Geometry of Immersions, Submersions, Harmonic Maps and of Estimation on Statistical Manifolds

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by

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Abstract

Information geometry emerged from the geometric study of a statistical model of probability distributions. A statistical model equipped with a Riemannian metric and a pair of dual affine connections is called a statistical manifold. Various geometric aspects of statistical manifolds were studied by many researchers. The main objective of this thesis is to explore certain geometric properties of statistical manifolds and the geometry of estimation.

In Chapter 2, we discuss the geometry of immersions and statistical manifolds. In Section 2.1, we discuss definitions and basic results related to affine immersions. In Proposition (2.1) detailed proof is given for the result that a simply connected statistical manifold can be realized in $\mathbf{R}^{n+1}$ if and only if it is 1-conformally flat. In Section 2.2, we first discuss about the statistical submanifolds and the fundamental equations associated with it. Then in Theorem (2.4) we prove a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Statistical immersion is defined and in Theorem (2.5) we prove a necessary condition for a statistical manifold to be a statistical hypersurface. Also, we prove its converse in Theorem (2.6). Then, in Theorem (2.10) a necessary and sufficient condition for a statistical immersion into a dually flat statistical manifold of codimension one to be minimal is obtained. Also, in Theorem (2.11) a necessary condition is obtained for minimal statistical immersion of statistical manifolds equipped with α-connections. In Section 2.3, centro-affine immersion into $\mathbf{R}^{n+2}$ and the fundamental equations of it are discussed first. Also, in Proposition (2.3) and in Proposition (2.4) a detailed proof of 1-conformal equivalence and (-1)-conformal equivalence of statistical manifold structures in the case of centroaffine immersions into $\mathbf{R}^{n+2}$ are given, respectively. We define centro-affine immersions of codimension two into a dually flat statistical manifold and in Theorem (2.13) we give a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. In Theorem (2.14) we show that the inherited statistical manifold structure is conformally-projectively flat in the case of non-degenerate, centro-affine, equiaffine immersion into a dually flat statistical manifold of codimension two. In Section 2.4, we first discuss the affine fundamental form and relations between curvature tensors for affine immersions of general codimension. Then, we define the transversal volume element map for equiaffine statistical immersion of gen-

eral codimension and certain properties are also proved in Lemma (2.3) and in Proposition (2.6).

In Chapter 3, we discuss the geometry of submersions and statistical manifolds. In Section 3.1, definitions of submersion and semi-Riemannian submersion and certain basic results are given. We summarize the definition and basic results of affine submersions with horizontal distribution in Section 3.2. Also, discuss the theorem by Abe and Hasegawa on geodesics comparison for an affine submersion with horizontal distribution. In Section 3.3, we first introduce the concept of a conformal submersion with horizontal distribution for Riemannian manifolds, which is a generalization of the affine submersion with horizontal distribution. Then, in Theorem (3.3) a necessary condition for the existence of such a map is proved. In Theorem (3.6) a necessary and sufficient condition is obtained for $\pi \circ \sigma$ to be a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$ for a conformal submersion with horizontal distribution. Then, in Proposition (3.5) we prove a necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic. Also, in corollary (3.3) we give a necessary condition for the connection on $\mathbf{B}$ to be complete when the connection on $\mathbf{M}$ is complete for a conformal submersion with horizontal distribution $\pi : \mathbf{M} \longrightarrow \mathbf{B}$. In Section 3.4, we first discuss the affine submersion with horizontal distribution and statistical manifolds. A statistical structure is obtained on the manifold $\mathbf{B}$ induced by the affine submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$. In the case of conformal submersion with horizontal distribution in Theorem (3.9) we prove a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold. Also, in Proposition (3.7) we prove $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla}^*)$ is a conformal submersion with horizontal distribution.

Chapter 4 deals with the statistical structures on tangent bundles, harmonic maps between statistical manifolds and between tangent bundles. In Theorem (4.3) of Section 4.1 we prove a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with the complete lift connection and the Sasaki lift metric. In Section 4.2, we first give a detailed description of the harmonic map using tension field. In Theorem (4.4) we prove a necessary and sufficient condition for the harmonicity of identity map for conformally-projectively equivalent statistical manifolds. Then, conformal statistical submersion is defined which is a generalization of the statistical submersion and in Theorem (4.5) we prove that harmonicity and conformality cannot coexist. In Section 4.3, certain properties of the differential of the tangent map is given first. For statistical manifolds, in Theorem (4.7) we prove that a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is harmonic with respect to ∇ and ∇^* if and only if it is harmonic with respect to the conjugate connections $\overline{\nabla}$ and $\overline{\nabla^*}$. Then, in Theorem (4.8) given a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles. Also, in Proposition (4.8) we prove a necessary and sufficient condition for the tangent map to be a statistical submersion.

In Chapter 5, estimation of parameters in statistical manifolds, exponential family and its submanifolds, estimation of parameters in the curved exponential family and Fisher-Neyman sufficient statistic for parametrized models are discussed. In Section 5.1, short account of the statistical properties of an estimator is given. In Theorem (5.2) of Section 5.2 we show that if all ∇^1-autoparallel proper submanifolds of a ±1-flat statistical manifold M are exponential then M is an exponential family. Also, in Theorem (5.3) we prove that if submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold. In the theory of estimation in curved exponential family we give a short account of Amari's geometric conditions for the consistency and efficiency of an estimator in a curved exponential family using ancilliary manifolds. Then discuss the MLE algorithm for estimating parameters in the curved exponential family obtained by Cheng et al. In Section 5.3 we show that the Fisher-Neyman sufficient statistic is invariant under the isostatistical immersions of statistical manifolds in Theorem (5.5).