Study of Higher-Order Fitted Mesh Methods for Singularly Perturbed Parabolic PDEs with Smooth and Nonsmooth Data

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by

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Abstract

This thesis addresses various higher-order accurate fitted mesh methods (FMMs) for solving singularly perturbed linear and nonlinear parabolic partial differential equations (PDEs) with smooth and nonsmooth data. The diffusion coefficient of those PDEs is generally considered as a small parameter ε, called "singular perturbation parameter ". Depending on the smooth and nonsmooth data, generally singularly perturbed differential equations (SPDEs) exhibit boundary or/and interior layers. These are thin regions in the vicinity of the boundary line of the given domain or/and the line of discontinuity of the given data where the gradients of the solution change rapidly as the perturbation parameter ε gets smaller; and hereby, the analytical solution of SPDEs inherently adapts to the multi-scale character.

Several real-life problems are often modeled as linear and nonlinear PDEs involving space and time variables, which can be viewed as SPDEs with smooth and nonsmooth data. Indeed, computational analysis of those PDEs that appeared, particularly in mathematical biology, has become incredibly significant for an understanding of the various biological processes and also for the application of such models to the medical sciences. As a prominent example in the context of mathematical biology, one can consider the chemo-taxis model, which arises in the mathematical modeling of tumor angiogenesis. On the other hand, the drift-diffusion equations are the most widely used mathematical models for describing semiconductor devices, which usually represent SPDEs with discontinuous source term. Henceforth, from the application point of view as well as for the multi-scale character of the analytical solution, the construction of effective numerical techniques are essential and challenging for analyzing SPDEs. It is well-known that devising FMMs constituted on appropriate layer-resolving non-uniform meshes is of natural and prime interest to the scientific community so as to achieve "parameter-robust"(also familiar as "ε-uniformly convergent ") numerical solution that converges independent of the parameter ε.

The major objective of the thesis is to devise, analyze, and compute parameter-robust, cost-effective high-order numerical approximations for a class of singularly perturbed linear parabolic PDEs of the convection-diffusion type with smooth and nonsmooth data; and their extensions to the semilinear parabolic PDEs.

Our investigation in this thesis begins with developing an ε-uniformly convergent robust numerical algorithm for solving a class of singularly perturbed one-dimensional linear parabolic convection-diffusion initial-boundary-value problems (IBVPs) with time-dependent convection coefficient and possessing a regular boundary layer. The current numerical algorithm consists of two parts. The first one is the development of a new hybrid FMM for higher-order numerical approximation in the spatial variable; the other one is the implementation of the Richardson extrapolation technique solely in the temporal direction (called temporal Richardson extrapolation) for enhancing the temporal accuracy. The idea behind the newly developed algorithm is further extended for the cost-effective higher-order numerical approximation of two-dimensional singularly perturbed linear parabolic problems with time-dependent boundary conditions by proposing a new fractional-step fitted mesh method (FSFMM) and, later, by the temporal Richardson extrapolation. Next, a complete convergence analysis is provided towards higher-order numerical approximations for a class of singularly perturbed one-dimensional semilinear parabolic convection-diffusion IBVPs exhibiting a regular boundary layer

by proposing two novels FMMs (the fully-implicit FMM and the implicit-explicit (IMEX) FMM) followed by the extrapolation technique. Our further investigation involves cost-effective higher-order numerical approximations of two-dimensional semilinear singularly perturbed parabolic convection-diffusion problem with non-homogeneous boundary data by developing two new fractional-step fitted mesh methods (FSFMMs) (the fully-implicit FSFMM and the IMEX-FSFMM); and later, by the extrapolation technique. Next, we turn our attention to investigating singularly perturbed PDEs with nonsmooth data. In this regard, efficient numerical methods are proposed and analyzed for two different classes of model problems with nonsmooth data. The first one is the singularly perturbed parabolic PDEs exhibiting strong interior layers, and the other one is the singularly perturbed parabolic PDEs exhibiting both boundary and weak interior layers. Finally, we focus our attention on devising and analyzing a higher-order time-accurate FMM for a class of singularly perturbed semilinear parabolic convection-diffusion IBVPs exhibiting both boundary and weak interior layers.