B[T] \xrightarrow{\mu_0} B,$$

where d/dT denotes the derivative with respect to T.

We claim that $D_\phi \in \mathrm{LND}_k(B)$.

One can check that $\mu_0 \circ \phi = \mathrm{id}_B$ and $\phi^0 = \mathrm{id}_{B[X,Y]}$ (see Definition A.1.2). D_ϕ being additive follows from the fact that all of μ_0, $\dfrac{d}{dT}$ and ϕ are additive. Now for the Leibniz rule,

$$\begin{aligned} D_\phi(xy) &= \mu_0\left(\frac{d}{dT}(\phi(x)\phi(y))\right) \\ &= \mu_0\left(\frac{d}{dT}(\phi(x)).\phi(y) + \phi(x).\frac{d}{dT}(\phi(y))\right) \\ &= \mu_0\left(\frac{d}{dT}(\phi(x))\right)\mu_0(\phi(y)) + \mu_0(\phi(x))\mu_0\left(\frac{d}{dT}(\phi(y))\right) \\ &= D_\phi(x)y + xD_\phi(y) \end{aligned}$$

Therefore, $D_\phi \in \mathrm{Der}_k(B)$. We now show that the following diagram commutes

$$\begin{array}{ccc} B[T] & \xrightarrow{d/dT} & B[T] \\ \uparrow{\scriptstyle \phi} & & {\scriptstyle \phi}\uparrow \\ B & \xrightarrow{D_\phi} & B \end{array} \tag{A.3}$$

Let $b \in B$ and write $\phi(b) = \sum_{i \in \mathbb{N}} b_i T^i \in B[T]$. Then,

$$
\begin{aligned}
\sum_{i\in\mathbb{N}} \phi^Y(b_i)X^i &= \phi^Y\left(\sum_{i\in\mathbb{N}} b_i X^i\right) \\
&= \phi^Y(\phi^X(b)) \\
&= \phi^{X+Y}(b) \\
&= \sum_{n\in\mathbb{N}} b_n (X+Y)^n \\
&= \sum_{n\in\mathbb{N}} b_n \left(\sum_{i+j=n} \binom{n}{i} X^i Y^j\right) \\
&= \sum_{i\in\mathbb{N}} \left(\sum_{j\in\mathbb{N}} b_{i+j}\binom{i+j}{i} Y^j\right) X^i
\end{aligned}
$$

Thus we get $\phi^Y(b_i) = \sum_{j\in\mathbb{N}} b_{i+j}\binom{i+j}{i} Y^j \quad \forall i \in \mathbb{N}$. Now since we have $\phi^Y \circ \nu(b_i) = \mu_Y \circ \phi(b_i)$ it follows that $\phi(b_i) = \sum_{j\in\mathbb{N}} b_{i+j}\binom{i+j}{i} T^j \quad \forall i \in \mathbb{N}$. But, since by definition $D_\phi(b) = b_1$, we have

$$\phi(D_\phi(b)) = \phi(b_1) = \sum_{i\in\mathbb{N}} b_{i+1}(i+1)T^{i+1} = \frac{d}{dT}\sum_{i\in\mathbb{N}} b_i T^i = \frac{d}{dT}(\phi(b)).$$

Thus it follows that the square A.3 commutes and thus by recursive usage of the commutative square A.3 it follows that the diagram

$$
\begin{array}{ccc}
B[T] & \xrightarrow{(d/dT)^n} & B[T] \\
\uparrow \phi & & \phi \uparrow \\
B & \xrightarrow{D_\phi^n} & B
\end{array}
\tag{A.4}
$$

also commutes. Now since d/dT is locally nilpotent it follows that D_ϕ is also locally nilpotent from the commutative square A.4. Hence $D_\phi \in \mathrm{LND}_k(B)$ and thus we have a well defined map from the set of exponential maps Γ to the set of all locally nilpotent k-derivations as $\phi \mapsto D_\phi$.

Now for the map in the converse direction let $D \in \mathrm{LND}_k(B)$ be given, consider the map $\phi : B \longrightarrow B[T]$ which is defined by $b \mapsto \sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!}T^n$. Since, D^n is additive $\forall n \in \mathbb{N}$ and respects multiplication by scalars as $k \subseteq \ker(D)$, it follows that the above maps are k-homomorphisms.

Claim : $\phi \in \Gamma$.

Define $\Delta : B[X,Y] \longrightarrow B[X,Y]$ by $\sum_{i,j}(b_{ij}X^iY^j) \mapsto \sum_{i,j}(D(b_{ij})X^iY^j$. It is easy to verify directly that $\Delta \in \mathrm{LND}_{k[X,Y]}(B[X,Y])$ by using $D \in \mathrm{LND}_k(B)$. Now for every $h \in k[X,Y]$ since $h \in \ker(\Delta)$, we have that $h\Delta \in \mathrm{LND}_{k[X,Y]}(B[X,Y])$.

Claim: $e^{(h\Delta)} \in \mathrm{Aut}_{k[X,Y]}(B[X,Y])$.
Clearly $e^{(h\Delta)}$ is a $k[X,Y]$-homomorphism by the direct application of Leibniz rule for $h\Delta$. Now since $h_1\Delta \circ h_2\Delta = h_2\Delta \circ h_1\Delta = (h_1+h_2)\Delta$ for all $h_1, h_2 \in k[X,Y]$ it follows that the map $e^{(h_1+h_2)\Delta} : B[X,Y] \longrightarrow B[X,Y]$ is well defined and for $g \in B[X,Y]$

$$\begin{aligned}
(e^{h_1\Delta} \circ e^{h_2\Delta})(g) &= e^{h_1\Delta}\left(\sum_{j\in\mathbb{N}} \frac{(h_2\Delta)^j(g)}{j!}\right) \\
&= \left(\sum_{j\in\mathbb{N}} \frac{e^{h_1\Delta}((h_2\Delta)^j(g))}{j!}\right) \\
&= \sum_{j\in\mathbb{N}} \frac{1}{j!}\left(\sum_{i\in\mathbb{N}} \frac{(h_1\Delta)^i((h_2\Delta)^j(g))}{i!}\right) \\
&= \left(\sum_{i,j\in\mathbb{N}} \frac{(h_1\Delta)^i \circ (h_2\Delta)^j(g)}{i!j!}\right) \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{i+j=n} \binom{n}{i} \left((h_1\Delta)^i \circ (h_2 j\Delta)^j\right)(g) \\
&= e^{(h_1+h_2)\Delta}(g)
\end{aligned}$$

From this it follows that $e^{h\Delta} \circ e^{-h\Delta} = e^0 = \mathrm{id}_{B[X,Y]}$ proving that $e^{h\Delta}$ is a $k[X,Y]$-automorphism.

We now prove that $e^{(h\Delta)} = \phi^h$, and for which it is enough to show the commutativity of diagram below

$$\begin{array}{ccc}
B[X,Y] & \xrightarrow{e^{h\Delta}} & B[X,Y] \\
\uparrow \nu & & \mu_h \uparrow \\
B & \xrightarrow{\phi} & B[T]
\end{array} \tag{A.5}$$

as it is trivial to observe that $e^{h\Delta}(X) = X$, $e^{h\Delta(Y)} = Y$. Note that for any $b \in B$ we have

$$(e^{h\Delta} \circ \nu)(b) = e^{h\Delta}(b) = \sum_{n\in\mathbb{N}} \frac{(h\Delta)^n(b)}{n!} = \sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!}h^n = \mu_h\left(\sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!}T^n\right) = \mu_h(\phi(b))$$

which establishes that the square A.5 commutes. Hence we have $\phi^{X+Y} = e^{X\Delta+Y\Delta} = e^{Y\Delta} \circ e^{X\Delta} = \phi^Y \circ \phi^X$ so that $\phi \in \Gamma$ as claimed.

Now we shall show that the above constructions establish two bijective maps between the set of exponential maps on B and the set of k-LNDs on B which are inverses to each other.

Let $D \in \mathrm{LND}_k(B)$. Define $\phi_D : B \longrightarrow B[T]$ as discussed above, i.e., $b \mapsto \sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!}T^n$. Now, if we construct back D_{ϕ_D}, then

$$D_{\phi_D}(b) = \mu_0(\frac{d}{dT}(\phi_D(b))) = \mu_0(\frac{d}{dT}(\sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!}T^n)) = \mu_0(\sum_{n=1}^{\infty} \frac{D^n(b)}{(n-1)!}T^{n-1})) = D(b)$$

and hence $D_{\phi_D} = D$.

On the other direction, let $\phi \in \Gamma$ and define D_ϕ as given in the construction above. Let $\psi : B \longrightarrow B[T]$ be the map defined by $b \mapsto \sum_{n\in\mathbb{N}} \frac{D_\phi^n(b)}{n!}T^n$. We need to show that $\phi = \psi$. Write $\phi(b) = \sum_{n\in\mathbb{N}} b_n T^n$. Since the diagram A.4 commutes, we have $\frac{d}{dT}^m(\phi(b)) = \phi(D_\phi^m(b))$ for all $m \in \mathbb{N}$. Letting $T \mapsto 0$, we have $m!b_m = \phi(D_\phi^m(b))|_{T=0} = \mu_0(\phi(D_\phi^m(b))) \in B$. Now, since $\mu_h \circ \phi(x) = \phi^h(x)$ for all $x \in B$ and by hypothesis $\phi^0 = id_{B[X,Y]}$, we have $\mu_0(\phi(D_\phi^m(b))) = \phi^0(D_\phi^m(b)) = D_\phi^m(b)$. Therefore, $b_m = \frac{D_\phi^m(b)}{m!}$ for all $m \in \mathbb{N}$, which proves that $\phi = \psi$.

Therefore, from the discussions in sections A.1 and A.2 it is clear that when k is a algebraically closed field of characteristic zero, there is a bijection between the $\mathbb{G}_a$-actions and the exponential maps as well as a bijection between the exponential maps and the locally nilpotent derivations.

Bibliography

[AB97] T. Asanuma and S. M. Bhatwadekar. "Structure of $\mathbf{A}^2$-fibrations over one-dimensional Noetherian domains". In: *J. Pure Appl. Algebra* 115.1 (1997), pp. 1–13. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(96)00005-9. URL: https://doi.org/10.1016/S0022-4049(96)00005-9.

[AEH72] Shreeram S. Abhyankar, Paul Eakin, and William Heinzer. "On the uniqueness of the coefficient ring in a polynomial ring". In: *J. Algebra* 23 (1972), pp. 310–342.

[AK70] Allen Altman and Steven Kleiman. *Introduction to Grothendieck duality theory*. Lecture Notes in Mathematics, Vol. 146. Berlin: Springer-Verlag, 1970, pp. ii+185.

[Asa87] Teruo Asanuma. "Polynomial fibre rings of algebras over Noetherian rings". In: *Invent. Math.* 87.1 (1987), pp. 101–127.

[BCW76] H. Bass, E. H. Connell, and D. L. Wright. "Locally polynomial algebras are symmetric algebras". In: *Invent. Math.* 38.3 (1976/77), pp. 279–299. ISSN: 0020-9910. DOI: 10.1007/BF01403135. URL: https://doi.org/10.1007/BF01403135.

[BD21] Janaki Raman Babu and Prosenjit Das. "Structure of $\mathbb{A}^2$-fibrations having fixed point free locally nilpotent derivations". In: *J. Pure Appl. Algebra* 225.12 (2021), Paper No. 106763, 12. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2021.106763. URL: https://doi.org/10.1016/j.jpaa.2021.106763.

[BD94] S. M. Bhatwadekar and Amartya Kumar Dutta. "On affine fibrations". In: *Commutative algebra (Trieste, 1992)*. World Sci. Publ., River Edge, NJ, 1994, pp. 1–17.

[BD97] S. M. Bhatwadekar and Amartya K. Dutta. "Kernel of locally nilpotent R-derivations of $R[X,Y]$". In: *Trans. Amer. Math. Soc.* 349.8 (1997), pp. 3303–3319. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-97-01946-6. URL: https://doi.org/10.1090/S0002-9947-97-01946-6.

[BDL21] Janaki Raman Babu, Prosenjit Das, and Swapnil A. Lokhande. "Rank and rigidity of locally nilpotent derivations of affine fibrations". In: *Comm. Algebra* 49.12 (2021), pp. 5214–5228. ISSN: 0092-7872. DOI: 10.1080/00927872.2021.1939360. URL: https://doi.org/10.1080/00927872.2021.1939360.

[BEM01] Joost Berson, Arno van den Essen, and Stefan Maubach. "Derivations having divergence zero on $R[X,Y]$". In: *Israel J. Math.* 124 (2001), pp. 115–124. ISSN: 0021-2172. DOI: 10.1007/BF02772610. URL: https://doi.org/10.1007/BF02772610.

[Bha88] S. M. Bhatwadekar. "Generalized epimorphism theorem". In: *Proc. Indian Acad. Sci. Math. Sci.* 98.2-3 (1988), pp. 109–116.

[CML08] Anthony J. Crachiola and Leonid G. Makar-Limanov. "An algebraic proof of a cancellation theorem for surfaces". In: *J. Algebra* 320.8 (2008), pp. 3113–3119. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2008.03.037. URL: https://doi.org/10.1016/j.jalgebra.2008.03.037.

[Dai10] Daniel Daigle. "Introduction to locally nilpotent derivations". In: http:// aix1.uottawa.ca/ ddaigle: Informal lecture notes, 2010.

[Dai96] Daniel Daigle. "A necessary and sufficient condition for triangulability of derivations of $k[X,Y,Z]$". In: *J. Pure Appl. Algebra* 113.3 (1996), pp. 297–305.

[Dai97] D. Daigle. "On some properties of locally nilpotent derivations". In: *J. Pure Appl. Algebra* 114.3 (1997), pp. 221–230. ISSN: 0022-4049. DOI: 10.1016/0022-4049(95)00173-5. URL: https://doi.org/10.1016/0022-4049(95)00173-5.

[Das15] Prosenjit Das. "On cancellation of variables of the form $bT^n - a$ over affine normal domains". In: *J. Pure Appl. Algebra* 219.12 (2015), pp. 5280–5288. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2015.05.011. URL: https://doi.org/10.1016/j.jpaa.2015.05.011.

[DD14] Prosenjit Das and Amartya K. Dutta. "A note on residual variables of an affine fibration". In: *J. Pure Appl. Algebra* 218.10 (2014), pp. 1792–1799. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2014.02.005. URL: https://doi.org/10.1016/j.jpaa.2014.02.005.

[DER02] H. Derksen, A. van den Essen, and P. van Rossum. *An extension of the Miyanishi–Sugie cancellation theorem to Dedekind rings*. Technical Report. 2002.

[DF10] Daniel Daigle and Gene Freudenburg. "Families of affine fibrations". In: *Symmetry and spaces*. Vol. 278. Progr. Math. Birkhäuser Boston, Boston, MA, 2010, pp. 35–43. DOI: 10.1007/978-0-8176-4875-6_3. URL: https://doi.org/10.1007/978-0-8176-4875-6_3.

[DF98] Daniel Daigle and Gene Freudenburg. "Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of $k[X_1, \cdots, X_n]$". In: *J. Algebra* 204.2 (1998), pp. 353–371. ISSN: 0021-8693. DOI: 10.1006/jabr.1998.7465. URL: https://doi.org/10.1006/jabr.1998.7465.

[DF99] Daniel Daigle and Gene Freudenburg. "A counterexample to Hilbert's fourteenth problem in dimension 5". In: *J. Algebra* 221.2 (1999), pp. 528–535. ISSN: 0021-8693. DOI: 10.1006/jabr.1999.7987. URL: https://doi.org/10.1006/jabr.1999.7987.

[Dix77] Jacques Dixmier. *Enveloping algebras*. North-Holland Mathematical Library, Vol. 14. Translated from the French. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, pp. xvi+375.

[EH73] Paul Eakin and William Heinzer. "A cancellation problem for rings". In: *Conference on Commutative Algebra (Univ. Kansas, Lawrence, Kan., 1972)*. Berlin: Springer, 1973, 61–77. Lecture Notes in Math., Vol. 311.

[EK04] M'Hammed El Kahoui. "Constants of derivations in polynomial rings over unique factorization domains". In: *Proc. Amer. Math. Soc.* 132.9 (2004), pp. 2537–2541. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-04-07313-7. URL: https://doi.org/10.1090/S0002-9939-04-07313-7.

[EKO12] M'hammed El Kahoui and Mustapha Ouali. "Fixed point free locally nilpotent derivations of $\mathbb{A}^2$-fibrations." In: *J. Algebra* 372 (2012), pp. 480–487. ISSN: 0021-8693.

[EKO14] M'hammed El Kahoui and Mustapha Ouali. "The cancellation problem over Noetherian one-dimensional domains". In: *Kyoto J. Math.* 54.1 (2014), pp. 157–165. ISSN: 2156-2261. DOI: 10.1215/21562261-2400301. URL: https://doi.org/10.1215/21562261-2400301.

[EKO16] M'hammed El Kahoui and Mustapha Ouali. "A triviality criterion for $\mathbb{A}^2$-fibrations over a ring containing $\mathbb{Q}$". In: *J. Algebra* 459 (2016), pp. 272–279. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2016.03.047. URL: https://doi.org/10.1016/j.jalgebra.2016.03.047.

[ER01] Arno van den Essen and Peter van Rossum. "A class of counterexamples to the cancellation problem for arbitrary rings". In: vol. 76. 1-2. Polynomial automorphisms and related topics (Kraków, 1999). 2001, pp. 89–93. DOI: 10.4064/ap76-1-8. URL: https://doi.org/10.4064/ap76-1-8.

[Ess00] Arno van den Essen. *Polynomial automorphisms and the Jacobian conjecture*. Vol. 190. Progress in Mathematics. Birkhäuser Verlag, Basel, 2000, pp. xviii+329.

[Ess07] Arno van den Essen. "Around the cancellation problem". In: *Affine algebraic geometry*. Osaka Univ. Press, Osaka, 2007, pp. 463–481.

[Ess92] Arno van den Essen. "Locally finite and locally nilpotent derivations with applications to polynomial flows and polynomial morphisms". In: *Proc. Amer. Math. Soc.* 116.3 (1992), pp. 861–871.

[Ess94] Arno van den Essen. "Locally finite and locally nilpotent derivations with applications to polynomial flows, morphisms and $\mathbb{G}_a$-actions. II". In: *Proc. Amer. Math. Soc.* 121.3 (1994), pp. 667–678.

[Fre00] G. Freudenburg. "A counterexample to Hilbert's fourteenth problem in dimension six". In: *Transform. Groups* 5.1 (2000), pp. 61–71. ISSN: 1083-4362. DOI: 10.1007/BF01237178. URL: https://doi.org/10.1007/BF01237178.

[Fre09] Gene Freudenburg. "Derivations of $R[X, Y, Z]$ with a slice". In: *J. Algebra* 322.9 (2009), pp. 3078–3087. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2008.05.007. URL: https://doi.org/10.1016/j.jalgebra.2008.05.007.

[Fre17] Gene Freudenburg. *Algebraic theory of locally nilpotent derivations*. Second. Vol. 136. Encyclopaedia of Mathematical Sciences. Invariant Theory and Algebraic Transformation Groups, VII. Springer-Verlag, Berlin, 2017, pp. xxii+319. ISBN: 978-3-662-55348-0; 978-3-662-55350-3. DOI: 10.1007/978-3-662-55350-3. URL: https://doi.org/10.1007/978-3-662-55350-3.

[Fre95] Gene Freudenburg. "Triangulability criteria for additive group actions on affine space". In: *J. Pure Appl. Algebra* 105.3 (1995), pp. 267–275.

[Fuj79] Takao Fujita. "On Zariski problem". In: *Proc. Japan Acad. Ser. A Math. Sci.* 55.3 (1979), pp. 106–110. ISSN: 0386-2194. URL: http://projecteuclid.org/euclid.pja/1195517399.

[Gup14a] Neena Gupta. "On the cancellation problem for the affine space $\mathbb{A}^3$ in characteristic p". In: *Invent. Math.* 195.1 (2014), pp. 279–288. ISSN: 0020-9910. DOI: 10.1007/s00222-013-0455-2. URL: https://doi.org/10.1007/s00222-013-0455-2.

[Gup14b] Neena Gupta. "On the family of affine threefolds $x^m y = F(x, z, t)$". In: *Compos. Math.* 150.6 (2014), pp. 979–998. ISSN: 0010-437X. DOI: 10.1112/S0010437X13007793. URL: https://doi.org/10.1112/S0010437X13007793.

[Gup14c] Neena Gupta. "On Zariski's cancellation problem in positive characteristic". In: *Adv. Math.* 264 (2014), pp. 296–307. ISSN: 0001-8708. DOI: 10.1016/j.aim.2014.07.012. URL: https://doi.org/10.1016/j.aim.2014.07.012.

[Ham75] Eloise Hamann. "On the R-invariance of $R[x]$". In: *J. Algebra* 35 (1975), pp. 1–16. ISSN: 0021-8693. DOI: 10.1016/0021-8693(75)90031-9. URL: https://doi.org/10.1016/0021-8693(75)90031-9.

[Hoc72] M. Hochster. "Nonuniqueness of coefficient rings in a polynomial ring". In: *Proc. Amer. Math. Soc.* 34 (1972), pp. 81–82. ISSN: 0002-9939. DOI: 10.2307/2037901. URL: https://doi.org/10.2307/2037901.

[Jun42] Heinrich W. E. Jung. "Über ganze birationale Transformationen der Ebene". In: *J. Reine Angew. Math.* 184 (1942), pp. 161–174. ISSN: 0075-4102. DOI: 10.1515/crll.1942.184.161. URL: https://doi.org/10.1515/crll.1942.184.161.

[Kam75] T. Kambayashi. "On the absence of nontrivial separable forms of the affine plane". In: *J. Algebra* 35 (1975), pp. 449–456.

[KEKO22] A. Ben Kaddah, M'hammed El Kahoui, and Mustapha Ouali. "The freeness property for locally nilpotent derivations of $R^{[2]}$". In: *Transformation Groups* (2022). DOI: 10.1007/s00031-022-09705-z. URL: https://doi.org/10.1007/s00031-022-09705-z.

[KL14] Manoj K. Keshari and Swapnil A. Lokhande. "A note on rigidity and triangulability of a derivation". In: *J. Commut. Algebra* 6.1 (2014), pp. 95–100.

[Kur04] Shigeru Kuroda. "A counterexample to the fourteenth problem of Hilbert in dimension four". In: *J. Algebra* 279.1 (2004), pp. 126–134.

[Kur05] Shigeru Kuroda. "A counterexample to the fourteenth problem of Hilbert in dimension three". In: *Michigan Math. J.* 53.1 (2005), pp. 123–132. ISSN: 0026-2285. DOI: 10.1307/mmj/1114021089. URL: https://doi.org/10.1307/mmj/1114021089.

[Mat80] Hideyuki Matsumura. *Commutative algebra*. Second. Vol. 56. Mathematics Lecture Note Series. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980, pp. xv+313. ISBN: 0-8053-7026-9.

[Miy07] Masayoshi Miyanishi. *Recent developments in affine algebraic geometry*. 2007.

[Miy73] Masayoshi Miyanishi. "Some remarks on polynomial rings". In: *Osaka Math. J.* 10 (1973), pp. 617–624. ISSN: 0388-0699. URL: http://projecteuclid.org/euclid.ojm/1200694568.

[Miy78] Masayoshi Miyanishi. *Curves on rational and unirational surfaces*. Vol. 60. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Tata Institute of Fundamental Research, Bombay; Narosa Publishing House, New Delhi, 1978, pp. ii+302.

[ML96] L. Makar-Limanov. “On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbf{C}^4$ or a $\mathbf{C}^3$-like threefold which is not $\mathbf{C}^3$”. In: *Israel J. Math.* 96.part B (1996), pp. 419–429. ISSN: 0021-2172. DOI: 10.1007/BF02937314. URL: https://doi.org/10.1007/BF02937314.

[ML98] Leonid G. Makar-Limanov. *Locally nilpotent derivations, a new ring invariant and applications*. Lecture Notes. 1998.

[MS80] Masayoshi Miyanishi and Tohru Sugie. “Affine surfaces containing cylinder-like open sets”. In: *J. Math. Kyoto Univ.* 20.1 (1980), pp. 11–42. ISSN: 0023-608X. DOI: 10.1215/kjm/1250522319. URL: https://doi.org/10.1215/kjm/1250522319.

[Nag60] Masayoshi Nagata. “On the fourteenth problem of Hilbert”. In: *Proc. Internat. Congress Math. 1958*. Cambridge Univ. Press, New York, 1960, pp. 459–462.

[NG67] Y. Nouazé and P. Gabriel. “Idéaux premiers de l’algèbre enveloppante d’une algèbre de Lie nilpotente”. In: *J. Algebra* 6 (1967), pp. 77–99.

[Ono84] Nobuharu Onoda. “Subrings of finitely generated rings over a pseudogeometric ring”. In: *Japan. J. Math. (N.S.)* 10.1 (1984), pp. 29–53. ISSN: 0289-2316.

[Ray68] Michèle Raynaud. “Modules projectifs universels”. In: *Invent. Math.* 6 (1968), pp. 1–26. ISSN: 0020-9910. DOI: 10.1007/BF01389829. URL: https://doi.org/10.1007/BF01389829.

[Ren68] Rudolf Rentschler. “Opérations du groupe additif sur le plan affine”. In: *C. R. Acad. Sci. Paris Sér. A-B* 267 (1968), A384–A387. ISSN: 0151-0509.

[Rob90] Paul Roberts. “An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert’s fourteenth problem”. In: *J. Algebra* 132.2 (1990), pp. 461–473. ISSN: 0021-8693. DOI: 10.1016/0021-8693(90)90141-A. URL: https://doi.org/10.1016/0021-8693(90)90141-A.

[RS79] Peter Russell and Avinash Sathaye. “On finding and cancelling variables in $k[X, Y, Z]$”. In: *J. Algebra* 57.1 (1979), pp. 151–166.

[Sat83] Avinash Sathaye. “Polynomial ring in two variables over a DVR: a criterion”. In: *Invent. Math.* 74.1 (1983), pp. 159–168.

[Sno89] Dennis M. Snow. “Unipotent actions on affine space”. In: *Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988)*. Vol. 80. Progr. Math. Birkhäuser Boston, Boston, MA, 1989, pp. 165–176.

[Sus82] A. A. Suslin. “Mennicke symbols and their applications in the K-theory of fields”. In: *Algebraic K-theory, Part I (Oberwolfach, 1980)*. Vol. 966. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 334–356.

[Swa80] Richard G. Swan. "On seminormality". In: *J. Algebra* 67.1 (1980), pp. 210–229. ISSN: 0021-8693.

[V01] Stéphane Vénéreau. "Automorphismes et variables de l'anneau de polynômes $A[y_1, y_2, \cdots, y_n]$". In: *Ph.D. Thesis, Institut Fourier des mathématiques, Grenoble* (2001).

[VD74] B. Ju. Veĭsfeĭler and I. V. Dolgačev. "Unipotent group schemes over integral rings". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 38 (1974), pp. 757–799. ISSN: 0373-2436.

[Win90] Jörg Winkelmann. "On free holomorphic $\mathbf{C}$-actions on $\mathbf{C}^n$ and homogeneous Stein manifolds". In: *Math. Ann.* 286.1-3 (1990), pp. 593–612.

[Wri81] David Wright. "On the Jacobian conjecture". In: *Illinois J. Math.* 25.3 (1981), pp. 423–440. ISSN: 0019-2082.

[Yan81] Joe Yanik. "Projective algebras". In: *J. Pure Appl. Algebra* 21.3 (1981), pp. 339–358. ISSN: 0022-4049. DOI: 10.1016/0022-4049(81)90022-0. URL: https://doi.org/10.1016/0022-4049(81)90022-0.

[Zar54] O. Zariski. "Interprétations algébrico-géométriques du quatorzième problème de Hilbert". In: *Bull. Sci. Math. (2)* 78 (1954), pp. 155–168. ISSN: 0007-4497.