STUDIES ON CONTROLLABILITY OF DYNAMICAL SYSTEMS WITH IMPULSES AND TIME-DELAY CONTROLS

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in partial fulfillment for the award of the degree of

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by

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supervised by

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29 March 2019

Certificate

This is to certify that the thesis titled ***STUDIES ON CONTROLLABILITY OF DYNAMICAL SYSTEMS WITH IMPULSES AND TIME-DELAY CONTROLS*** submitted by **VIJAYAKUMAR S. MUNI**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** in **Mathematics**, is a bona fide record of the original work carried out by him under my supervision. The contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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To my parents, for everything...

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Abstract

One of the fundamental properties of the dynamical systems is “controllability”, introduced by R. Kalman in 1960s. Since then it has become an active topic of research in the modern control theory. This thesis is devoted to explore the controllability issues for certain classes of finite-dimensional continuous dynamical control systems possessing impulses in state and time-delay in controls. The main reason for considering these types of systems is that, many of the evolution processes which occur in real life, like medicine, biology, computer networking, neural networks, information science, artificial intelligence, telecommunications, robotics etc., are modelled by such systems involving state functions which exhibit an abrupt changes at certain moments of time, that in the form of impulses. Similarly in many systems, like chemical process systems, hydraulically actuated systems, combustion systems, population dynamics etc., the past values of the control function exerts its influence on the present, and hence on the future of the state function, and these phenomena are modelled by the time-delay control systems. Though some research has been conducted on the controllability of certain classes of impulsive and delay systems, but it is not fully explored, especially when it comes to nonlinear systems, networked systems and systems modelled by Lyapunov equations. Further, many of the anomalous processes shows a very complex behaviour which can be studied if their dynamics are modelled by fractional-order differential equations. In case of fractional systems also, like the classical derivative models, the controllability property is not fully examined, in particular for the systems possessing time-delay controls. Based on all the observations, the objectives of this thesis is in the establishment of the controllability properties of the following classes of dynamical control systems:

1. Impulsive systems with time-delay controls.
2. Systems described by the fractional derivatives with delays in control.
3. Systems modelled by the Lyapunov matrix equations with impulses and time-delays in the control function.
4. Finite-dimensional linear systems carrying impulses in state.
5. Networked control systems exhibiting impulses.

We use the tools of linear and nonlinear operator theory such as fixed-point theorems—Schauder’s fixed-point theorem, Banach contraction principle—and spectral theory, to obtain the controllability results. Numerical examples are provided to substantiate the theoretical results.

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List of notations and symbols

1. The vectors [vector valued functions] are denoted by the lower case bold Latin letters, e.g. $\mathbf{v}$ $\big[\mathbf{f}(\cdot)\big]$, the matrices [matrix valued functions] are denoted by the upper case Latin bold letters, e.g. $\mathbf{A}$ $\big[\mathbf{A}(\cdot)\big]$, and scalars (real numbers or complex numbers) by the lower case Latin letters a, b etc.

2. The transpose of any vector or operator is denoted by $(\cdot)^T$ and the Hermitian adjoint by $(\cdot)^*$. Note that for real vectors (real operators), both are same.

3. The set of all positive integers is denoted by $\mathbb{N}$, the set of all real [complex] numbers is denoted by $\mathbb{R}\,[\mathbb{C}]$.

4. For any $n \in \mathbb{N}$, we define the $n-$dimensional real Euclidean space by

$$\mathbb{R}^n := \Big\{\mathbf{v} \,\Big|\, \mathbf{v} = [v_1 \; v_2 \cdots v_n]^T,\; v_i \in \mathbb{R},\; i = 1, 2, \ldots, n\Big\}.$$

   This also represents an $n-$dimensional real Banach space endowed with an Euclidean norm (or $2-$norm) $\|\cdot\|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|\mathbf{v}\|_{\mathbb{R}^n} := \left(\sum_{j=1}^{n} |v_j|^2\right)^{\frac{1}{2}}.$$

5. For any $m, n \in \mathbb{N}$, the set of all $(m \times n)-$real matrices is denoted and defined as

$$\mathbb{R}^{m\times n} := \Big\{\mathbf{A} \,\big|\, \mathbf{A} = \big(a_{ij}\big)_{1\le i\le m,\; 1\le j\le n},\; a_{ij} \in \mathbb{R}\Big\}.$$

   This also represents a real Banach space endowed with a Frobenius norm $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ defined by

$$\|\mathbf{A}\| := \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}.$$

6. The identity $(n \times n)-$matrix is denoted by $\mathbf{I}_n$ and the diagonal $(n \times n)-$matrix with diagonal entries $a_{11}, \ldots, a_{nn}$ by $\mathrm{diag}(a_{11}, \ldots, a_{nn})$.

7. The set of all continuous functions from set A to set B is denoted by $\mathcal{C}(A; B)$.

8. $\mathcal{L}^2\big([t_0,\infty);\mathbb{R}^n\big) := \Big\{\mathbf{f}(\cdot)\,\Big|\,\mathbf{f}(\cdot) : [t_0,\infty) \to \mathbb{R}^n,\ \int_{t_0}^{\infty} \big\|\mathbf{f}(t)\big\|_{\mathbb{R}^n}^2 \mathrm{d}t < \infty\Big\}$ denotes a real Hilbert space of square-integrable functions endowed with a norm $\|\cdot\|_{\mathcal{L}^2} : \mathcal{L}^2\big([t_0,\infty);\mathbb{R}^n\big) \to \mathbb{R}$ defined by

$$\|\mathbf{f}(\cdot)\|_{\mathcal{L}^2} := \left(\int_{t_0}^{\infty} \big\|\mathbf{f}(t)\big\|_{\mathbb{R}^n}^2 \mathrm{d}t\right)^{\frac{1}{2}} = \left(\int_{t_0}^{\infty} \sum_{j=1}^{n} |f_j(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}},$$

for any $\mathbf{f}(t) = \Big[f_1(t)\ f_2(t) \cdots f_n(t)\Big]^T,\quad \forall\, t \in [t_0,\infty)$.

9. The augmented matrix of $\mathbf{A}$ and $\mathbf{B}$ is written as $[\mathbf{A},\, \mathbf{B}]$.

10. The zero matrix denoted by $\mathbf{O}$, whereas zero vector by $\mathbf{0}$.

Chapter 1
Introduction

1.1 Preliminary background

Here we give a brief description of the systems we consider for the controllability analysis.

(a) Impulsive systems. During the recent decades, the controllability analysis and synthesis of dynamical impulsive systems have drawn tremendous amount of attention among the researchers across the world, due to their various applications in science, engineering, technology, economics, sociology, medicine, and in almost all disciplines. Because of abrupt changes in states of the impulsive systems at certain moments of time, the establishment of the controllability criteria requires a careful observation in the behaviour of their trajectory. Nevertheless, a vast amount of effort has been made to derive the controllability criteria for various kinds of linear impulsive systems, see [45, 49, 70, 73, 114, 115], whereas for the nonlinear impulsive systems mostly fixed-point approach has been followed to obtain the controllability conditions, refer [41, 48, 71, 72, 84, 117]. Without employing fixed-point theorems, the controllability of nonlinear impulsive systems is investigated in [17, 18] under the boundedness assumption on nonlinearities in the systems. Further it is to be noted here that, in most of these articles, the null controllability is established, but for the impulsive systems the null controllability need not implies controllability (refer Chapter 6).

(b) Delay systems. The systems involved in chemical processes, combustions, gas pressurized biopropellant rockets, agricultural economics, population dynamics, harmonic oscillator etc., where past values of the control function influences the present, and hence the future of the state of the system [35, 38, 68]. Such processes are modelled by delay differential equations with time-delays in the control function. Further, various kinds of time-delays are considered in the literature, like constant delays [27]–[29], [33, 63, 64, 87, 100], variable delays [7, 8, 32, 56, 58, 65], distributed delays [6, 53, 57, 59, 60, 104], and correspondingly obtained the controllability results for the systems of both linear and nonlinear types. Many systems encountered in practice involve impulses as well as delays in control. Under such situation, the establishment of their controllability becomes complex, because of the coexistence of impulses and delays. However, the linear case of this scenario was explored in [74] and [75], and not much work is reported on nonlinear systems.

(c) Fractional-order systems. Though several phenomena are modelled by the classical derivatives and integrals, but the experiments and reality confirms that, many anomalous processes shows a complex behaviour, and due to this, their dynamics cannot be characterized by classial derivative models. In these situations, the usage of the fractional-order derivatives and integrals provides a better understanding of such models [52, 54, 92]. In the last few decades, when many researchers came to know that, the fractional-order derivatives and integrals will be used in the modelling of viscoelastic materials, kinetics of anomalous diffusion, fractional wave equations, fractional Brownian motion, electrochemical process, feedback amplifiers, electrical circuits, biological systems etc., there started a growing interest on the study of controllability of fractional-order dynamical systems, which resulted in the publishing of many papers in this area. For the controllability of linear fractional-order dynamical systems, refer [9, 19, 42, 80] and for nonlinear fractional systems, see [12, 43], [97]–[99].

(d) Matrix differential systems. Several problems in control theory and game theory are required to be modelled by the matrix differential equations, and these systems have numerous applications in stability analysis and optimal control [39]. Hence it is natural to consider the controllability issues of such dynamical systems. This consideration has attracted few control theorists, and it resulted in the matrix rank conditions for the controllability of the linear matrix Lyapunov systems [83], semilinear matrix Lyapunov systems [36], impulsive matrix Lyapunov systems [37]; but we take the case of impulsive matrix Lyapunov systems with time-delay controls to study the controllability.

(e) Networked systems. A networked system is a collection of dynamic units that interact over an information exchange network for its operation. Studies on the controllability of networked systems is greatly inspired by their ubiquitous behaviour in diverse areas of science and engineering, such as physiological systems, gene networks, large scale energy systems, formation control of satellite clusters, multiagent robotics, data networks and many others. Despite of developing the controllability criteria for various kinds of systems, the difficulty arises when these controllability properties are directly applied to the large scale networks. This is because of their structural complexity. Hence the subject of controllability of the networked systems is an emerging area of research for many scientists. Noticeably, some results are available on this, refer [76, 77, 94, 116] etc. However, a complete settlement of the controllability verification for the large scale networked systems still requires further investigations. In [108] Wang et al analyzed networked MIMO-systems and established their controllability criteria. But considering the case where less transmitted information is more economical, further simplifications are performed in [109]. The research is yet to happen on the controllability of impulsive networked systems and delay-systems.

1.2 Dynamical systems

The time evolution of a function in some geometrical space is represented by a mathematical equation called as dynamical system. At any given time, a dynamical system has a state given by a tuple of real (complex) numbers that can be represented by a point in an appropriate state-space, and the function representing the evolution rule of such a dynamical system describes the future state from the current state.

Let us explain the modelling of some time evolution process by a deterministic continuous dynamical system with an example of a population growth of the bacteria in a petri dish over a small duration of time as proposed by Thomas Malthus in 1798 (see pp. 21 of [118]). Let $p(t)$ denotes the total population of the bacteria at time t. Assuming that the rate at which the population grows at a certain time is proportional to the total population at that time, then this assumption can be represented by a differential equation:

$$\frac{dp(t)}{dt} = k\,p(t),$$

where k is the constant of proportionality. Here $p(t)$ is called as the state of the system. Similar to this example, several other can be found in—modelling of radioactive decay, spread of diseases, chemical reactions, falling objects, suspendable cables, and many more. In some of the situations, time evolution processes modelled by the dynamical systems requires large number of state variables, for example, earths global climate, transportation and communication systems, the behaviour of neurons in human brain etc.

1.2.1 Fractional-order systems

Many of the experiments and reality confirms that, the time-evolution processes in the context of viscoelastic materials, anomalous diffusion, fractional wave equation, electrochemical process, Brownian motion, biological systems etc., have a complex behaviour, due to which their dynamics cannot be characterized by the classical derivatives. In such cases, the usage of the fractional-order derivatives will give a better understanding of such models. In this way, the fractional derivatives has an advantage over the classical derivatives (the derivatives of integer-orders). Note here that, the word fractional in this context refers to any arbitrary real or complex number. Various mathematicians and scientists—Euler, Lagrange, Laplace, Fourier, Liouville, Riemann, Laurent, Krug, and others—subsequently mentioned the fractional derivatives in some context of their work over a period of years during 18th and the early decades of 19th century. The theory took an advanced development after 1950s, when many monographs and research papers have been published on fractional dynamical systems. Different definitions for the fractional derivatives and integrals—Riemann–Liouville fractional derivatives, Liouville fractional derivative, Caputo fractional derivative, Erdelyi–Kober type fractional derivatives, Hadamard type fractional derivatives, Grunwald–Letnikov fractional derivatives, and others—proposed over a period of years which came certainly as a generalization of various formulas. The first book devoted exclusively for the study of fractional calculus is by Oldham and Spaniar [88] published in 1974. Later, the monographs by Podlubny [92] in 1999, Kilbas et al [54] in 2006 and many others give an insightful into the analysis and synthesis of various properties of dynamical systems modelled by the fractional-order derivatives and integrals.

1.3 Control systems

The mathematical notion of controllability begun to develop by the contributions by U. S. scientists R. Bellman in the context of dynamic programming, R. Kalman in filtering techniques and the algebraic approach to linear systems, and the Russian mathematician L. Pontryagin with the maximum principle for nonlinear optimal control problems [25]. Now control theory is considered to be an interdisciplinary branch of engineering and mathematics that deals with influence behaviour of dynamical systems. The word control means to act, to put things in order to guarantee that the system behaves as desired. In this thesis, unless otherwise specified, the controllability means state controllability. Other types of controllability are also reported in the literature, for example output controllability [77, 86], controllability in the behavioural framework [93] etc.

To explain a control problem in mathematical terms, consider a dynamical system governed by the state equation:

$$\dot{x}(t) + A(x) = f(u). \tag{1.3.1}$$

Here x is the state function describes a time-dependence of a point in some geometrical shape usually belongs to some vector space V and is unknown of the system (1.3.1) that we are willing to control so that it behaves as we desire. On the other hand u is the control, which belongs to a set of admissible controls, say $\mathcal{U}_{ad}$. This is the variable that we can choose freely in $\mathcal{U}_{ad}$ to act on the system (1.3.1).

Here we assume that $A : \mathcal{D}(A) \subset V \to V$ and $f : \mathcal{U}_{ad} \to V$ are the given two linear or nonlinear mappings. The operator A determines the equation that must be satisfied by the state function x, according to the laws of Physics. The function f indicates the way the control u acts on the system governing the state. It is necessary to have the state equation (1.3.1) possesses exactly one solution in a vector space V, called as the trajectory of the system (1.3.1) for each control $u \in \mathcal{U}_{ad}$. To control the dynamical system (1.3.1) means, roughly speaking, it is possible to steer the system (1.3.1) from arbitrary initial state to arbitrary desired final state in V by using a suitable $u \in \mathcal{U}_{ad}$.

It should be mentioned that, there are various notions of controllability (i.e. state controllability) in the literature that strongly depends on the class of dynamical systems on one hand, and the form of admissible controls on the other hand. In recent years, many research papers are published on various types of controllability for varieties of dynamical systems—linear, semilinear and nonlinear systems. But the controllability of finite-dimensional, continuous linear dynamical control systems modelled by the first-order ordinary differential equations (see (1.3.2) below) is well established in the literature and many monographs are available on this, for instance see [23, 55, 105]. For this system, the most natural space to be considered for admissible control functions $\mathcal{U}_{ad}$ is the Hilbert space $\mathcal{L}^2(\cdot)$.

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad t \in [t_0, \infty), \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned}\right\} \tag{1.3.2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(\cdot) \in \mathcal{L}^2\big([t_0, \infty); \mathbb{R}^m\big)$ is an admissible control, $\mathbf{A}(\cdot) : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is an $(n \times n)-$dimensional real matrix-valued function with elements $a_{ij}(\cdot) \in \mathcal{L}^2\big([t_0, \infty); \mathbb{R}\big)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, $\mathbf{B}(\cdot) : [t_0, \infty) \to \mathbb{R}^{n\times m}$ is an $(n \times m)-$dimensional real matrix valued function with elements $b_{ij} \in \mathcal{L}^2\big([t_0, \infty); \mathbb{R}\big)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

For any given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ and an admissible control function $\mathbf{u}(\cdot) \in \mathcal{L}^2\big([t_0, \infty); \mathbb{R}^m\big)$, there exists a unique solution to the system (1.3.2) which is absolutely continuous in $[t_0, \infty)$ (for reference, see [55]).

Let $\phi_j(t) \in \mathbb{R}^n$, $j = 1, \ldots, n$, denotes the $n-$linearly independent solutions to the homogeneous system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, then

$$\mathbf{\Phi}(t) = \Big[\phi_1(t) \;\; \phi_2(t) \cdots \phi_n(t)\Big]$$

is a fundamental matrix (of size $n \times n$) solution to this homogeneous system which satisfies $\dot{\mathbf{\Phi}}(t) = \mathbf{A}(t)\mathbf{\Phi}(t)$, and this matrix is invertible. Then the state-transition matrix generated by $\mathbf{A}(t)$ is defined by $\mathbf{\Phi}(t, s) := \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$, $t_0 \le s \le t < \infty$, and it satisfies the following properties:

1. $\mathbf{\Phi}(t, t) = \mathbf{I}_n$, $\forall\, t \in [t_0, \infty)$, an identity matrix of size $(n \times n)$.
2. $\mathbf{\Phi}(\cdot, \cdot)$ satisfies the semigroup property

$$\mathbf{\Phi}(t, s) = \mathbf{\Phi}(t, \tau)\mathbf{\Phi}(\tau, s), \;\; \forall\, t_0 \le s \le \tau \le t < \infty.$$

3. $\frac{\partial \mathbf{\Phi}(t,s)}{\partial t} = \mathbf{A}(t)\mathbf{\Phi}(t, s)$.
4. $\frac{\partial \mathbf{\Phi}(t,s)}{\partial s} = -\mathbf{\Phi}(t, s)\mathbf{A}(s)$.

5. $\mathbf{\Phi}(t,s)$ is a non-singular (and bounded) matrix such that $\mathbf{\Phi}(t,s) = \mathbf{\Phi}^{-1}(s,t), \ \forall\, t_0 \leq s \leq t < \infty$.

Remark 1.3.1. *A series expansion for the state-transition matrix is given by the following Peano–Baker series (see [23])*

$$\begin{aligned}\mathbf{\Phi}(t,s) = \mathbf{I}_n + \int_s^t \mathbf{A}(\sigma_1)\mathrm{d}\sigma_1 + \int_s^t \mathbf{A}(\sigma_1)\int_s^{\sigma_1}\mathbf{A}(\sigma_2)\mathrm{d}\sigma_2\mathrm{d}\sigma_1 \\ + \int_s^t \mathbf{A}(\sigma_1)\int_s^{\sigma_1}\mathbf{A}(\sigma_2)\int_s^{\sigma_2}\mathbf{A}(\sigma_3)\mathrm{d}\sigma_3\mathrm{d}\sigma_2\mathrm{d}\sigma_1 + \cdots\end{aligned}$$

This series converges uniformly and absolutely for all $t_0 \leq s \leq t < \infty$. If $\mathbf{A}(t) = \mathbf{A}$, a constant matrix, then the above series converges to $e^{\mathbf{A}(t-s)}$.

By the method of variation of parameters, the solution to the system (1.3.2) at any time $t \in [t_0, \infty)$ is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \int_{t_0}^t \mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s. \tag{1.3.3}$$

Though there are many notions of controllability, we consider the following definition of controllability.

Definition 1.3.1. *The system* (1.3.2) *is said to be controllable over $\mathbb{R}^n$ on $[t_0, t_f], \ t_f < \infty$, if for every pair of vectors $\mathbf{x}_0, \ \mathbf{x}_f \in \mathbb{R}^n$ there exists at least one function $\mathbf{u}(\cdot) \in \mathcal{L}^2\big([t_0,t_f];\mathbb{R}^m\big)$, such that the solution given in eq* (1.3.3) *to the system* (1.3.2) *with this $\mathbf{u}(\cdot)$ and initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, satisfies the condition: $\mathbf{x}(t_f) = \mathbf{x}_f$. In this case, the function $\mathbf{u}(\cdot)$ is called a control function that steers the state of the system* (1.3.2) *from $\mathbf{x}_0$ to $\mathbf{x}_f$ during $[t_0, t_f]$.*

Remark 1.3.2. *The above notion of controllability on $[t_0, t_f]$ over $\mathbb{R}^n$ is also called as complete controllability on $[t_0, t_f]$ over $\mathbb{R}^n$.*

Remark 1.3.3. *In the above notion of controllability on $[t_0, t_f]$ over $\mathbb{R}^n$, if $\mathbf{x}_f = \mathbf{0}$, then the system* (1.3.2) *is said to be null* (*or zero*) *controllable on $[t_0, t_f]$ over $\mathbb{R}^n$.*

Note here that, if a system is controllable, then such system is null controllable also. For the linear systems (1.3.2) *the reverse notion is also true which we prove shortly. But in general, the null controllability need not implies controllability* [*refer Chapter 6 for more details*].

Now from the solution given in eq (1.3.3) we can say that, the system (1.3.2) is controllable over $\mathbb{R}^n$ on $[t_0, t_f], \ t_f < \infty$, if and only if $\exists$ at least one $\mathbf{u}(\cdot) \in \mathcal{L}^2\big([t_0,t_f];\mathbb{R}^m\big)$ such that

$$\mathbf{x}_f - \mathbf{\Phi}(t_f,t_0)\mathbf{x}_0 = \int_{t_0}^{t_f} \mathbf{\Phi}(t_f,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s, \quad \forall\, \mathbf{x}_0, \ \mathbf{x}_f \in \mathbb{R}^n.$$

Let us introduce a linear operator $\mathrm{C} : \mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big) \to \mathbb{R}^n$ defined by

$$\mathrm{C}\mathbf{u}(\cdot) := \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s. \tag{1.3.4}$$

The linear operator C defines its adjoint $\mathrm{C}^* : \mathbb{R}^n \to \mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big)$ in the following way:

$$\begin{aligned}
\Big\langle \mathbf{u}(\cdot),\, \mathrm{C}^*\mathbf{v} \Big\rangle_{\mathcal{L}^2([t_0,t_f];\,\mathbb{R}^m)} &= \Big\langle \mathrm{C}\mathbf{u}(\cdot),\, \mathbf{v} \Big\rangle_{\mathbb{R}^n} \\
&= \left\langle \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,\;\; \mathbf{v} \right\rangle_{\mathbb{R}^n} \\
&= \mathbf{v}^* \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&= \int_{t_0}^{t_f} \mathbf{v}^*\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&= \int_{t_0}^{t_f} \Big\langle \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s),\;\mathbf{v} \Big\rangle_{\mathbb{R}^n} \mathrm{d}s \\
&= \int_{t_0}^{t_f} \Big\langle \mathbf{u}(s),\;\mathbf{B}^*(s)\mathbf{\Phi}^*(t_f, s)\mathbf{v} \Big\rangle_{\mathbb{R}^m} \mathrm{d}s \\
&= \Big\langle \mathbf{u}(\cdot),\;\mathbf{B}^*(\cdot)\mathbf{\Phi}^*(t_f, \cdot)\mathbf{v} \Big\rangle_{\mathcal{L}^2([t_0,t_f];\,\mathbb{R}^m)}.
\end{aligned}$$

Therefore, the adjoint of C is the linear operator $\mathrm{C}^* : \mathbb{R}^n \to \mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big)$ defined by

$$\mathrm{C}^*\mathbf{v} := \mathbf{B}^*(\cdot)\mathbf{\Phi}^*(t_f, \cdot)\mathbf{v},\;\; \forall\, \mathbf{v} \in \mathbb{R}^n. \tag{1.3.5}$$

Further, the composition of linear operators C^* and C defines a linear operator $\mathrm{CC}^* : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathrm{CC}^*\mathbf{v} := \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{B}^*(s)\mathbf{\Phi}^*(t_f, s)\mathbf{v}\, \mathrm{d}s,\;\; \forall\, \mathbf{v} \in \mathbb{R}^n. \tag{1.3.6}$$

Obviously the linear operator CC^* is realized as a $(n \times n)-$matrix, and is called as controllability Grammian of the system (1.3.2), denoted by $\mathbf{W}$.

The theorem given below demonstrates how the controllability of the system (1.3.2) and the linear operators C, C^*, CC^* are related with each other.

Theorem 1.3.1. *The following statements are equivalent:*

1. *The system* (1.3.2) *is controllable on* $[t_0, t_f],\ t_f < \infty,$ *over* $\mathbb{R}^n$.

2. *The linear operator* C *is surjective.*

3. *The linear operator* C^* *is injective.*

4. The controllability Grammian $\mathbf{W} = \mathrm{C}\mathrm{C}^*$ *is positive definite.*

Proof. Let us begin by showing (1) $\iff$ (2). From eq (1.3.4), C is a bounded linear operator and Range space(C) is a subspace of $\mathbb{R}^n$. As $\mathbf{x}_0$, $\mathbf{x}_f$ are arbitrary vectors of $\mathbb{R}^n$, so the system (1.3.2) is controllable on $[t_0, t_f],\ t_f < \infty$, over $\mathbb{R}^n$ if and only if the linear operator C is surjective, i. e.

$$\text{Range space(C)} = \mathrm{C}\Big(\mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big)\Big) = \mathbb{R}^n.$$

Now we show that (2) $\Longrightarrow$ (3). Let C be surjective and $\mathrm{C}^*\mathbf{v} = \mathbf{o}(\cdot)$, the zero element of $\mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big)$, for some $\mathbf{v} \in \mathbb{R}^n$. Consider

$$\begin{aligned}\Big\langle \mathrm{C}\mathbf{u}(\cdot),\, \mathbf{v}\Big\rangle_{\mathbb{R}^n} &= \Big\langle \mathbf{u}(\cdot),\, \mathrm{C}^*\mathbf{v}\Big\rangle_{\mathcal{L}^2([t_0,t_f];\, \mathbb{R}^m)} \\ &= \Big\langle \mathbf{u}(\cdot),\, \mathbf{o}(\cdot)\Big\rangle_{\mathcal{L}^2([t_0,t_f];\, \mathbb{R}^m)} \\ &= 0.\end{aligned}$$

Since C is surjective, so there exists some $\mathbf{u}(\cdot) \in \mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big)$ such that $\mathrm{C}\mathbf{u}(\cdot) = \mathbf{v}$. Hence the above condition becomes $\langle \mathbf{v},\, \mathbf{v}\rangle_{\mathbb{R}^n} = 0 \Longrightarrow \mathbf{v} = \mathbf{0}$, and hence C^* is injective.

To prove that (3) $\Longrightarrow$ (4), let C^* be injective and for some $\mathbf{v} \in \mathbb{R}^n$, $\mathrm{C}\mathrm{C}^*\mathbf{v} = \mathbf{0}$. Then $\big\langle \mathrm{C}\mathrm{C}^*\mathbf{v},\, \mathbf{w}\big\rangle_{\mathbb{R}^n} = 0,\ \forall \mathbf{w} \in \mathbb{R}^n$. In particular, $\big\langle \mathrm{C}\mathrm{C}^*\mathbf{v},\, \mathbf{v}\big\rangle_{\mathbb{R}^n} = 0$. Therefore $\big\langle \mathrm{C}^*\mathbf{v},\, \mathrm{C}^*\mathbf{v}\big\rangle_{\mathcal{L}^2([t_0,t_f];\, \mathbb{R}^m)} = 0 \Longrightarrow \big\|\mathrm{C}^*\mathbf{v}\big\|_{\mathcal{L}^2} = 0 \Longrightarrow \mathrm{C}^*\mathbf{v} = \mathbf{o}(\cdot) \Longrightarrow \mathbf{v} = \mathbf{0}$, as C^* is injective. Therefore $\mathbf{W} = \mathrm{C}\mathrm{C}^*$ is injective. By rank–nullity theorem, $\mathrm{C}\mathrm{C}^*$ is bijective. Moreover $\mathrm{C}\mathrm{C}^*$ is positive definite, as $\langle \mathrm{C}\mathrm{C}^*\mathbf{v},\, \mathbf{v}\rangle_{\mathbb{R}^n} > 0$ for every $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^n$.

Now it is left to show that (4) $\Longrightarrow$ (1). As $\mathbf{W} = \mathrm{C}\mathrm{C}^*$ is positive definite, so it is an invertible matrix. Consider a control function $\mathbf{u}(t) = \mathbf{B}^*(t)\mathbf{\Phi}^*(t_f, t)\mathbf{W}^{-1}\big[\mathbf{x}_f - \mathbf{\Phi}(t_f, t_0)\mathbf{x}_0\big]$. Then the solution of the system (1.3.2) given in eq (1.3.3) at $t = t_f$ with this $\mathbf{u}(t)$ satisfies the condition $\mathbf{x}(t_f) = \mathbf{x}_f$, proving that the system (1.3.2) is controllable over $\mathbb{R}^n$ on $[t_0, t_f],\ t_f < \infty$. □

Remark 1.3.4. *There are various possibilities to design a control function doing the same job, i.e. steering the state of system* (1.3.2) *from* $\mathbf{x}_0$ *to* $\mathbf{x}_f$*, and one such control is given by*

$$\mathbf{u}(t) = \mathbf{B}^*(t)\mathbf{\Phi}^*(t_f, t)\mathbf{W}^{-1}\big[\mathbf{x}_f - \mathbf{\Phi}(t_f, t_0)\mathbf{x}_0\big].$$

Remark 1.3.5. *Range space of* C *is also called as reachable set or attainable set of the system* (1.3.2)*, and is denoted by* $K(t_0, t_1)$*. Hence the system* (1.3.2) *is controllable over* $\mathbb{R}^n$ *on* $[t_0, t_f],\ t_f < \infty$*, if and only if*

$$K(t_0, t_1) = \text{Range space(C)} = \mathrm{C}\Big(\mathcal{L}^2\big([t_0, t_f];\, \mathbb{R}^m\big)\Big) = \mathbb{R}^n.$$

One could think, is there any way to further simplify the conditions of the Theorem 1.3.1 for the time-invariant case of the system (1.3.2), i.e. for $\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{B}(t) = \mathbf{B}$? The answer is yes. It is a tribute to the genius of an U. S. scientist Rudolf Kalman who recognized this for the first time, and gave the controllability condition of the system (1.3.2) directly in terms of system matrices **A** and **B**. The condition is named after him as Kalman's rank condition for the controllability of linear autonomous system.

Corollary 1.3.1. *If the system* (1.3.2) *is linear time-invariant* (*LTI*)*, i.e. autonomous,* $\mathbf{A}(t) = \mathbf{A}$ *and* $\mathbf{B}(t) = \mathbf{B}$, *then the system* (1.3.2) *is controllable if and only if the augmented matrix of size* $(n \times nm)$ *given by*

$$\mathrm{Q} := \big([\mathbf{B},\ \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]\big)$$

has full rank $= n$.

Proof. Let the system (1.3.2) be controllable, then by Theorem 1.3.1, the linear operator C is surjective, i.e. Range space(C) $= \mathbb{R}^n$. First of all note that $\mathrm{Q} : \mathbb{R}^{mn} \to \mathbb{R}^n$ is a bounded linear operator such that Range space(Q) $\subset \mathbb{R}^n$. Now if we can prove that $\mathbb{R}^n \subset$ Range space(Q), then the necessary condition is established. For this, choose a vector $\mathbf{v} \in \mathbb{R}^n =$ Range space(C), and hence there exists some $\mathbf{u}(\cdot) \in \mathcal{L}^2([t_0, t_f]; \mathbb{R}^m)$ such that $\mathrm{C}\mathbf{u}(\cdot) = \mathbf{v}$. Therefore

$$\begin{aligned} &\int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}\mathbf{u}(s)\mathrm{d}s = \mathbf{v} \\ \Longrightarrow &\int_{t_0}^{t_f} e^{\mathbf{A}(t_f - s)}\mathbf{B}\mathbf{u}(s)\mathrm{d}s = \mathbf{v}. \end{aligned}$$

Using Cayley–Hamilton theorem, the above equation can be written as

$$\int_{t_0}^{t_f} \big[\mathcal{P}_0(s)\mathbf{I}_n + \mathcal{P}_1(s)\mathbf{A} + \cdots + \mathcal{P}_{n-1}(s)\mathbf{A}^{n-1}\big]\mathbf{B}\mathbf{u}(s)\mathrm{d}s = \mathbf{v},$$

where each $\mathcal{P}_i(s)$ is a polynomial function of 's' that appears during the expansion of $e^{\mathbf{A}(t_f - s)}$. This shows that $\mathbf{v} \in$ Range space$\big([\mathbf{B},\ \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]\big) =$ Range space(Q). Since $\mathbf{v}$ is an arbitrary vector of $\mathbb{R}^n$, hence $\mathbb{R}^n \subset$ Range space(Q). Therefore we have

$$\text{Range space(Q)} = \mathbb{R}^n \Longrightarrow \text{rank(Q)} = n.$$

Conversely, let rank(Q) $= n$, i.e. Range space(Q) $= \mathbb{R}^n$, but assume that system (1.3.2) is not controllable. Then by Theorem 1.3.1, the controllability Grammian $\mathbf{W} = \mathrm{CC}^*$ is singular, by which we find some $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^n$ satisfying $\mathbf{W}\mathbf{v} = \mathbf{0} \Longrightarrow \langle \mathbf{W}\mathbf{v},\, \mathbf{v}\rangle_{\mathbb{R}^n} = 0 \Longrightarrow$

$\mathbf{v}^*\mathbf{W}\mathbf{v} = 0$. Therefore

$$\int_{t_0}^{t_f} \mathbf{v}^* e^{\mathbf{A}(t_f - s)} \mathbf{B}\mathbf{B}^* e^{\mathbf{A}^*(t_f - s)} \mathrm{d}s\, \mathbf{v} = 0$$
$$\implies \int_{t_0}^{t_f} \left\| \mathbf{B}^* e^{\mathbf{A}^*(t_f - s)} \mathbf{v} \right\|_{\mathbb{R}^m}^2 \mathrm{d}s = 0.$$

Since $\mathbf{B}^* e^{\mathbf{A}^*(t_f - s)} \mathbf{v}$ is a continuous function on $[t_0, t_f]$, so the above equation implies

$$\mathbf{B}^* e^{\mathbf{A}^*(t_f - s)} \mathbf{v} = \mathbf{0} \in \mathbb{R}^m, \ \ \forall\, s \in [t_0, t_f]$$
$$\implies \mathbf{v}^* e^{\mathbf{A}(t_f - s)} \mathbf{B} = \mathbf{0} \in \mathbb{R}^{1\times m}, \ \ \forall\, s \in [t_0, t_f].$$

At $s = t_f$, the above condition gives $\mathbf{v}^*\mathbf{B} = \mathbf{0} \in \mathbb{R}^{1\times m}$. Further, differentiating with respect to 's' and putting $s = t_f$ each time yields

$$\mathbf{v}^*\mathbf{A}\mathbf{B} = \mathbf{v}^*\mathbf{A}^2\mathbf{B} = \cdots = \mathbf{v}^*\mathbf{A}^{n-1}\mathbf{B} = \mathbf{0} \in \mathbb{R}^{1\times m}.$$

But then $\mathbf{v} \perp$ Range space$\big([\mathbf{B},\, \mathbf{A}\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]\big)$ = Range space(Q) $= \mathbb{R}^n$. In particular $\mathbf{v} \perp \mathbf{v} \implies \langle \mathbf{v},\, \mathbf{v} \rangle_{\mathbb{R}^n} = 0 \implies \mathbf{v} = \mathbf{0}$, which is a contradiction to our assumption that $\mathbf{W}$ is singular. Hence the controllability Grammian $\mathbf{W}$ is positive definite, therefore by Theorem 1.3.1, system (1.3.2) is controllable. □

Let us illustrate how one could apply the Kalman's rank condition to check the controllability by considering an example of a rocket launching control system: Let $x_1(t)$ and $x_2(t)$ denote the altitude and velocity of the rocket relative to the earth at the time $t \geq 0$ and $\mathbf{u}(t) = \frac{f(t)}{m(t)} - g$ be the control applied to this rocket system; here $f(t)$ is the thrust force at the time $t \geq 0$ generated by the propulsion system of the rocket responsible for its motion through an application of Newton's third law of motion, $m(t)$ is the mass of the rocket at the time $t \geq 0$ and g is the acceleration due to gravity which depends on the altitude (and hence on the time). The differential equation governing the dynamics for the vertical motion of the rocket is given by

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t), \quad t \in [0, \infty).$$

Note here that $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By Kalman's rank condition: $\operatorname{rank}([\mathbf{B},\, \mathbf{A}\mathbf{B}]) = 2$, showing that this rocket system is controllable in $(x_1, x_2)-$space on every finite-time interval.

As we commented earlier, now we show that the null controllability of the linear system (1.3.2) implies its controllability in the following

Theorem 1.3.2. *If the linear system* (1.3.2) *is null controllable on* $[t_0, t_f]$, $t_f < \infty$, *over* $\mathbb{R}^n$, *then it is controllable.*

Proof. Let the system (1.3.2) be null controllable. Then for a vector $\mathbf{w}_0 = \mathbf{x}_0 - \mathbf{\Phi}(t_0, t_f)\mathbf{x}_f$ (where $\mathbf{x}_0$, $\mathbf{x}_f \in \mathbb{R}^n$ are arbitrary), there exists some $\mathbf{u}(\cdot) \in \mathcal{L}^2([t_0, t_f]; \mathbb{R}^m)$ such that this $\mathbf{u}(\cdot)$ steers the system (1.3.2) from $\mathbf{w}_0$ to $\mathbf{0}$. Then the solution of the system (1.3.2) given in eq (1.3.3) becomes,

$$\begin{aligned}
\mathbf{0} &= \mathbf{\Phi}(t_f, t_0)\mathbf{w}_0 + \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&= \mathbf{\Phi}(t_f, t_0)\big[\mathbf{x}_0 - \mathbf{\Phi}(t_0, t_f)\mathbf{x}_f\big] + \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
\Longrightarrow \mathbf{x}_f &= \mathbf{\Phi}(t_f, t_0)\mathbf{x}_0 + \int_{t_0}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s = \mathbf{x}(t_f),
\end{aligned}$$

proving that the state of the system (1.3.2) moves from $\mathbf{x}_0$ to $\mathbf{x}_f$, and hence system (1.3.2) is controllable on $[t_0, t_f]$. □

The literature is very rich on the controllability issues for linear dynamical systems and many monographs are devoted towards the study of such systems, for example [16, 23, 105] etc. Unlike the linear systems, the bibliography is not very broad when it comes to the semilinear and nonlinear dynamical control systems, especially with different types of impulses, delays in state and control variables, and with constrained controls. For an overview, refer the survey papers [24, 66]. In the last few decades, much attention has been paid by many people on these systems, and proposed various sufficient conditions on the system parameters leading to the investigation of the controllability of semilinear and nonlinear systems [107], and mostly the fixed-point approaches have been followed in this direction [62]. Let us recall, the semilinear systems are those in which the linear part of the system is separated from its nonlinear term in the differential state equations unlike the nonlinear systems. Further, our literature survey on the controllability of semilinear systems reveals that, many authors have established their controllability under the assumptions—the linear part is controllable and the nonlinear part satisfies certain conditions, and to establish these, the linearization methods and the fixed-point theorems—Schauder's fixed-point theorem, Banach contraction principle, Brouwer fixed-point theorem etc., are used.

1.4 Impulsive systems

Differential equations are used in the modelling of dynamics of many real world phenomena. There are evolution processes which experiences an abrupt changes in their states at certain moments of time. These phenomena involves short term perturbations from their continuous and smooth dynamics, and their duration is negligible in comparison with the duration of the whole process. The dynamics of such behaviour are modelled by impulsive differential equations, i.e. the differential equations exhibiting impulsive effects at certain times in their states and such processes naturally occurs in the modelling of biological phenomena involving thresholds—like the drug resistance models, infectious diseases, ecosystem management, systems with automatic regulation, disturbances in cellular neural networks, industrial robotics, pharmacokinetics, optimal control problems, population dynamics problems, chemical processes, financial systems—like shock changes of the prices in the closed market etc.

The history of impulsive differential equations goes back to the early 20th century, when Pavlidis [89] proposed a dynamical description of pulse frequency modulation involving impulsive effects. The richness in the applications of impulsive differential equations to many of the real world phenomena have attracted the scientific community, and consequently many monographs and research papers appeared on the fundamental and qualitative theory of these systems in the early eighties and then. For a study on elementary theory of differential equations with impulsive effects, see the monographs [20, 69] and references therein.

Let us first describe the most general form of impulsive differential equation in which we are interested. Assume that the law of evolution of the process is described by an $n-$dimensional ordinary differential equation:

$$\left.\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{f}(t, \mathbf{x}(t)), & t &\in [t_0, \infty) \setminus \{t_k : k = 1, 2, \ldots\}, \\ \Delta(\mathbf{x}(t_k)) &:= \mathbf{x}(t_k^+) - \mathbf{x}(t_k^-) = \mathbf{g}(\mathbf{x}(t_k^-)), & t &= t_k, \end{aligned}\right\} \tag{1.4.1}$$

where t is the time, $\mathbf{x}(t)$ is the state, $\mathbf{f}(\cdot) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, t_k, $k \in \mathbb{N}$, such that $t_0 < t_1 < t_2 < t_3 < \cdots$ are the fixed times at which the system shows impulsive behaviour, $\mathbf{g}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ denotes the impulse mapping of the solution to the above system before the impulse, $\mathbf{x}(t_k^-)$, to after the impulse, $\mathbf{x}(t_k^+)$. It is convenient to assume that, the solution to this system is left continuous at each impulse times t_k (refer [1]), i.e. $\mathbf{x}(t_k^-) := \lim_{t \uparrow t_k} \mathbf{x}(t) = \mathbf{x}(t_k)$, $\forall\, k \in \mathbb{N}$. It may also be possible that, the dynamics of the impulsive system is characterized by different impulse functions $\mathbf{g}_k(\cdot)$ at each impulsive time t_k.

To understand how impulses play the role in the dynamical system, consider an example from the literature in which such systems have been used in the applied context: A simple two compartment model for drug distribution in the human body proposed by Kruger–Thiemer is described in [69]. Assuming after the drug is administered orally, it is absorbed into the gastro-intestinal tract. The drug is then absorbed into the so called apparent volume of distribution (a lumped compartment which accounts for blood, muscle, tissue etc.), and finally is eliminated from the body by the kidneys. Let $x_1(t)$ and $x_2(t)$ denote the amount of drug at the time t in the gastro-intestinal tract and apparent volume of distribution with rate constant k_1 and k_2, respectively. The dynamical system of this model is then

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{1.4.2}$$

Let us postulate that the drug is given orally in amounts $\delta_0, \delta_1, \dots, \delta_M$ at the times $t_0 < t_1 < \cdots < t_M < T$ respectively, so that we have

$$\begin{bmatrix} \Delta(x_1(t_i)) \\ \Delta(x_2(t_i)) \end{bmatrix} := \begin{bmatrix} x_1(t_i^+) - x_1(t_i^-) \\ x_2(t_i^+) - x_2(t_i^-) \end{bmatrix} = \begin{bmatrix} \delta_i \\ 0 \end{bmatrix}, \ i = 1, \dots, M. \tag{1.4.3}$$

To achieve a desired therapeutic effect, it is required that the amount of drug in the apparent volume of distribution never goes below a constant level or plateau during the time-interval. System (1.4.2) together with eq (1.4.3) represents an impulsive differential equation. For other models of impulsive differential equations, see [26, 34, 81].

The study of controllability of impulsive systems begun by the works of Leela et al [70] in 1993. After this, several authors explored the investigation of the controllability of different types of impulsive systems. Various sufficient and necessary criteria have been discovered till date, by using both continuous control and impulsive control in the impulsive systems. In the modelling of impulsive control systems, an impulse function $\mathbf{g}(\cdot)$ given in (1.4.1) may also depend on the control function. Such a control function is named as impulsive control, as this control is applied only at that impulsive times. The necessity of introducing the impulsive controls in the dynamics of impulsive control systems are given below [73, 95, 113]:

1. Impulsive controls may be simpler and involves cheaper control mechanisms. For example, in a chemical process system, if we want to control the process of certain chemical reaction in a chemical reactor where the quantities of different chemicals serve as state variables, one can add some chemicals instantaneously to change some of the state variables. In this sense, one can use impulsive control here.

2. In certain cases the plant cannot be controlled by using continuous control. For example, a government cannot change the saving rates of its central bank everyday.

Therefore there are several reasons why impulsive control systems are favoured over continuous differential control systems with continuous perturbations. Our objective is to explore on the controllability properties of impulsive differential equations of particular interest.

1.5 Time-delay systems

Of course many of the dynamical control systems governed by a principle of casuality, i.e. the future state of the system is independent of the past and determined solely by the present. But there are processes whose realistic model also involves the past history of the system, i.e. the involvement of time-delays. A time-delay occurs because a finite-time is required to sense information and then react to it. A simple example is, an illness caused to the human body because of certain parasites. It would take a few days to few months to regain the natural immune system of the body after consuming the right dose of medicines. Another example is, after adding the sugar to a glass of water, it would take at least few seconds to reflect the effect of sugar, i.e. to show the sweetness. Like these many of the processes, both natural and man made, in biology, chemistry, physics, engineering, economics etc., almost certainly involves time-delays which we cannot ignore in the study of their dynamics. Note here that, time-delay occurs from short duration, like few seconds, to very long duration, like years, depending on the phenomena we consider. The dynamics of such evolution behaviour are modelled by a new class of differential equations, called as delay differential equations (DDE).

The systematic study of DDE's begun after the first world war, because of the development and use of automatic control systems. In the context of dynamical control systems of both continuous time-scale and discrete time-scale, time-delay is one of the inevitable problem. There are various forms of delays involves in the control systems—constant delays, variable delays, distributed delays etc., and these can be either internal, i.e. appearing in the state function or they can be external, i.e. appearing in the control input, or in both, depending on the nature of the system. Examples include a delay models in physiology in the context of dynamic diseases, electrodynamics problems, delayed epidemic models, cyclic behaviours, population dynamics, prey–predator population models, fluid dynamics, nonlinear optics, economics, rocket systems, mechanical engineering etc. Among these, the dynamical control systems experiencing the time-delays in control are of specific interest, in which the future of the state is influenced not only by the present value of the control, but also by the past values of it. Such cases arise in gas pressurized biopropellant rocket systems, agricultural economics, population dynamics, harmonic oscillator, and some chemical process systems. For example, an equation of harmonic oscillator with a delayed forcing term is represented by

$$\ddot{x}(t) + k^2 x(t) = u(t) + u(t-h), \quad k^2 > 0;$$

an equation arising in population dynamics is given by

$$\dot{x}(t) = x(t) + \int_0^\infty e^{-\sigma} u(t-\sigma)\mathrm{d}\sigma$$

etc. For many other practical examples where time-delays are involved in control, one can see [3, 4, 102] etc.

Owing to the abundance of mathematical models of dynamical control systems with different types of delays in control, the controllability problem for such systems is very essential. In this respect, several articles have been published over the decades on the controllability properties of the systems involving constant delays, variable delays and distributed delays in control, both on a continuous time-scale and discrete time-scale [85]. Further many of these research papers employed some mapping theorems taken from functional analysis and linear approximation theory to derive the controllability results. Nevertheless, the controllability is not fully investigated on the systems with delays, in particular for the semilinear and nonlinear systems and with constrained controls.

1.6 Thesis outline and contributions overview

This thesis investigates the controllability properties of dynamical systems in a finite-dimensional space on a continuous time-scale, for the impulsive and time-delay systems of the following classes: (i) semilinear impulsive systems with multiple constant time-delays in control, (ii) fractional-order systems of order $\alpha \in (0, 1)$ with multiple constant time-delays in control, (iii) systems modelled by matrix Lyapunov ordinary differential equations possessing impulses and multiple delays in control and (iv) networked impulsive systems.

The objectives of Chapter 3 is to determine the controllability conditions for a class of semilinear dynamical systems modelled by a first-order impulsive ordinary differential systems having multiple constant time-delays in control. For three different classes of nonlinearities and impulse functions, we establish the controllability results under some assumptions on the system parameters. We adopt Schauder's fixed-point theorem and Banach contraction principle to accomplish this task. Numerical examples are provided to show the effectiveness of the theoretical results.

Chapter 4 investigates the controllability issues for a class of dynamical systems modelled by a fractional order $\alpha \in (0, 1)$ semilinear systems possessing multiple constant time-delays in control. Here also for three different classes of nonlinearities, the controllability is investigated under certain assumptions by employing Schauder's fixed-point theorem and Banach contraction principle like in Chapter 3. To support our theoretical results, examples are provided.

Chapter 5 is concerned with the dynamical control systems characterized by matrix

Lyapunov ordinary differential systems whose state function experiences impulses and control function possesses multiple constant delays. The controllability is investigated for two different classes of admissible control functions. The behaviour of the controlled trajectory and control functions are analyzed with a numerical example.

In Chapter 6, the linear impulsive systems is considered for which various necessary and sufficient algebraic criteria for controllability, including the matrix rank conditions are established. These conditions are further synthesized for the time-invariant case of the system, and under some special properties, controllability conditions in terms of the eigenvalues of the system matrix is established. Further, it is shown that, for the impulsive systems, the null controllability need not imply controllability, unlike the linear systems without impulses. Numerical examples are given.

In Chapter 7, the networked impulsive systems are considered for which the controllability result is established, in terms of system matrices using the results of Chapter 6. An easy to verify sufficient condition is obtained in terms of two algebraic matrix equations to determine their controllability. The obtained results are verified with some examples.

Finally some conclusions have been drawn based on our contributions, and thesis concludes with plan for future work.

Chapter 2

Preliminaries

In this chapter, we give some important definitions and recall some lemmas which are used in the thesis.

2.1 Basic definitions

Definition 2.1.1. *A property $\mathcal{P}$ is said to holds almost everywhere (a. e.) on a set A if the following conditions are satisfied:*

(i) The property $\mathcal{P}$ holds on a subset B of A.

(ii) If the property $\mathcal{P}$ fails to satisfy on $A \setminus B$, then the Lebesgue measure of the set $A \setminus B$ is zero.

Definition 2.1.2. *For any $(m \times n)-$matrix* $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ *we define*

$$vec\,\mathbf{A} := \begin{bmatrix} a_{11} & a_{21} \cdots a_{m1} & a_{12} & a_{22} \cdots a_{m2} \cdots a_{1n} & a_{2n} \cdots a_{mn} \end{bmatrix}^T_{mn \times 1}.$$

Then $\|vec\,\mathbf{A}\|_{\mathbb{R}^{mn}} = \|\mathbf{A}\|$, the Frobenius norm of matrix $\mathbf{A}$.

Definition 2.1.3. *Let* $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$ *and $\mathbf{B} \in \mathbb{R}^{p \times q}$ be given two matrices, then the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is denoted and defined by the partitioned* *matrix*

$$\mathbf{A} \otimes \mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}_{mp\times nq} \in \mathbb{R}^{mp\times nq}.$$

The Kronecker product satisfies the following properties [47]:

(i) $(\mathbf{A} \otimes \mathbf{B})^* = (\mathbf{A}^* \otimes \mathbf{B}^*)$,

(ii) $(\mathbf{A} \otimes \mathbf{B})^{-1} = (\mathbf{A}^{-1} \otimes \mathbf{B}^{-1})$,

(iii) $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC} \otimes \mathbf{BD})$, *provided the dimension of the these matrices are compatible with the matrix product,*

(iv) $\frac{d(\mathbf{A}(t)\otimes\mathbf{B}(t))}{dt} = \frac{d(\mathbf{A}(t))}{dt} \otimes \mathbf{B}(t) + \mathbf{A}(t) \otimes \frac{d(\mathbf{B}(t))}{dt}$,

(v) *if* $\mathbf{A}$ *and* $\mathbf{X}$ *are the matrices of order* $n \times n$, *then*

(vi) $vec(\mathbf{AX}) = (\mathbf{I}_n \otimes \mathbf{A})vec(\mathbf{X})$,

(vii) $vec(\mathbf{XA}) = (\mathbf{A}^* \otimes \mathbf{I}_n)vec(\mathbf{X})$,

(viii) $vec(\mathbf{AXB}) = (\mathbf{B}^* \otimes \mathbf{A})vec(\mathbf{X})$.

Definition 2.1.4. *The Gamma function is the simple generalization of the factorial for all real numbers. The definition of the Gamma function is given by*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t, \text{ for all } z \in \mathbb{C}, \ \Re(z) > 0. \tag{2.1.1}$$

The Gamma function has the following recurrence relation:

$$\Gamma(z+1) = z\Gamma(z), \text{ for all } z \in \mathbb{C}, \ \Re(z) > 0.$$

Definition 2.1.5. [54] *The Caputo fractional derivative of order* $\alpha \in (n-1, n)$, $n \in \mathbb{N}$, *for a suitable function* $f(\cdot)$ *is defined as*

$$\left({}^cD_{0+}^{\alpha}f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{d^nf(s)}{ds^n}\mathrm{d}s. \tag{2.1.2}$$

In particular, if $0 < \alpha < 1$, *then*

$$\left({}^cD_{0+}^{\alpha}f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)\mathrm{d}s. \tag{2.1.3}$$

For brevity, the Caputo fractional derivative $\left({}^cD_{0+}^{\alpha}\right)$ *is denoted as* $\left({}^cD_t^{\alpha}\right)$.

The state-transition matrix can be generalized to Mittag-Leffler matrix.

Definition 2.1.6. [54] *The Mittag–Leffler function for an arbitrary $z \in \mathbb{C}$ is*

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \;\; \alpha, \beta > 0, \\ E_{\alpha,1}(z) &= E_{\alpha}(z) \textit{ with } \beta = 1. \end{aligned} \tag{2.1.4}$$

Thus the Mittag–Leffler function is an entire function and it converges for all values of the argument z. For an $(n \times n)-$ matrix $\mathbf{A}$, the matrix extension of the above Mittag–Leffler function is

$$\begin{aligned} E_{\alpha,\beta}(\mathbf{A}t^{\alpha}) &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^{\alpha k}}{\Gamma(\alpha k + \beta)}, \\ E_{\alpha,1}(\mathbf{A}t^{\alpha}) &= E_{\alpha}(\mathbf{A}t^{\alpha}) \textit{ with } \beta = 1. \end{aligned} \tag{2.1.5}$$

The function $t \to E_{\alpha,\beta}(\mathbf{A}t^{\alpha})$ is continuous on $[0, T]$ and there exists a positive constant C such that $E_{\alpha,\beta}(\mathbf{A}t^{\alpha}) \leq C$, for every $t \in [0, T]$. For more information on Mittag–Leffler functions, see [2, 79].

2.2 Basic lemmas

Lemma 2.2.1 (Strong version of Schauder fixed-point theorem [82])**.** *Let $\mathcal{X}$ be a Banach space and let $\mathcal{B} \subset \mathcal{X}$ be a nonempty, closed and convex subset. If $\mathcal{K}$ is a continuous operator from $\mathcal{B}$ into a compact subset of $\mathcal{B}$, then $\mathcal{K}$ has at least one fixed-point in $\mathcal{B}$.*

Lemma 2.2.2 (Banach contraction principle [40])**.** *If $\mathcal{X}$ is a complete metric space and the operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a contraction, then $\mathcal{K}$ has a unique fixed-point in $\mathcal{X}$.*

Chapter 3

Controllability of semilinear impulsive control system with multiple constant time-delays in control

3.1 Introduction

The concept of impulsive systems grew out from the study of evolution processes which occur in physics, chemistry, biology, population dynamics, engineering, information science etc., which are characterized by the fact that, at certain moments of time, the state function experiences a sudden change, in the form of impulses. There has been a significant development in the impulsive theory in the past three decades. For a detailed study on impulsive differential equations, see [69] and the references therein. The study of controllability of impulsive systems has begun in 1993 by the work of Leela et al [70]. The controllability of various types of linear impulsive systems is well established, and many references are available on this in the literature; perhaps, one can see [45, 49, 73, 114].

Unlike the linear systems, not much work has been done for nonlinear systems. However in [41], the authors obtained the controllability conditions of such systems by employing Banach contraction principle. Some other references which are available in the literature in this respect are [84, 117] etc., in which Schaefer's fixed-point theorem was used to obtain the controllability results. But in all these papers, the nonlinear term and the impulse functions involved in the system, depends on time and the state function, but not on the control parameter. In [71], authors studied this case by assuming system's nonlinear term and the impulse functions depends also on the control parameter and obtained the controllability conditions by employing Rothe's type fixed-point theorem. The extension of this result has appeared in [72] for the systems with nonlocal conditions. The controllability of impulsive systems with nonlocal conditions was also studied in [48] and [51] using Monch fixed-point theorem, and in [96] by using Schauder's fixed-point theorem. In [17] and [18], authors have shown that without employing the fixed-point theorem, one can obtain the controllability results for semilinear systems under the boundedness assumption on the nonlinearity.

As we mentioned, there are some chemical process systems, hydraulically actuated systems, combustion systems, population dynamics, harmonic oscillator etc., in which the present value of control function depends upon the past values of it (see [35, 38, 68] and the references therein). Such processes are modelled by delay differential equations having time-delays in the control function. Several mathematicians contributed in the development of controllability of the linear systems involving time-delays in control, for example refer [27]–[29], [53], [56]–[59], [87, 100]. For the nonlinear systems, one can refer [6, 8, 32, 60], [63]–[65], [104].

If an impulsive system involves time-delays in control, the establishment of the controllability of such systems becomes much more complex, because of the coexistence of impulses and delays. However, the linear case of this scenario was explored in [75] and the controllability results was given in terms of a matrix rank condition, which is easy to check whether the system is controllable or not. But we know that most of the problems occurring in real world are not linear in nature. In the existing literature, there is no work reported on the controllability of the nonlinear (in particular semilinear) impulsive system with delays in control. Motivated by this fact, in this chapter we establish the conditions for the controllability of a semilinear impulsive system possessing multiple constant time-delays in control. We obtain the sufficient conditions of controllability for three different classes of the nonlinearities and impulse functions involved in the system. Schauder's fixed-point theorem and Banach contraction principle have been used to establish the results.

In Section 3.2 of this chapter, we formulate the controllability problem of a semilinear impulsive system with multiple constant time-delays in control. A necessary and sufficient condition for the controllability of the corresponding linear system without impulses and with multiple constant time-delays in control in terms of a matrix rank condition is established in Section 3.3. In Section 3.4, we prove that under some sufficient conditions, the corresponding semilinear system is controllable for certain classes of nonlinearities and impulse functions with the help of Schauder's fixed-point theorem and Banach contraction principle. In Section 3.5, numerical examples are given to demonstrate the effectiveness of the proposed results. Finally in Section 3.6, some conclusions have been drawn based on the theoretical results obtained.

3.2 System description

We consider the following dynamical control system modelled by an $n-$dimensional semilinear impulsive ordinary differential equations whose control function experiencing a multiple constant time-delays as

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_i(t)\mathbf{u}(t-h_i) + \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \\ &\qquad\qquad \text{for } t \in [t_0, T] \setminus \{t_k : k = 1, 2, \ldots, M\}, \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \\ \Delta(\mathbf{x}(t_k)) &:= \mathbf{x}(t_k^+) - \mathbf{x}(t_k) = \mathbf{g}_k(\mathbf{x}(t_k), \mathbf{u}(t_k)), \\ \mathbf{u}(t) &= \mathbf{u}_0(t), \;\; t \in [t_0 - h_N, \; t_0), \end{aligned}\right\} \tag{3.2.1}$$

where,

(i) for each t, the state $\mathbf{x}(t) \in \mathbb{R}^n$ with a given initial state $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$,

(ii) for each t, the control $\mathbf{u}(t) \in \mathbb{R}^m$,

(iii) $\mathbf{A}(\cdot) \in \mathcal{C}\big([t_0, T]; \mathbb{R}^{n\times n}\big)$ and $\mathbf{B}_i(\cdot) \in \mathcal{C}\big([t_0, T]; \mathbb{R}^{n\times m}\big)$ are the given matrix valued functions,

(iv) $t_0 \le t_1 \le t_2 \le \cdots \le t_M < T$, t_k's are the fixed times at which the state function $\mathbf{x}(\cdot)$ experiences impulses and are state independent,

(v) $0 \le h_1 \le h_2 \le \cdots \le h_N \le \min\big\{(t_1 - t_0), (t_2 - t_1), \ldots, (t_M - t_{M-1}), (T - t_M)\big\}$, h_i's are the known constant time-delays in the control function $\mathbf{u}(\cdot)$,

(vi) $\Delta(\mathbf{x}(t_k))$ is an impulse in the state function $\mathbf{x}(\cdot)$ at the time t_k,

(vii) $\mathbf{u}_0(\cdot) \in \mathcal{C}\big([t_0 - h_N, \; t_0); \mathbb{R}^m\big)$ denotes a given initial control function (and is assumed to be bounded on its domain) applied to the system (3.2.1),

(viii) the function $\mathbf{f}(\cdot, \cdot, \cdot) \in \mathcal{C}([t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ is nonlinear in its second argument and each $\mathbf{g}_k(\cdot, \cdot) \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ is either linear or nonlinear.

Before proceeding, we will make sure that this system is solvable for any initial state $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ and for a given function $\mathbf{u}(\cdot)$. Note that the solvability of the system (3.2.1) is similar to the solvability of the following initial value problem (3.2.2) having no impulses:

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}(t, \mathbf{x}(t)), \quad t \in [t_0, T], \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned}\right\} \tag{3.2.2}$$

where $\mathbf{F}(\cdot, \cdot) \in \mathcal{C}([t_0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. This is because, when system (3.2.2) possesses a unique solution, then its corresponding impulsive system with finite impulses also admits a unique solution, but having jumps in the state function at the impulse times (refer [69], Corollary 2.2.1, pp. 64).

First let us discuss the solvability of system (3.2.2) on $[t_0, T]$. In (3.2.2), if we assume that $\mathbf{F}(\cdot, \cdot)$ is a continuous on $[t_0, T] \times \mathbb{R}^n$ and satisfies Lipschitz condition with respect to the second argument $\mathbf{x}(t)$, then (3.2.2) has a unique solution on the interval $[t_0, T]$ (refer [30], Theorem 4, pp. 252). But note here that, these are just sufficient conditions, not necessary, for the existence of a unique solution to the system (3.2.2) on $[t_0, T]$.

Now, for the system (3.2.1), if we assume $\mathbf{f}(\cdot, \cdot, \cdot)$ satisfies a Lipschitz condition with respect to the second argument on its domain $[t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ (then the right hand side of (3.2.1) also satisfy a Lipschitz condition with respect to $\mathbf{x}$), then (3.2.1) has a unique solution on $[t_0, T]$ for a given $\mathbf{u}(\cdot)$. Note that Lipschitz functions are either bounded or unbounded on their domain. However there exist some bounded continuous functions $\mathbf{f}(\cdot, \cdot, \cdot)$ which do not satisfy a Lipschitz condition (then of course the right hand side of $\dot{\mathbf{x}}(t)$ also do not satisfy a Lipschitz condition), but still the system (3.2.1) admits a unique solution on $[t_0, T]$ for a given $\mathbf{u}(\cdot)$. Similarly, there exist some linear growth continuous functions $\mathbf{f}(\cdot, \cdot, \cdot)$ which are unbounded and do not satisfy a Lipschitz condition, but (3.2.1) still has a unique solution on $[t_0, T]$ for a given $\mathbf{u}(\cdot)$. In this chapter, we consider all the three cases and assume that our system (3.2.1) admits a unique solution on $[t_0, T]$ for a given $\mathbf{u}(\cdot)$.

We now introduce the solution space for the system (3.2.1) as

$$\begin{aligned}\mathcal{B}_1 :=\Big\{ &\mathbf{x}(\cdot) \,\big|\, \mathbf{x}(\cdot) : [t_0, T] \to \mathbb{R}^n,\ \mathbf{x}(\cdot) \text{ is a continuous on } [t_0, T] \setminus \{t_k : k = 1, 2, \ldots, M\} \\ &\text{and differentiable a.e. on } [t_0, T] \text{ such that } \exists \text{ a left limit } \mathbf{x}(t_k^-) := \lim_{t \uparrow t_k} \mathbf{x}(t) \text{ and a right} \\ &\text{limit } \mathbf{x}(t_k^+) := \lim_{t \downarrow t_k} \mathbf{x}(t) \text{ with } \mathbf{x}(t_k^-) = \mathbf{x}(t_k), \forall\, k \text{ and } \mathbf{x}(t_0) = \lim_{t \downarrow t_0} \mathbf{x}(t) \Big\}\end{aligned}$$

and we define a space for admissible control functions as

$$\mathcal{B}_2 := \Big\{ \mathbf{u}(\cdot) \,\big|\, \mathbf{u}(\cdot) : [t_0, T] \to \mathbb{R}^m, \mathbf{u}(\cdot) \text{ is continuous a.e. and bounded on } [t_0, T] \Big\}.$$

One can readily check that these two spaces are real Banach spaces endowed with the norms

$$\|\mathbf{x}(\cdot)\|_{\mathcal{B}_1} := \sup_{t \in [t_0, T]} \|\mathbf{x}(t)\|_{\mathbb{R}^n} \text{ and } \|\mathbf{u}(\cdot)\|_{\mathcal{B}_2} := \sup_{t \in [t_0, T]} \|\mathbf{u}(t)\|_{\mathbb{R}^m},$$

respectively.

Let us recall the controllability for the system (3.2.1) as

Definition 3.2.1. *The system* (3.2.1) *is said to be controllable over* $\mathbb{R}^n$ *on* $[t_0, T]$*, if for every pair of vectors* $(\mathbf{x}_0, \mathbf{x}_T) \in \mathbb{R}^n \times \mathbb{R}^n$ *and for every bounded function* $\mathbf{u}_0(\cdot) \in \mathcal{C}\big([t_0 - h_N, t_0)\,; \mathbb{R}^m\big)$ *there exists at least one control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$ *such that, with this control function on* $[t_0, T]$*, the corresponding solution to the system* (3.2.1) *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *and* $\mathbf{u}(t) = \mathbf{u}_0(t)$*,* $t \in [t_0 - h_N, t_0)$*, satisfies the condition* $\mathbf{x}(T) = \mathbf{x}_T$*.*

3.3 Controllability of the linear system without impulses and with multiple constant time-delays in control

In this section, we derive a necessary and sufficient condition for the controllability of the corresponding linear system (3.3.1) without impulses and with multiple constant time-delays in the control function. The associated linear system of (3.2.1) without impulses is given by

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_i(t)\mathbf{u}(t-h_i),\ t \in [t_0, T], \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \\ \mathbf{u}(t) &= \mathbf{u}_0(t), \quad t \in [t_0 - h_N,\ t_0). \end{aligned}\right\} \tag{3.3.1}$$

Let $\mathbf{\Phi}(t)$ be the fundamental matrix solution of the homogeneous system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, and hence $\mathbf{\Phi}(t,s) := \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$ is the state-transition matrix generated by $\mathbf{A}(t)$. Now, the solution to the linear system (3.3.1) at any time $t \in [t_0, T]$ is given by (refer [23])

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t,s) \sum_{i=1}^{N} \mathbf{B}_i(s)\mathbf{u}(s-h_i)\mathrm{d}s \\ &= \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \mathbf{\Phi}(t,t_0) \sum_{i=1}^{N} \int_{t_0-h_i}^{t_0} \mathbf{\Phi}(t_0, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}_0(s)\mathrm{d}s \\ &\quad + \sum_{i=1}^{N} \int_{t_0}^{t-h_i} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s. \end{aligned}$$

Let us denote

$$\sum_{i=1}^{N} \int_{t_0-h_i}^{t_0} \mathbf{\Phi}(t_0, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}_0(s)\mathrm{d}s = \mathbf{a}_0 \in \mathbb{R}^n, \tag{3.3.2}$$

therefore we have,

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \sum_{i=1}^{N} \int_{t_0}^{t-h_i} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathbf{d}s. \tag{3.3.3}$$

Now let us simplify the summation given in equation (3.3.3) as

$$\begin{aligned}\sum_{i=1}^{N}\int_{t_0}^{t-h_i}&\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s\\ &=\int_{t_0}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s\\ &\quad+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s.\end{aligned} \tag{3.3.4}$$

Using eq (3.3.4) in eq (3.3.3), the solution to the system (3.3.1) can be written as

$$\begin{aligned}\mathbf{x}(t)=\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)&+\int_{t_0}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s\\ +\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}&\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s.\end{aligned} \tag{3.3.5}$$

Now we introduce the following matrices which helps in the establishment of the controllability criteria for the system (3.3.1):

$$\begin{aligned}\mathbf{W}_l:=\mathbf{W}_l(T)&=\int_{T-h_{l+1}}^{T-h_l}\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]\\ &\quad\times\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]^*\mathrm{d}s,\\ \mathbf{W}_N:=\mathbf{W}_N(T)&=\int_{t_0}^{T-h_N}\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]\\ &\quad\times\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]^*\mathrm{d}s,\end{aligned} \tag{3.3.6}$$

where $l=1,2,\ldots,(N-1)$. Let us investigate the properties on ranks of these matrices. This is furnished in the following

Lemma 3.3.1. *Each $\mathbf{W}_i$ given in eqs (3.3.6) is a positive semidefinite symmetric matrix of size-$(n\times n)$ and* $\operatorname{rank}([\mathbf{W}_1,\ \mathbf{W}_2,\ldots,\mathbf{W}_N])=\operatorname{rank}(\mathbf{W}_1+\mathbf{W}_2+\cdots+\mathbf{W}_N)$.

Proof. Let $\mathbf{P}(s)$ be any $(n\times m)-$matrix valued function with each of its entry is a real valued continuous function of s. Denote $\mathbf{P}^*(s)$ for the transpose of this matrix. Then for each fixed $s\in[t_0,T]$, for all $\mathbf{v}\in\mathbb{R}^n$ and under the usual inner product on $\mathbb{R}^n$, we have

$$\Big\langle\mathbf{P}(s)\mathbf{P}^*(s)\mathbf{v},\ \ \mathbf{v}\Big\rangle_{\mathbb{R}^n}=\Big\langle\mathbf{P}^*(s)\mathbf{v},\ \ \mathbf{P}^*(s)\mathbf{v}\Big\rangle_{\mathbb{R}^m}=\big\|\mathbf{P}^*(s)\mathbf{v}\big\|_{\mathbb{R}^m}^2\geq 0,$$

which shows that $\mathbf{P}(s)\mathbf{P}^*(s)$ is positive semidefinite $(n \times n)-$symmetric matrix for each $s \in [t_0, T]$. Now for $\alpha < \beta$, let us consider

$$\left\langle \int_\alpha^\beta \mathbf{P}(s)\mathbf{P}^*(s)\mathrm{d}s\, \mathbf{v},\ \mathbf{v} \right\rangle_{\mathbb{R}^n} = \int_\alpha^\beta \big(\mathbf{P}^*(s)\mathbf{v}\big)^*\big(\mathbf{P}^*(s)\mathbf{v}\big)\mathrm{d}s = \int_\alpha^\beta \big\|\mathbf{P}^*(s)\mathbf{v}\big\|_{\mathbb{R}^m}^2 \mathrm{d}s \geq 0,$$

which easily shows that $\int_\alpha^\beta \mathbf{P}(s)\mathbf{P}^*(s)\mathrm{d}s$ is positive semidefinite symmetric $(n \times n)-$matrix. Therefore each $\mathbf{W}_i$ given in eqs (3.3.6) is a positive semidefinite symmetric $(n \times n)-$matrix. Also we know that

$$\Big\langle (\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)\mathbf{v},\ \mathbf{v} \Big\rangle_{\mathbb{R}^n} = \big\langle \mathbf{W}_1\mathbf{v},\ \mathbf{v}\big\rangle_{\mathbb{R}^n} + \big\langle \mathbf{W}_2\mathbf{v},\ \mathbf{v}\big\rangle_{\mathbb{R}^n} + \cdots + \big\langle \mathbf{W}_N\mathbf{v},\ \mathbf{v}\big\rangle_{\mathbb{R}^n} \geq 0,$$

for all $\mathbf{v} \in \mathbb{R}^n$, which shows that $(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)$ is a positive semidefinite symmetric $(n \times n)-$matrix.

It remains to prove that $\mathrm{rank}(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N) = \mathrm{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N])$. This follows from the following fact:

$$\begin{aligned}
\mathbf{v} \in \ker\big((\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)^*\big) &\Longleftrightarrow (\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)^*(\mathbf{v}) = \mathbf{0} \\
&\Longleftrightarrow \mathbf{W}_i^*(\mathbf{v}) = \mathbf{0},\ \text{for all } i,\ \text{as each } \mathbf{W}_i \text{ is a positive} \\
&\qquad\qquad\qquad \text{semidefinite matrix} \\
&\Longleftrightarrow \mathbf{v} \in \ker(\mathbf{W}_i^*),\ \text{for all } i \\
&\Longleftrightarrow \mathbf{v} \in \ker \begin{bmatrix} \mathbf{W}_1^* \\ \mathbf{W}_2^* \\ \vdots \\ \mathbf{W}_N^* \end{bmatrix}_{Nn \times n} \\
&\Longleftrightarrow \mathbf{v} \in \ker([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N]^*).
\end{aligned}$$

Therefore by using the rank–nullity theorem, we have

$$\begin{aligned}
&\ker\big((\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)^*\big) = \ker([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N]^*) \\
\Longrightarrow\ & n - \mathrm{rank}\big((\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)^*\big) = n - \mathrm{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N]^*) \\
\Longrightarrow\ & \mathrm{rank}(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N) = \mathrm{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N]),
\end{aligned}$$

which completes the proof. □

Now we will be able to establish the controllability condition for the linear system (3.3.1).

Theorem 3.3.1. *The system* (3.3.1) *is controllable over* $\mathbb{R}^n$ *on* $[t_0, T]$ *if and only if*

$$\mathrm{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N]) = n.$$

Proof. In order to show the sufficiency, let us assume that $\text{rank}(\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_N) = n$. Therefore from Lemma 3.3.1, it is clear that the matrix $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N$ is positive definite. Let us define a control function as follows:

$$\mathbf{u}(t) := \begin{cases} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i)\right]^* \mathbf{W}^{-1}[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0)], \\ \qquad\qquad \text{for } t \in [t_0, T-h_N], \\ \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i)\right]^* \mathbf{W}^{-1}[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0)], \\ \qquad\qquad \text{for } t \in (T-h_{l+1}, T-h_l], \\ \mathbf{0}, \qquad\qquad \text{for } t \in (T-h_1, T], \end{cases} \tag{3.3.7}$$

where $l = 1, 2, \dots, (N-1)$. The state $\mathbf{x}(t)$ given in eq (3.3.5) at $t = T$ becomes

$$\mathbf{x}(T) = \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{T-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s$$
$$+ \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s.$$

Substituting $\mathbf{u}(t)$ from eq (3.3.7) in the above expression, we get

$$\begin{aligned} \mathbf{x}(T) &= \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \Bigg\{ \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right] \\ &\quad \times \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s + \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right] \\ &\quad \times \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0)] \\ &= \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \big\{\mathbf{W}_N + \cdots + \mathbf{W}_1\big\}\mathbf{W}^{-1}[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0)] \\ &= \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \mathbf{W}\mathbf{W}^{-1}[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0)] = \mathbf{x}_T. \end{aligned}$$

Hence the system (3.3.1) is controllable over $\mathbb{R}^n$ on $[t_0, T]$.

The converse can be proved by contradiction. Let the system (3.3.1) be controllable on $[t_0, T]$, but assume that $\text{rank}([\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_N]) < n$. Then from Lemma 3.3.1, we know that $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N$ is a singular matrix. Thus, there exists at least one non-zero vector, say $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{W}\mathbf{v} = \mathbf{0}$, i.e.

$$(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N)\mathbf{v} = \mathbf{0} \Longrightarrow \mathbf{W}_1\mathbf{v} + \mathbf{W}_2\mathbf{v} + \cdots + \mathbf{W}_N\mathbf{v} = \mathbf{0}.$$

Hence $\mathbf{W}_i\mathbf{v} = \mathbf{0}$ for all i (since each $\mathbf{W}_i$ is positive semidefinite matrix). This shows that each $\mathbf{W}_i$ is a singular matrix and $\langle \mathbf{W}_i\mathbf{v}, \mathbf{v}\rangle_{\mathbb{R}^n} = 0$, for all i, i.e.

$$\begin{cases} \left\langle \displaystyle\int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s\, \mathbf{v},\ \mathbf{v}\right\rangle_{\mathbb{R}^n} = 0, \\ \left\langle \displaystyle\int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s\, \mathbf{v},\ \mathbf{v}\right\rangle_{\mathbb{R}^n} = 0. \end{cases}$$

$$\Longrightarrow \begin{cases} \displaystyle\int_{T-h_{l+1}}^{T-h_l} \left\| \sum_{i=1}^{l} \mathbf{B}_i^*(s+h_i)\mathbf{\Phi}^*(T, s+h_i)\mathbf{v}\right\|_{\mathbb{R}^m}^2 \mathrm{d}s = 0, \text{ for all } l = 1, 2, \ldots, (N-1), \\ \displaystyle\int_{t_0}^{T-h_N} \left\| \sum_{i=1}^{N} \mathbf{B}_i^*(s+h_i)\mathbf{\Phi}^*(T, s+h_i)\mathbf{v}\right\|_{\mathbb{R}^m}^2 \mathrm{d}s = 0. \end{cases}$$

Since each $\mathbf{B}_i^*(\cdot)$ and $\mathbf{\Phi}^*(\cdot,\cdot)$ are continuous functions, so the above integrals implies that

$$\mathbf{v}^*\left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right] = \mathbf{0},\ \forall\, l = 1, 2, \ldots, N \text{ and some } \mathbf{v} \neq \mathbf{0} \in \mathbb{R}^n. \qquad (3.3.8)$$

We assumed that the system (3.3.1) is controllable on $[t_0, T]$, in particular the system is null controllable. Now, let us choose an initial state $\mathbf{x}_0 = -\mathbf{a}_0 + \mathbf{\Phi}^{-1}(T, t_0)\mathbf{v}$ and a final state $\mathbf{x}(T) = \mathbf{0}$. Then with some control $\mathbf{u}(\cdot)$, the state of the system (3.3.1) given in eq (3.3.5) satisfies $\mathbf{x}(T) = \mathbf{0}$, i.e.

$$\begin{aligned} \mathbf{0} = \mathbf{x}(T) &= \mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &= \mathbf{\Phi}(T, t_0)\mathbf{\Phi}^{-1}(T, t_0)\mathbf{v} + \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbf{v} = &- \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &- \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s. \end{aligned}$$

Premultiply the above equation with $\mathbf{v}^*$, we get

$$\begin{aligned}\mathbf{v}^*\mathbf{v} = &-\int_{t_0}^{T-h_N} \mathbf{v}^* \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &-\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l} \mathbf{v}^* \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]\mathbf{u}(s)\mathrm{d}s.\end{aligned}$$

Using eq (3.3.8) in the above equation, we get $\mathbf{v}^*\mathbf{v} = 0 \implies \left\|\mathbf{v}\right\|^2_{\mathbb{R}^n} = 0 \implies \mathbf{v} = \mathbf{0}$, which is a contradiction. Hence our assumption that $\mathrm{rank}([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_N]) < n$ cannot hold true. Further we have $\mathrm{rank}([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_N]) = n$. □

For the time-invariant system, the controllability condition in terms of system matrices is given by a generalized Kalman's rank condition, and is proved in the following

Theorem 3.3.2. *If the linear system* (3.3.1) *is autonomous, i.e. time-invariant, then the necessary and sufficient condition for the controllability of this system* (3.3.1) *given by*

$$\mathrm{rank}(\mathrm{Q}) = n,$$

where

$$\mathrm{Q} := \left[\mathbf{B}_1,\, \mathbf{A}\mathbf{B}_1, \ldots, \mathbf{A}^{n-1}\mathbf{B}_1,\, \mathbf{B}_2,\, \mathbf{A}\mathbf{B}_2, \ldots, \mathbf{A}^{n-1}\mathbf{B}_2, \ldots, \mathbf{B}_N,\, \mathbf{A}\mathbf{B}_N, \ldots, \mathbf{A}^{n-1}\mathbf{B}_N\right].$$

Proof. First of all note that $\mathrm{Q} : \mathbb{R}^{Nnm} \to \mathbb{R}^n$ is a linear operator. We first show that the condition is necessary. Let the system be controllable on $[t_0, T]$ and choose a vector $\mathbf{v} \in \mathbb{R}^n$. Then with an initial state $\mathbf{x}_0 = -\mathbf{a}_0$ and a final state $\mathbf{x}(T) = \mathbf{v}$, there exists a control function for the system (3.3.1), which steers the state from $-\mathbf{a}_0$ to $\mathbf{v}$. Hence the state given in eq (3.3.3) at $t = T$ becomes:

$$\begin{aligned}\mathbf{v} = \mathbf{x}(T) &= \int_{t_0}^{T-h_N} \sum_{i=1}^{N} e^{\mathbf{A}(T-t-h_i)}\mathbf{B}_i\mathbf{u}(t)\mathrm{d}t + \sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} e^{\mathbf{A}(T-t-h_i)}\mathbf{B}_i\mathbf{u}(t)\mathrm{d}t \\ &= \sum_{i=1}^{N}\int_{t_0}^{T-h_i} e^{\mathbf{A}(T-t-h_i)}\mathbf{B}_i\mathbf{u}(t)\mathrm{d}t.\end{aligned}$$

We expand each $e^{\mathbf{A}(T-t-h_i)}$ by Cayley–Hamilton's theorem to obtain

$$\mathbf{v} = \sum_{i=1}^{N}\int_{t_0}^{T-h_i} \left[\mathcal{P}_0^{(i)}(t)\mathbf{I}_n + \mathcal{P}_1^{(i)}(t)\mathbf{A} + \mathcal{P}_2^{(i)}(t)\mathbf{A}^2 + \cdots + \mathcal{P}_{n-1}^{(i)}(t)\mathbf{A}^{n-1}\right]\mathbf{B}_i\mathbf{u}(t)\mathrm{d}t,$$

where each $\mathcal{P}_j^{(i)}(t)$ is a polynomial function of 't' that appears during the expansion of $e^{\mathbf{A}(T-t-h_i)}$. The above equation shows that $\mathbf{v} \in$ Range space(Q). Hence

$\mathbb{R}^n \subset$ Range space(Q) $\implies$ rank(Q) $= n$.

We prove the sufficiency condition by contradiction. Let rank(Q) $= n$, and assume that the system (3.3.1) is not controllable on $[t_0, T]$. Then by Theorem 3.3.1 we have

$$\text{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_N]) < n,$$

and we arrive at eq (3.3.8). But for an autonomous system, eq (3.3.8) reduces to

$$\left.\begin{aligned} &\mathbf{v}^* e^{\mathbf{A}(T-t-h_1)}\mathbf{B}_1 = \mathbf{0},\ \forall\, t \in (T-h_2, T-h_1], \\ &\mathbf{v}^* \left[\sum_{i=1}^{2} e^{\mathbf{A}(T-t-h_i)}\mathbf{B}_i\right] = \mathbf{0},\ \forall\, t \in (T-h_3, T-h_2], \\ &\qquad\vdots \\ &\mathbf{v}^* \left[\sum_{i=1}^{N} e^{\mathbf{A}(T-t-h_i)}\mathbf{B}_i\right] = \mathbf{0},\ \forall\, t \in [t_0, T-h_N]. \end{aligned}\right\} \tag{3.3.9}$$

Let us first consider $\mathbf{v}^* e^{\mathbf{A}(T-t-h_1)}\mathbf{B}_1 = \mathbf{0},\ \ \forall\, t \in (T-h_2, T-h_1]$. For $t = T - h_1$, we get $\mathbf{v}^*\mathbf{B}_1 = \mathbf{0}$. Now differentiating $\mathbf{v}^* e^{\mathbf{A}(T-t-h_1)}\mathbf{B}_1 = \mathbf{0}$ with respect to t for $(n-1)-$times and then putting $t = T - h_1$, we obtain $\mathbf{v}^*(\mathbf{A}\mathbf{B}_1) = \mathbf{v}^*(\mathbf{A}^2\mathbf{B}_1) = \cdots = \mathbf{v}^*(\mathbf{A}^{n-1}\mathbf{B}_1) = \mathbf{0}$. Next consider $\mathbf{v}^*\big[e^{\mathbf{A}(T-t-h_1)}\mathbf{B}_1 + e^{\mathbf{A}(T-t-h_2)}\mathbf{B}_2\big] = \mathbf{0},\ \ \forall\, t \in (T-h_3, T-h_2]$. For $t = T-h_2$, we obtain $\mathbf{v}^*\big[e^{\mathbf{A}(h_2-h_1)}\mathbf{B}_1 + \mathbf{B}_2\big] = \mathbf{0}$. Using $\mathbf{v}^*\mathbf{B}_1 = \mathbf{v}^*(\mathbf{A}\mathbf{B}_1) = \cdots = \mathbf{v}^*(\mathbf{A}^{n-1}\mathbf{B}_1) = \mathbf{0}$, we find $\mathbf{v}^*\mathbf{B}_2 = \mathbf{0}$. Differentiating $\mathbf{v}^*\big[e^{\mathbf{A}(T-t-h_1)}\mathbf{B}_1 + e^{\mathbf{A}(T-t-h_2)}\mathbf{B}_2\big] = \mathbf{0}$ with respect to t for $(n-1)-$times and putting $t = T - h_2$ and again using $\mathbf{v}^*(\mathbf{A}\mathbf{B}_1) = \mathbf{v}^*(\mathbf{A}^2\mathbf{B}_1) = \cdots = \mathbf{v}^*(\mathbf{A}^{n-1}\mathbf{B}_1) = \mathbf{0}$, we get $\mathbf{v}^*(\mathbf{A}\mathbf{B}_2) = \mathbf{v}^*(\mathbf{A}^2\mathbf{B}_2) = \cdots = \mathbf{v}^*(\mathbf{A}^{n-1}\mathbf{B}_2) = \mathbf{0}$. Continuing this process, we finally obtain $\mathbf{v}^*\mathbf{B}_N = \mathbf{v}^*(\mathbf{A}\mathbf{B}_N) = \cdots = \mathbf{v}^*(\mathbf{A}^{n-1}\mathbf{B}_N) = \mathbf{0}$. Therefore

$$\mathbf{v} \perp \text{Range space}\big(\mathbf{B}_1,\ \mathbf{A}\mathbf{B}_1, \ldots, \mathbf{A}^{n-1}\mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{A}\mathbf{B}_2, \ldots, \mathbf{A}^{n-1}\mathbf{B}_2, \ldots, \mathbf{B}_N,\ \mathbf{A}\mathbf{B}_N, \ldots, \mathbf{A}^{n-1}\mathbf{B}_N\big),$$

i.e. $\mathbf{v} \perp \mathbb{R}^n$. In particular, $\mathbf{v} \perp \mathbf{v}$. Hence $\langle \mathbf{v}, \mathbf{v}\rangle_{\mathbb{R}^n} = 0 \implies \|\mathbf{v}\|^2_{\mathbb{R}_n} = 0 \implies \mathbf{v} = \mathbf{0}$. This is a contradiction and hence the system (3.3.1) is controllable on $[t_0, T]$ if and only if rank(Q) $= n$. □

Remark 3.3.1. *In the system* (3.3.1)*, if delays are absent in the control, i.e.* $h_i = 0,\ \forall\, i$, *then* $\mathbf{W}_1 = \cdots = \mathbf{W}_{N-1} = \mathbf{O}$ *and* $\mathbf{W}_N = \int_{t_0}^{T} \left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{B}_i(s)\right]\left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{B}_i(s)\right]^* \mathrm{d}s$. *Then we have* $\mathbf{W} = \sum_{i=1}^{N}\mathbf{W}_i = \mathbf{W}_N$*, and this matrix is called the controllability Grammian of the linear system* (3.3.1) *with no delays, and such system is controllable on* $[t_0, T]$ *if and only if* rank($\mathbf{W}$) = rank($[\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{N-1},\ \mathbf{W}_N]$) = rank($\mathbf{W}_N$) $= n$, *i.e. the controllability Grammian* $\mathbf{W}$ *is positive definite.*

Corollary 3.3.1. *It can be easily shown that, our necessary and sufficient condition given in Theorem 3.3.1 for controllability of the linear system* (3.3.1) *is equivalent to the condition given in Corollary 3.9 of Liu and Zhou* [75]*, in which, the authors provide controllability condition:*

$$\operatorname{rank}(\mathcal{C}_1) = n,$$

where

$$\begin{aligned}\mathcal{C}_1 = \Bigg(& \int_{t_0}^{t_1-h_N} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\ \int_{t_1-h_N}^{t_1-h_{N-1}} \mathbf{V}\mathbf{V}^* \mathrm{d}s, \dots, \int_{t_1-h_2}^{t_1-h_1} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\ \int_{t_1-h_1}^{t_2-h_N} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\\ & \int_{t_2-h_N}^{t_2-h_{N-1}} \mathbf{V}\mathbf{V}^* \mathrm{d}s, \dots, \int_{t_2-h_2}^{t_2-h_1} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\ \int_{t_{M-1}-h_1}^{t_M-h_N} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\ \int_{t_M-h_N}^{t_M-h_{N-1}} \mathbf{V}\mathbf{V}^* \mathrm{d}s, \dots,\\ & \int_{t_M-h_2}^{t_M-h_1} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\ \int_{t_M-h_1}^{T-h_N} \mathbf{V}\mathbf{V}^* \mathrm{d}s,\ \mathbf{W}_{N-1},\ \mathbf{W}_{N-2}, \dots, \mathbf{W}_1 \Bigg),\end{aligned}$$

where $\mathbf{V} = \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)$*. As we are dealing with the linear delay system without impulses, the matrix* $\mathcal{C}_1$ *reduces to* $[\mathbf{W}_N,\ \mathbf{W}_{N-1}, \dots, \mathbf{W}_1]$*.*

Remark 3.3.2. *It is always true that, if at least one of the* $\mathbf{W}_i$ *is of full rank* n*, then* $\operatorname{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \dots, \mathbf{W}_N]) = n$*. Thus* $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N$ *is positive definite, and by Theorem 3.3.1, system* (3.3.1) *is controllable on* $[t_0, T]$*. But if*

$$\operatorname{rank}([\mathbf{W}_1,\ \mathbf{W}_2, \dots, \mathbf{W}_N]) = n,$$

then it may be possible that none of $\mathbf{W}_i$ *has full rank* n*. For example, let* $\mathbf{W}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$*,* $\mathbf{W}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$*, and both have rank* $= 1$*, but the augmented matrix* $[\mathbf{W}_1,\ \mathbf{W}_2] = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$ *has full rank* $= 2$*. Therefore, we conclude that even if none of* $\mathbf{W}_i$ *are positive definite matrices, the system* (3.3.1) *is still controllable on* $[t_0, T]$*.*

3.4 Main controllability results for the nonlinear system

This last section is devoted for the investigation of the controllability for the semilinear impulsive delay system (3.2.1). For the sake of convenience this is divided into three subsections; in each one, we present controllability results for the system (3.2.1) under the different classes of nonlinearities $\mathbf{f}(\cdot,\cdot,\cdot)$ and impulse functions $\mathbf{g}_k(\cdot,\cdot)$. Before proceeding it is advisable to assume that the semilinear impulsive delay system (3.2.1) admits one and only one solution for any initial state $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ and for any $\mathbf{u}(\cdot) \in \mathcal{B}_2$, and the linear delay system (3.3.1) is controllable over $\mathbb{R}^n$ on $[t_0, T]$. Here for a given initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and for a given $\mathbf{u}(\cdot)$, the solution to the system (3.2.1) satisfies the following equation:

$$\mathbf{x}(t) = \begin{cases} \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \displaystyle\int_{t_0}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s \\ + \displaystyle\sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s \\ + \displaystyle\int_{t_0}^{t} \mathbf{\Phi}(t,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s, \text{ for all } t \in [t_0, t_1], \\ \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \displaystyle\int_{t_0}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s \\ + \displaystyle\sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\mathbf{u}(s)\mathrm{d}s \\ + \displaystyle\int_{t_0}^{t} \Phi(t,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s \\ + \displaystyle\sum_{j=1}^{k} \mathbf{\Phi}(t,t_j)\mathbf{g}_j(\mathbf{x}(t_j),\mathbf{u}(t_j)), \text{ for all } t \in (t_k, t_{k+1}], \end{cases} \tag{3.4.1}$$

where $k = 1, 2, \ldots, M$ and $t_{M+1} = T$.

Now we define a real Banach space by

$$\mathcal{X} := \mathcal{B}_1 \times \mathcal{B}_2 = \big\{(\mathbf{x},\mathbf{u}) : \mathbf{x} \in \mathcal{B}_1, \;\; \mathbf{u} \in \mathcal{B}_2\big\},$$

endowed with the norm

$$\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} := \|\mathbf{x}\|_{\mathcal{B}_1} + \|\mathbf{u}\|_{\mathcal{B}_2}.$$

Further defining an operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{K}(\mathbf{x},\mathbf{u}) := \big(\mathcal{K}_1(\mathbf{x},\mathbf{u}), \mathcal{K}_2(\mathbf{x},\mathbf{u})\big) = (\boldsymbol{y}, \boldsymbol{v}), \tag{3.4.2}$$

where $\mathcal{K}_1 : \mathcal{X} \to \mathcal{B}_1$ is defined by

$$\begin{aligned}
&\mathcal{K}_1(\mathbf{x},\mathbf{u})(t) = \boldsymbol{y}(t) \\
&:= \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \Bigg\{ \int_{t_0}^{t-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) \\
&+ \int_{t_0}^{t} \mathbf{\Phi}(t,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathbf{d}s, \text{ for all } t \in [t_0, t_1],
\end{aligned} \tag{3.4.3}$$

$$\begin{aligned}
&\mathcal{K}_1(\mathbf{x},\mathbf{u})(t) = \boldsymbol{y}(t) \\
&:= \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \Bigg\{ \int_{t_0}^{t-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) \\
&+ \int_{t_0}^{t} \mathbf{\Phi}(t,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s + \sum_{j=1}^{k} \mathbf{\Phi}(t,t_j)\mathbf{g}_j(\mathbf{x}(t_j),\mathbf{u}(t_j)), \text{for all } t \in (t_k, t_{k+1}],
\end{aligned} \tag{3.4.4}$$

and $\mathcal{K}_2 : \mathcal{X} \to \mathcal{B}_2$ is defined by

$$\begin{aligned}
&\mathcal{K}_2(\mathbf{x},\mathbf{u})(t) = \boldsymbol{v}(t) \\
&:= \begin{cases} \left[\sum\limits_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}), & t \in [t_0, T-h_N], \\ \left[\sum\limits_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}), & t \in (T-h_{l+1}, T-h_l], \\ \mathbf{0}, & t \in (T-h_1, T], \end{cases}
\end{aligned} \tag{3.4.5}$$

where $l = 1, 2, \ldots, (N-1)$, and the operator $\mathcal{L} : \mathcal{X} \to \mathbb{R}^n$ is defined by

$$\begin{aligned}
\mathcal{L}(\mathbf{x},\mathbf{u}) := \mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0) - \int_{t_0}^{T} \mathbf{\Phi}(T,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))ds \\
- \sum_{j=1}^{M} \mathbf{\Phi}(T,t_j)\mathbf{g}_j(\mathbf{x}(t_j),\mathbf{u}(t_j)).
\end{aligned} \tag{3.4.6}$$

For the establishment of the controllability of the system (3.2.1), the following theorem is used.

Theorem 3.4.1. *The system* (3.2.1) *is controllable over* $\mathbb{R}^n$ *on* $[t_0, T]$ *if and only if for every initial state* $\mathbf{x}_0 \in \mathbb{R}^n$ *and a final state* $\mathbf{x}_T \in \mathbb{R}^n$*, the operator* $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ *given in equations* (3.4.2)–(3.4.6) *has a fixed-point, i.e. there exists some* $(\mathbf{x}, \mathbf{u}) \in \mathcal{X}$ *such that* $\mathcal{K}(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \mathbf{u})$.

Proof. Let the system (3.2.1) be controllable on $[t_0, T]$, then there exists a control function $\mathbf{u}(\cdot) \in \mathcal{B}_2$, which steers the state of the system given in eq (3.4.1) from $\mathbf{x}_0$ to $\mathbf{x}(T) = \mathbf{x}_T$. That is,

$$\begin{aligned}\mathbf{x}_T =&\mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s + h_i)\mathbf{B}_i(s + h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s + h_i)\mathbf{B}_i(s + h_i)\mathbf{u}(s)\mathrm{d}s + \int_{t_0}^{T} \Phi(T, s)\mathbf{f}(s, \mathbf{x}(s), \mathbf{u}(s))\mathrm{d}s \\ &+ \sum_{j=1}^{M} \mathbf{\Phi}(T, t_j)\mathbf{g}_j(\mathbf{x}(t_j), \mathbf{u}(t_j)).\end{aligned}$$

Combining the above equation with eq (3.4.6), we get

$$\begin{aligned}\mathcal{L}(\mathbf{x}, \mathbf{u}) =& \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s + h_i)\mathbf{B}_i(s + h_i)\right]\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s + h_i)\mathbf{B}_i(s + h_i)\mathbf{u}(s)\mathrm{d}s.\end{aligned} \tag{3.4.7}$$

We choose a function $\mathbf{u}(\cdot)$ satisfying eq (3.4.7) as

$$\mathbf{u}(t) = \begin{cases} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, t + h_i)\mathbf{B}_i(t + h_i)\right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}), & t \in [t_0, T - h_N], \\ \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, t + h_i)\mathbf{B}_i(t + h_i)\right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}), & t \in (T - h_{l+1}, T - h_l], \\ \mathbf{0}, & t \in (T - h_1, T]. \end{cases} \tag{3.4.8}$$

Now if we compare eq (3.4.8) with eq (3.4.5), it can be easily seen that $\mathcal{K}_2(\mathbf{x}, \mathbf{u}) = \mathbf{u}$. Furthermore, with this control function, the corresponding solution given in eq (3.4.1) reduces to eq (3.4.3) and eq (3.4.4). Hence we have $\mathcal{K}_1(\mathbf{x}, \mathbf{u}) = \mathbf{x}$. Therefore $\mathcal{K}(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \mathbf{u})$, i.e. $\mathcal{K}$ has a fixed-point.

For the converse, let us assume that the operator $\mathcal{K}$ has a fixed-point, i.e. $\mathcal{K}(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \mathbf{u})$, for some $(\mathbf{x}, \mathbf{u}) \in \mathcal{X}$. Our purpose is to show that there exists some control function $\mathbf{u}(\cdot) \in \mathcal{B}_2$ such that $\mathbf{x}(T) = \mathbf{x}_T$. Since $\mathcal{K}(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \mathbf{u})$, from eq (3.4.4) and eq (3.4.5), we obtain the following equations:

$$
\begin{aligned}
\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)(\mathbf{x}_0 + \mathbf{a}_0) &+ \Bigg\{ \int_{t_0}^{t-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \\
&+ \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}) \\
&+ \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{f}(s, \mathbf{x}(s), \mathbf{u}(s))\mathrm{d}s + \sum_{j=1}^{k} \mathbf{\Phi}(t, t_j)\mathbf{g}_j(\mathbf{x}(t_j), \mathbf{u}(t_j)), \; \forall\, t \in (t_k, t_{k+1}],
\end{aligned}
\tag{3.4.9}
$$

and

$$
\mathbf{u}(t) = \begin{cases}
\left[\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}), & t \in [t_0, T-h_N], \\
\left[\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}), & t \in (T-h_{l+1}, T-h_l], \\
0, & t \in (T-h_1, T].
\end{cases}
\tag{3.4.10}
$$

In order to get $\mathbf{x}(T) = \mathbf{x}_T$, let us put $t = T$ in eq (3.4.9) and use eq (3.4.6) to obtain

$$
\begin{aligned}
\mathbf{x}(T) = \mathbf{x}_T - \mathcal{L}(\mathbf{x}, \mathbf{u}) &+ \Bigg\{ \int_{t_0}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s + \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right] \\
&\times \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}) \\
= \mathbf{x}_T - \mathcal{L}(\mathbf{x}, \mathbf{u}) &+ \mathbf{W}\mathbf{W}^{-1}\mathcal{L}(\mathbf{x}, \mathbf{u}) = \mathbf{x}_T.
\end{aligned}
$$

Hence the system (3.2.1) is controllable on $[t_0, T]$. □

The following notations help us in the proof of upcoming theorems.

$$
M_1 := \sup_{t_0 \le s \le t \le T} \|\mathbf{\Phi}(t, s)\|,
$$

$$M_2 := \|\mathbf{x}_0 + \mathbf{a}_0\|_{\mathbb{R}^n},$$

$$M_3 := \sup_{t\in[t_0,T]} \left\| \int_{t_0}^{t-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\right] \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s \right.$$
$$\left. + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i)\right] \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s \right\|,$$

$$M_4 := \max_{l=1,\ldots,(N-1)} \left\{ \sup_{[t_0,T-h_N]} \left\| \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i)\right]^* \right\|, \right.$$
$$\left. \sup_{(T-h_{l+1},T-h_l]} \left\| \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i)\right]^* \right\| \right\}.$$

We consider the following three types of verifiable assumptions on $\mathbf{f}(\cdot,\cdot,\cdot)$ and $\mathbf{g}_k(\cdot,\cdot)$ under which the system (3.2.1) is controllable.

(i) The class of bounded functions:
$\mathfrak{B}_1 := \Big\{\mathbf{f}(\cdot,\cdot,\cdot) \mid \mathbf{f}(\cdot,\cdot,\cdot) : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ is continuous and bounded$\Big\}$ and
$\mathfrak{B}_2 := \Big\{\mathbf{g}(\cdot,\cdot) \mid \mathbf{g}(\cdot,\cdot) : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ is continuous and bounded$\Big\}$.

(ii) The class of Lipschitz functions:
$\mathcal{L}ip_1 := \Big\{\mathbf{f}(\cdot,\cdot,\cdot) \mid \mathbf{f}(\cdot,\cdot,\cdot) : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfying a Lipschitz condition with respect to second and third arguments$\Big\}$ and
$\mathcal{L}ip_2 := \Big\{\mathbf{g}(\cdot,\cdot) \mid \mathbf{g}(\cdot,\cdot) : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfying a Lipschitz condition with respect to both the arguments$\Big\}$.

(iii) The class of linear growth functions:
$\mathcal{LG}_1 := \Big\{\mathbf{f}(\cdot,\cdot,\cdot) \mid \mathbf{f}(\cdot,\cdot,\cdot) : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ continuous and satisfying a linear growth condition with respect to second and third arguments$\Big\}$ and
$\mathcal{LG}_2 := \Big\{\mathbf{g}(\cdot,\cdot) \mid \mathbf{g}(\cdot,\cdot) : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfying a linear growth condition with respect to both the arguments$\Big\}$.

3.4.1 Controllability results for a class of bounded nonlinearities and bounded impulse functions

Here we establish controllability results of the system (3.2.1) for the case (i) given above using Schauder's fixed-point theorem.

Theorem 3.4.2. *Suppose that in system* (3.2.1) *we assume* *(i) the function* $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathfrak{B}_1$ *with bound* $K \geq 0$*, i.e.*

$$\|\mathbf{f}(t,\mathbf{v},\mathbf{w})\|_{\mathbb{R}^n} \leq K, \text{ for all } (t,\mathbf{v},\mathbf{w}) \in [t_0,T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

(ii) each function $\mathbf{g}_k(\cdot,\cdot) \in \mathfrak{B}_2$ *with bounds* $\vartheta_k \geq 0$*, i.e.*

$$\|\mathbf{g}_k(\mathbf{v},\mathbf{w})\|_{\mathbb{R}^n} \leq \vartheta_k, \text{ for all } (\mathbf{v},\mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^m,$$

then the semilinear impulsive delay system (3.2.1) *is controllable on* $[t_0, T]$.

Proof. For $r_0 > 0$, consider $\mathcal{B} = \{(\mathbf{x},\mathbf{u}) \in \mathcal{X} : 0 \leq \|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq r_0\}$ a nonempty, closed and convex subset of $\mathcal{X}$. Now if we can show that $\mathcal{K}$ defined in eqs (3.4.2)–(3.4.6) is a continuous operator from $\mathcal{B}$ into a compact subset of $\mathcal{B}$, then by the Schauder's fixed-point theorem, it implies that $\mathcal{K}$ has a fixed-point, which in turn implies the controllability of the system (3.2.1) by Theorem 3.4.1. To proceed, divide the proof into three steps:

Step 1: $\mathcal{K}$ is a continuous operator on $\mathcal{B}$.

This is accomplished by showing that $\mathcal{K}_1$ and $\mathcal{K}_2$ are continuous operators on $\mathcal{B}$. For this we choose $(\mathbf{x}_1,\mathbf{u}_1)$, $(\mathbf{x}_2,\mathbf{u}_2) \in \mathcal{B}$ such that $\|(\mathbf{x}_1,\mathbf{u}_1) - (\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} \to 0$ and prove that $\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1) - \mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} \to 0$ and $\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1) - \mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} \to 0$. Since $\mathbf{f}(\cdot,\cdot,\cdot)$ is a continuous function on its domain, so in particular it is continuous with respect to the second and third arguments, therefore

$$\sup_{t\in[t_0,T]} \|\mathbf{f}(t,\mathbf{x}_1(t),\mathbf{u}_1(t)) - \mathbf{f}(t,\mathbf{x}_2(t),\mathbf{u}_2(t))\|_{\mathbb{R}^n} \to 0,$$

as $\|(\mathbf{x}_1,\mathbf{u}_1) - (\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} \to 0$. Similarly, we have

$$\sup_{t\in[t_0,T]} \|\mathbf{g}_k(\mathbf{x}_1(t),\mathbf{u}_1(t)) - \mathbf{g}_k(\mathbf{x}_2(t),\mathbf{u}_2(t))\|_{\mathbb{R}^n} \to 0, \text{for all } k.$$

Then the continuity of $\mathcal{K}_1$ follows from the following argument:

$$\begin{aligned}
&\left\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1) - \mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)\right\|_{\mathcal{B}_1} \\
&= \sup_{t\in[t_0,T]} \left\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)(t) - \mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)(t)\right\|_{\mathbb{R}^n} \\
&= \max_k \left\{ \sup_{t\in[t_0,t_1]} \left\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)(t) - \mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)(t)\right\|_{\mathbb{R}^n}, \sup_{t\in(t_k,t_{k+1}]} \left\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)(t) - \mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)(t)\right\|_{\mathbb{R}^n} \right\} \\
&= \max_k \left\{ \sup_{t\in[t_0,t_1]} \left\| \left(\int_{t_0}^{t-h_N} \left[\sum_{i=1}^N \mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\right] \left[\sum_{i=1}^N \mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s \right.\right.\right. \\
&\qquad \left. + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^l \mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\right] \left[\sum_{i=1}^l \mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]^* \mathrm{d}s \right)
\end{aligned}$$

$$\times \mathbf{W}^{-1}\big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)+\int_{t_0}^{t}\mathbf{\Phi}(t,s)[\mathbf{f}(s,\mathbf{x}_1(s),\mathbf{u}_1(s))-\mathbf{f}(s,\mathbf{x}_2(s),\mathbf{u}_2(s))]\mathrm{d}s\Bigg\|_{\mathbb{R}^n},$$

$$\sup_{t\in(t_k,t_{k+1}]}\Bigg\|\Bigg(\int_{t_0}^{t-h_N}\left[\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\right]\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]^*\mathrm{d}s$$

$$+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\left[\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{B}_i(s+h_i)\right]\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right]^*\mathrm{d}s\Bigg)$$

$$\times \mathbf{W}^{-1}\big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)+\int_{t_0}^{t}\mathbf{\Phi}(t,s)[\mathbf{f}(s,\mathbf{x}_1(s),\mathbf{u}_1(s))-\mathbf{f}(s,\mathbf{x}_2(s),\mathbf{u}_2(s))]\mathrm{d}s$$

$$+\sum_{j=1}^{k}\mathbf{\Phi}(t,t_j)[\mathbf{g}_j(\mathbf{x}_1(t_j),\mathbf{u}_1(t_j))-\mathbf{g}_j(\mathbf{x}_2(t_j),\mathbf{u}_2(t_j))]\Bigg\|_{\mathbb{R}^n}\Bigg\}$$

$$\le M_3\|\mathbf{W}^{-1}\|\|\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathbb{R}^n}$$

$$+\sup_{t_0\le s\le t\le T}\|\mathbf{\Phi}(t,s)\|\,T\sup_{s\in[t_0,T]}\|\mathbf{f}(s,\mathbf{x}_1(s),\mathbf{u}_1(s))-\mathbf{f}(s,\mathbf{x}_2(s),\mathbf{u}_2(s))\|_{\mathbb{R}^n}$$

$$+\sup_{t_0\le s\le t\le T}\|\mathbf{\Phi}(t,s)\|\sum_{j=1}^{M}\|\mathbf{g}_j(\mathbf{x}_1(t_j),\mathbf{u}_1(t_j))-\mathbf{g}_j(\mathbf{x}_2(t_j),\mathbf{u}_2(t_j))\|_{\mathbb{R}^n}$$

$$\le M_1\big(1+M_3\|\mathbf{W}^{-1}\|\big)\Bigg(T\sup_{s\in[t_0,T]}\|\mathbf{f}(s,\mathbf{x}_1(s),\mathbf{u}_1(s))-\mathbf{f}(s,\mathbf{x}_2(s),\mathbf{u}_2(s))\|_{\mathbb{R}^n}$$

$$+\sum_{j=1}^{M}\|\mathbf{g}_j(\mathbf{x}_1(t_j),\mathbf{u}_1(t_j))-\mathbf{g}_j(\mathbf{x}_2(t_j),\mathbf{u}_2(t_j))\|_{\mathbb{R}^n}\Bigg).$$

The continuity of $\mathcal{K}_2$ follows from the following estimation:

$$\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{B}_2}$$

$$=\sup_{t\in[t_0,T]}\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)(t)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)(t)\|_{\mathbb{R}^m}$$

$$=\max_{l}\Bigg\{\sup_{t\in[t_0,T-h_N]}\Bigg\|\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,t+h_i)\mathbf{B}_i(t+h_i)\right]^*\mathbf{W}^{-1}\big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)\Bigg\|_{\mathbb{R}^m},$$

$$\sup_{t\in(T-h_{l+1},T-h_l]}\Bigg\|\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,t+h_i)\mathbf{B}_i(t+h_i)\right]^*\mathbf{W}^{-1}\big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)\Bigg\|_{\mathbb{R}^m}\Bigg\}$$

$$\le M_4\|\mathbf{W}^{-1}\|\|\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathbb{R}^n}$$

$$\le M_1M_4\|\mathbf{W}^{-1}\|\Bigg(T\sup_{s\in[t_0,T]}\|\mathbf{f}(s,\mathbf{x}_1(s),\mathbf{u}_1(s))-\mathbf{f}(s,\mathbf{x}_2(s),\mathbf{u}_2(s))\|_{\mathbb{R}^n}$$

$$+\sum_{j=1}^{M}\|\mathbf{g}_j(\mathbf{x}_1(t_j),\mathbf{u}_1(t_j))-\mathbf{g}_j(\mathbf{x}_2(t_j),\mathbf{u}_2(t_j))\|_{\mathbb{R}^n}\Bigg).$$

Finally, the continuity of $\mathcal{K}$ follows from the following estimate:

$$\begin{aligned}\|\mathcal{K}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} &= \|\big(\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1),\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)\big)-\big(\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2),\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\big)\|_{\mathcal{X}}\\ &= \|\big(\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2),\ \mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\big)\|_{\mathcal{X}}\\ &= \|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{B}_1}+\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{B}_2}.\end{aligned}$$

Step 2: $\mathcal{K}(\mathcal{B})$ is a compact subset.

In order to prove this, we first claim that $\mathcal{K}_1(\mathcal{B})=\{\mathcal{K}_1(\mathbf{x},\mathbf{u}):\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}\leq r_0\}\subset\mathcal{B}_1$ is equicontinuous set on each subinterval $[t_0,t_1]$, $(t_k,t_{k+1}], k=1,2,\ldots,M$ and uniformly bounded on $[t_0,T]$. Similarly, $\mathcal{K}_2(\mathcal{B})=\{\mathcal{K}_2(\mathbf{x},\mathbf{u}):\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}\leq r_0\}\subset\mathcal{B}_2$ is equicontinuous set on $[t_0,T-h_N]$, $(T-h_i,T-h_{i-1}],\ldots,(T-h_1,T]$, $i=2,\ldots,N$ and uniformly bounded on $[t_0,T]$. First we require the following inequality estimated from eq (3.4.6):

$$\|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n}\leq\|\mathbf{x}_T\|_{\mathbb{R}^n}+M_1M_2+M_1(TK+\vartheta), \tag{3.4.11}$$

where $\vartheta=\sum\limits_{k=1}^{M}\vartheta_k$. Let $s_1<s_2$, where $s_1,s_2\in[t_0,t_1]$ or $s_1,s_2\in(t_k,t_{k+1}]$ and $\mathcal{K}_1(\mathbf{x},\mathbf{u})\in\mathcal{K}_1(\mathcal{B})$ be any elements. Now consider the following estimation:

$$\begin{aligned}&\|\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_1)-\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^n}\\ &=\Bigg\|[\mathbf{\Phi}(s_1,t_0)-\mathbf{\Phi}(s_2,t_0)](\mathbf{x}_0+\mathbf{a}_0)\\ &\quad+\Bigg[\Bigg\{\int_{t_0}^{s_1-h_N}\left(\sum_{i=1}^{N}\mathbf{\Phi}(s_1,s+h_i)\mathbf{B}_i(s+h_i)\right)\left(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right)^*\mathrm{d}s\\ &\quad-\int_{t_0}^{s_2-h_N}\left(\sum_{i=1}^{N}\mathbf{\Phi}(s_2,s+h_i)\mathbf{B}_i(s+h_i)\right)\left(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\right)^*\mathrm{d}s\Bigg\}+\cdots\\ &\quad+\Bigg\{\int_{s_1-h_2}^{s_1-h_1}\mathbf{\Phi}(s_1,s+h_1)\mathbf{B}_1(s+h_1)\big(\mathbf{\Phi}(T,s+h_1)\mathbf{B}_1(s+h_1)\big)^*\mathrm{d}s\\ &\quad-\int_{s_2-h_2}^{s_2-h_1}\mathbf{\Phi}(s_2,s+h_1)\mathbf{B}_1(s+h_1)\big(\mathbf{\Phi}(T,s+h_1)\mathbf{B}_1(s+h_1)\big)^*\mathrm{d}s\Bigg\}\Bigg]\mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u})\\ &\quad+\int_{t_0}^{s_1}\mathbf{\Phi}(s_1,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s-\int_{t_0}^{s_2}\mathbf{\Phi}(s_2,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s\\ &\quad+\sum_{j=1}^{k}\big[\mathbf{\Phi}(s_1,t_j)-\mathbf{\Phi}(s_2,t_j)\big]\mathbf{g}_j(\mathbf{x}(t_j),\mathbf{u}(t_j))\Bigg\|_{\mathbb{R}^n}\\ &\leq\|\mathbf{\Phi}(s_1,t_0)-\mathbf{\Phi}(s_2,t_0)\|\|\mathbf{x}_0+\mathbf{a}_0\|_{\mathbb{R}^n}\\ &\quad+\Bigg\{\Bigg\|\int_{t_0}^{s_1-h_N}\left(\sum_{i=1}^{N}\big[\mathbf{\Phi}(s_1,s+h_i)-\mathbf{\Phi}(s_2,s+h_i)\big]\mathbf{B}_i(s+h_i)\right)\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left(\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right)^* \mathrm{d}s \\
&\quad - \int_{s_1-h_N}^{s_2-h_N} \left(\sum_{i=1}^{N} \mathbf{\Phi}(s_2, s+h_i)\mathbf{B}_i(s+h_i) \right) \left(\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right)^* \mathrm{d}s \Bigg\| + \cdots \\
&\quad + \Bigg\| \int_{s_1-h_2}^{s_1-h_1} \big(\mathbf{\Phi}(s_1, s+h_1)\mathbf{B}_1(s+h_1)\big)\big(\mathbf{\Phi}(T, s+h_1)\mathbf{B}_1(s+h_1)\big)^* \mathrm{d}s \\
&\quad - \int_{s_2-h_2}^{s_1-h_2} \big(\mathbf{\Phi}(s_2, s+h_1)\mathbf{B}_1(s+h_1)\big)\big(\mathbf{\Phi}(T, s+h_1)\mathbf{B}_1(s+h_1)\big)^* \mathrm{d}s \\
&\quad - \int_{s_1-h_2}^{s_1-h_1} \big(\mathbf{\Phi}(s_2, s+h_1)\mathbf{B}_1(s+h_1)\big)\big(\mathbf{\Phi}(T, s+h_1)\mathbf{B}_1(s+h_1)\big)^* \mathrm{d}s \\
&\quad - \int_{s_1-h_1}^{s_2-h_1} \big(\mathbf{\Phi}(s_2, s+h_1)\mathbf{B}_1(s+h_1)\big)\big(\mathbf{\Phi}(T, s+h_1)\mathbf{B}_1(s+h_1)\big)^* \mathrm{d}s \Bigg\| \Bigg\} \|\mathbf{W}^{-1}\| \\
&\quad \times \|\mathcal{L}(\mathbf{x}, \mathbf{u})\|_{\mathbb{R}^n} + \Bigg\| \int_{t_0}^{s_1} \big[\mathbf{\Phi}(s_1, s) - \mathbf{\Phi}(s_2, s)\big] \mathbf{f}(s, \mathbf{x}(s), \mathbf{u}(s)) \mathrm{d}s \\
&\quad - \int_{s_1}^{s_2} \mathbf{\Phi}(s_2, s)\mathbf{f}(s, \mathbf{x}(s), \mathbf{u}(s)) \mathrm{d}s \Bigg\|_{\mathbb{R}^n} \\
&\quad + \sum_{j=1}^{k} \|\mathbf{\Phi}(s_1, t_j) - \mathbf{\Phi}(s_2, t_j)\| \|\mathbf{g}_j(\mathbf{x}(t_j), \mathbf{u}(t_j))\|_{\mathbb{R}^n} \\
&\le M_2 \|\mathbf{\Phi}(s_1, t_0) - \mathbf{\Phi}(s_2, t_0)\| \\
&\quad + \Bigg\{ (s_1 - h_N) \sup_s \Bigg\| \sum_{i=1}^{N} \big(\mathbf{\Phi}(s_1, s+h_i) - \mathbf{\Phi}(s_2, s+h_i)\big)\mathbf{B}_i(s+h_i) \Bigg\| \\
&\quad \times \sup_s \Bigg\| \left(\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right)^* \Bigg\| \\
&\quad + (s_2 - s_1) \sup_s \Bigg\| \sum_{i=1}^{N} \mathbf{\Phi}(s_2, s+h_i)\mathbf{B}_i(s+h_i) \Bigg\| \sup_s \Bigg\| \left(\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right)^* \Bigg\| + \cdots \\
&\quad + (h_2 - h_1) \sup_s \|\mathbf{\Phi}(s_1, s+h_1) - \mathbf{\Phi}(s_2, s+h_1)\| \sup_s \|\mathbf{B}_1(s+h_1)\| \\
&\quad \times \sup_s \big\| \big(\mathbf{\Phi}(T, s+h_1)\mathbf{B}_1(s+h_1)\big)^* \big\| \\
&\quad + 2(s_2 - s_1) \sup_s \|\mathbf{\Phi}(s_2, s+h_1)\mathbf{B}_1(s+h_1)\big(\mathbf{\Phi}(T, s+h_1)\mathbf{B}_1(s+h_1)\big)^*\| \Bigg\} \\
&\quad \times \|\mathbf{W}^{-1}\| \big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_1 M_2 + M_1(TK + \vartheta) \big) \\
&\quad + s_1 \sup_s \|\mathbf{\Phi}(s_1, s) - \mathbf{\Phi}(s_2, s)\| K + (s_2 - s_1) M_1 K + \vartheta \sum_{j=1}^{k} \|\mathbf{\Phi}(s_1, t_j) - \mathbf{\Phi}(s_2, t_j)\|.
\end{aligned}
$$

Observe that, the right hand side of the above inequality is independent of the choice of $\mathbf{x}$

and $\mathbf{u}$. Also, if we take $s_1 \to s_2$, then we see that $\|\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_1(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^n} \to 0$, for all $\mathcal{K}_1(\mathbf{x},\mathbf{u}) \in \mathcal{K}_1(\mathcal{B})$. Therefore, $\mathcal{K}_1(\mathcal{B})$ is equicontinuous set on $[t_0, t_1]$, $(t_k, t_{k+1}]$, $\forall\, k$.

For uniform boundedness of $\mathcal{K}_1(\mathcal{B})$, we consider the following estimation:

$$
\begin{aligned}
&\|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1}\\
&= \sup_{t\in[t_0,T]} \|\mathcal{K}_1(\mathbf{x},\mathbf{u})(t)\|_{\mathbb{R}^n} = \max_k \left\{ \sup_{t\in[t_0,t_1]} \|\mathcal{K}_1(\mathbf{x},\mathbf{u})(t)\|_{\mathbb{R}^n}, \sup_{t\in(t_k,t_{k+1}]} \|\mathcal{K}_1(\mathbf{x},\mathbf{u})(t)\|_{\mathbb{R}^n} \right\}\\
&= \max_k \left\{ \sup_{t\in[t_0,t_1]} \left\| \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) \right.\right.\\
&\quad + \left(\int_{t_0}^{t-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \right.\\
&\quad \left. + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \right)\\
&\quad \left. \times \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) + \int_{t_0}^{t} \mathbf{\Phi}(t,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s \right\|_{\mathbb{R}^n},\\
&\quad \sup_{t\in(t_k,t_{k+1}]} \left\| \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) \right.\\
&\quad + \left(\int_{t_0}^{t-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \right.\\
&\quad \left. + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{B}_i(s+h_i) \right] \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{B}_i(s+h_i) \right]^* \mathrm{d}s \right)\\
&\quad \left.\left. \times \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) + \int_{t_0}^{t} \mathbf{\Phi}(t,s)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s + \sum_{j=1}^{k} \mathbf{\Phi}(t,t_j)\mathbf{g}_j(\mathbf{x}(t_j),\mathbf{u}(t_j)) \right\|_{\mathbb{R}^n} \right\}\\
&\leq \sup_{t\in[t_0,T]} \|\mathbf{\Phi}(t,t_0)\| \|\mathbf{x}_0 + \mathbf{a}_0\|_{\mathbb{R}^n} + M_3\|\mathbf{W}^{-1}\| \|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n}\\
&\quad + \sup_{t_0\leq s\leq t\leq T} \|\mathbf{\Phi}(t,s)\| T \sup_{s\in[t_0,T]} \|\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\|_{\mathbb{R}^n} + \sup_{t,s\in[t_0,T]} \|\mathbf{\Phi}(t,s)\| \sum_{j=1}^{M} \|\mathbf{g}_j(\mathbf{x}(t_j),\mathbf{u}(t_j))\|_{\mathbb{R}^n}\\
&\leq M_1M_2 + M_3\|\mathbf{W}^{-1}\| \big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_1M_2 + M_1(TK+\vartheta) \big) + M_1TK + M_1\vartheta,
\end{aligned}
$$

i.e.

$$
\|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} \leq \big(1 + M_3\|\mathbf{W}^{-1}\|\big)\big(M_1M_2 + M_1(TK+\vartheta)\big) + M_3\|\mathbf{W}^{-1}\| \|\mathbf{x}_T\|_{\mathbb{R}^n}. \qquad (3.4.12)
$$

Since the right hand side of the above inequality is independent of the choice of $\mathbf{x}$ and $\mathbf{u}$, so the above inequality holds for any $\mathcal{K}_1(\mathbf{x},\mathbf{u}) \in \mathcal{K}_1(\mathcal{B})$. Thus, the set $\mathcal{K}_1(\mathcal{B})$ is uniformly bounded on $[t_0, T]$.

To show the equicontinuity of $\mathcal{K}_2(\mathcal{B})$, choose the elements $s_1, s_2 \in [t_0, T-h_N]$ or $s_1, s_2 \in (T-h_i, T-h_{i-1}]$ or $s_1, s_2 \in (T-h_1, T]$ with $s_1 < s_2$, and for any $\mathcal{K}_2(\mathbf{x},\mathbf{u}) \in \mathcal{K}_2(\mathcal{B})$, consider the following estimation:

$$\begin{aligned}
&\|\mathcal{K}_2(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_2(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^m} \\
&\le \sum_i \left\| \Big(\mathbf{\Phi}(T, s_1+h_i)\mathbf{B}_i(s_1+h_i)\Big)^* - \Big(\mathbf{\Phi}(T, s_2+h_i)\mathbf{B}_i(s_2+h_i)\Big)^* \right\| \|\mathbf{W}^{-1}\| \|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n} \\
&\le \sum_i \left\| \Big(\mathbf{\Phi}(T, s_1+h_i)\mathbf{B}_i(s_1+h_i)\Big)^* - \Big(\mathbf{\Phi}(T, s_2+h_i)\mathbf{B}_i(s_2+h_i)\Big)^* \right\| \\
&\quad \times \|\mathbf{W}^{-1}\| \big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_1M_2 + M_1(TK+\vartheta)\big).
\end{aligned}$$

For uniform boundedness of $\mathcal{K}_2(\mathcal{B})$, see the following estimation:

$$\begin{aligned}
\|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} &= \sup_{t\in[t_0,T]} \|\mathcal{K}_2(\mathbf{x},\mathbf{u})(t)\|_{\mathbb{R}^m} \\
&= \max_l \Bigg\{ \sup_{t\in[t_0,T-h_N]} \left\| \left(\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right)^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) \right\|_{\mathbb{R}^m}, \\
&\qquad \sup_{t\in(T-h_{l+1},T-h_l]} \left\| \left(\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right)^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) \right\|_{\mathbb{R}^m} \Bigg\} \\
&\le \max_l \Bigg\{ \sup_{t\in[t_0,T-h_N]} \left\| \left(\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right)^* \right\|, \\
&\qquad \sup_{t\in(T-h_{l+1},T-h_l]} \left\| \left(\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{B}_i(t+h_i) \right)^* \right\| \Bigg\} \|\mathbf{W}^{-1}\| \|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n},
\end{aligned}$$

i.e.

$$\|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} \le M_4 \|\mathbf{W}^{-1}\| \big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_1M_2 + M_1(TK+\vartheta)\big). \tag{3.4.13}$$

Therefore

$$\mathcal{K}(\mathcal{B}) = \mathcal{K}_1(\mathcal{B}) \times \mathcal{K}_2(\mathcal{B}) = \big\{ \big(\mathcal{K}_1(\mathbf{x},\mathbf{u}),\, \mathcal{K}_2(\mathbf{x},\mathbf{u})\big) \,:\, \|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \le r_0 \big\}$$

is equicontinuous on $[t_0,\ t_1]$, $(t_1,\ t_2], \ldots, (t_{M-1},\ t_M]$, $(t_M,\ T-h_N]$, $(T-h_N,\ T-h_{N-1}], \ldots, (T-h_1,\ T]$ and uniformly bounded on $[t_0, T]$. Consequently, a sequence $\{(\mathcal{K}_1^n(\mathbf{x},\mathbf{u}),\ \mathcal{K}_2^n(\mathbf{x},\mathbf{u}))\} \subset \mathcal{K}(\mathcal{B})$ is uniformly bounded and equicontinuous on each interval and in particular on $[t_0,\ t_1]$, so by Arzela–Ascoli theorem, there exists a subsequence $\big\{\big(\mathcal{K}_1^{n_1}(\mathbf{x},\mathbf{u}),\ \mathcal{K}_2^{n_1}(\mathbf{x},\mathbf{u}\big)\big\}$ of $\{(\mathcal{K}_1^n(\mathbf{x},\mathbf{u}),\ \mathcal{K}_2^n(\mathbf{x},\mathbf{u}))\}$ which is uniformly convergent on $[t_0,\ t_1]$.

Consider the sequence $\big\{\big(\mathcal{K}_1^{n_1}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_1}(\mathbf{x},\mathbf{u}\big)\big\}$ which is equicontinuous and uniformly bounded on each interval, in particular on $(t_1, t_2]$, and, for the same reason, there exists a subsequence $\{(\mathcal{K}_1^{n_2}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_2}(\mathbf{x},\mathbf{u})\}$ of $\{(\mathcal{K}_1^{n_1}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_1}(\mathbf{x},\mathbf{u})\}$ which is uniformly convergent on $[t_0, t_2]$.

Continuing this process for the intervals $(t_2,\ t_3], \dots, (t_{M-1},\ t_M]$, $(t_M,\ T - h_N]$, $(T - h_N,\ T - h_{N-1}], \dots, (T - h_1,\ T]$, we see that the sequence $\{(\mathcal{K}_1^{n_{(M+N+1)}}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_{(M+N+1)}}(\mathbf{x},\mathbf{u}))\}$ is uniformly convergent on $[t_0, T]$. Thus $\{(\mathcal{K}_1^{n}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n}(\mathbf{x},\mathbf{u}))\}$ being an arbitrary sequence in $\mathcal{K}(\mathcal{B})$, has a converging subsequence $\{(\mathcal{K}_1^{n_{(M+N+1)}}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_{(M+N+1)}}(\mathbf{x},\mathbf{u}))\}$ on $[t_0, T]$. Hence $\mathcal{K}(\mathcal{B})$ is a compact set in $\mathcal{X}$.

Step 3: $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$.

Let $\mathcal{K}(\mathbf{x},\mathbf{u}) \in \mathcal{K}(\mathcal{B})$ be any element. We use the estimates (3.4.12) and (3.4.13) to get

$$
\begin{aligned}
\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} &= \|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} + \|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} \\
&\leq \left(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\right)\left(M_1 M_2 + M_1(TK + \vartheta)\right) + \|\mathbf{x}_T\|_{\mathbb{R}^n}(M_3 + M_4)\|\mathbf{W}^{-1}\|.
\end{aligned}
$$

Then, we see that $\lim\limits_{\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \to \infty} \frac{\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}}{\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}} = 0$. Therefore, for a fixed $\varepsilon \in (0,1)$, we have $\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq \varepsilon \|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}$ for sufficiently large value of $\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}$, say $r_0 > 0$. Hence, we obtain $\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq \varepsilon r_0 < r_0$. Therefore, we have $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$.

Now the Shauder's fixed-point theorem ensures the existence of a fixed-point for an operator $\mathcal{K}$ in $\mathcal{B} \subset \mathcal{X}$ and hence by Theorem 3.4.1, system (3.2.1) is controllable on $[t_0, T]$. □

Remark 3.4.1. *It is clear from the Theorem 3.4.2 that, if the semilinear impulsive delay system* (3.2.1) *possesses a unique solution on* $[t_0, T]$ *for any initial state* $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ *and for any control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$*, the linear delay system* (3.3.1) *is controllable on* $[t_0, T]$ *and the continuous functions* $\mathbf{f}(\cdot,\cdot,\cdot)$ *and each* $\mathbf{g}_k(\cdot,\cdot)$ *are bounded on their domain, then the system* (3.2.1) *is also controllable on* $[t_0, T]$.

3.4.2 Controllability results for a class of Lipschitz nonlinearities and Lipschitz impulse functions

A class of Lipschitz nonlinearities and Lipschitz impulses are considered here. We establish the controllability results for the nonlinear system (3.2.1) in the following theorem by using Banach contraction principle.

Theorem 3.4.3. *Let us assume that the semilinear impulsive delay system* (3.2.1) *satisfies the following conditions:*

(i) the function $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{L}ip_1$ *with Lipschitz constants* $\alpha_0,\ \beta_0 \geq 0$, *i.e.*

$$\|\mathbf{f}(t, \mathbf{v}_1, \mathbf{w}_1) - f(t, \mathbf{v}_2, \mathbf{w}_2)\|_{\mathbb{R}^n} \leq \alpha_0 \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{R}^n} + \beta_0 \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbb{R}^m},$$

for all $(t, \mathbf{v}_1, \mathbf{w}_1), (t, \mathbf{v}_2, \mathbf{w}_2) \in [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$,

(ii) each function $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{L}ip_2$ *with Lipschitz constants* $\alpha_k,\ \beta_k \geq 0$, *i.e.*

$$\|\mathbf{g}_k(\mathbf{v}_1, \mathbf{w}_1) - \mathbf{g}_k(\mathbf{v}_2, \mathbf{w}_2)\|_{\mathbb{R}^n} \leq \alpha_k \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{R}^n} + \beta_k \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbb{R}^m},$$

for all $(\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \in \mathbb{R}^n \times \mathbb{R}^m$,

(iii) $\delta = \left[M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\gamma\right] < 1$, *where*

$$\gamma := \max\left\{\left(T\alpha_0 + \sum_{k=1}^{M} \alpha_k\right), \left(T\beta_0 + \sum_{k=1}^{M} \beta_k\right)\right\}. \qquad (3.4.14)$$

Then the semilinear impulsive delay system (3.2.1) *is controllable on* $[t_0, T]$.

Proof. Here we prove that, the operator $\mathcal{K}$ has a unique fixed-point by using Banach contraction principle, and the controllability follows from Theorem 3.4.1. Start by considering

$$\|\mathcal{K}(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{X}} = \|\mathcal{K}_1(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}_1(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_1} + \|K_2(\mathbf{x}_1, \mathbf{u}_1) - K_2(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_2}.$$

Now, we use the estimates for $\|\mathcal{K}_1(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}_1(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_1}$ and $\|K_2(\mathbf{x}_1, \mathbf{u}_1) - K_2(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_2}$ obtained in Step 1 of the proof of Theorem 3.4.2.

$$\begin{aligned}
\|\mathcal{K}(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{X}} &\leq M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big) \\
&\quad \times \Bigg\{T \sup_{s\in[t_0,T]} \|\mathbf{f}(s, \mathbf{x}_1(s), \mathbf{u}_1(s)) - \mathbf{f}(s, \mathbf{x}_2(s), \mathbf{u}_2(s))\|_{\mathbb{R}^n} \\
&\quad + \sum_{k=1}^{M} \|\mathbf{g}_k(\mathbf{x}_1(t_k), \mathbf{u}_1(t_k)) - \mathbf{g}_k(\mathbf{x}_2(t_k), \mathbf{u}_2(t_k))\|_{\mathbb{R}^n}\Bigg\} \\
&\leq M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\gamma\big(\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathcal{B}_1} + \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{B}_2}\big) \\
&\leq \delta\ \|(\mathbf{x}_1, \mathbf{u}_1) - (\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{X}}.
\end{aligned}$$

Since $\delta < 1$, so $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a contraction, and hence by Banach contraction principle, $\mathcal{K}$ has a unique fixed-point in $\mathcal{X}$. Then by Theorem 3.4.1, the system (3.2.1) is controllable on $[t_0, T]$. □

Remark 3.4.2. *Theorem 3.4.3 shows that, if the semilinear impulsive delay system* (3.2.1) *possesses a unique solution on* $[t_0, T]$ *for any initial state* $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ *and for any control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$, *the linear delay system* (3.3.1) *is controllable on* $[t_0, T]$ *and the continuous functions* $\mathbf{f}(\cdot,\cdot,\cdot)$ *and each* $\mathbf{g}_k(\cdot,\cdot)$ *satisfies a Lipschitz condition as defined above,* *then the system* (3.2.1) *is also controllable on* $[t_0, T]$, *provided*

$$M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\gamma < 1,$$

where γ *is given in eq* (3.4.14).

Further by defining $(\mathbf{x}_n,\, \mathbf{u}_n) = \mathcal{K}(\mathbf{x}_{n-1},\, \mathbf{u}_{n-1})$, *the sequence* $(\mathbf{x}_n,\, \mathbf{u}_n)$ *converges to the controlled trajectory and steering controller, for arbitrary initial pair* $(\mathbf{x}_0,\, \mathbf{u}_0)$.

3.4.3 Controllability results for a class of nonlinearities and impulse functions satisfying the linear growth condition

This subsection contains the controllability results of the system (3.2.1) for a class of nonlinearities and impulse functions satisfying the linear growth condition. This is established in the following

Theorem 3.4.4. *Under the assumptions*

(i) the function $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{LG}_1$ *with growth constants* $a_0, b_0, c_0 \geq 0$, *i.e.*

$$\|\mathbf{f}(t, \mathbf{v}, \mathbf{w})\|_{\mathbb{R}^n} \leq a_0\|\mathbf{v}\|_{\mathbb{R}^n} + b_0\|\mathbf{w}\|_{\mathbb{R}^m} + c_0, \ \text{for all } (t, \mathbf{v}, \mathbf{w}) \in [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

(ii) each function $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{LG}_2$ *with growth constants* $a_k, b_k \geq 0$, *i.e.*

$$\|\mathbf{g}_k(\mathbf{v}, \mathbf{w})\|_{\mathbb{R}^n} \leq a_k\|\mathbf{v}\|_{\mathbb{R}^n} + b_k\|\mathbf{w}\|_{\mathbb{R}^m}, \ \text{for all } (\mathbf{v}, \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^m,$$

(iii) $M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\left(T(a_0 + b_0) + \sum\limits_{k=1}^{M}(a_k + b_k)\right) < 1,$

the semilinear impulsive delay system (3.2.1) *is controllable on* $[t_0, T]$.

Proof. Let $\mathcal{B}$ be a nonempty, closed and convex subset of $\mathcal{X}$ as defined earlier. The proof is similar to the proof of Theorem 3.4.2. We show that $\mathcal{K}$ is a continuous operator from $\mathcal{B}$ into a compact subset of $\mathcal{B}$, so that the existence of the fixed-point for $\mathcal{K}$ is guaranteed by Schauder's fixed-point theorem. The continuity of $\mathcal{K}$ on $\mathcal{B}$ is already established in the proof of Theorem 3.4.2. Now, only thing left to be proved here is that $\mathcal{K}(\mathcal{B})$ is compact subset of $\mathcal{B}$. For this, we first need the following estimate obtained from eq (3.4.6)

$$\begin{aligned}\|\mathcal{L}(\mathbf{x}, \mathbf{u})\|_{\mathbb{R}^n} \leq \|\mathbf{x}_T\|_{\mathbb{R}^n} + M_1M_2 + TM_1c_0 + M_1\Bigg\{\left(Ta_0 + \sum_{k=1}^{M} a_k\right)\|\mathbf{x}\|_{\mathcal{B}_1} \\ + \left(Tb_0 + \sum_{k=1}^{M} b_k\right)\|\mathbf{u}\|_{\mathcal{B}_2}\Bigg\}.\end{aligned} \tag{3.4.15}$$

Now, the equicontinuity of the set $\mathcal{K}_1(\mathcal{B})$ on $[t_0,t_1],(t_k,t_{k+1}]$ and its uniform boundedness on $[t_0,T]$ follow from the following estimates:

$$\begin{aligned}
&\|\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_1)-\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^n}\\
&\le M_2\|\mathbf{\Phi}(s_1,t_0)-\mathbf{\Phi}(s_2,t_0)\|\\
&\quad+\Bigg\{(s_1-h_N)\sup_s\Bigg\|\sum_{i=1}^{N}\big(\mathbf{\Phi}(s_1,s+h_i)-\mathbf{\Phi}(s_2,s+h_i)\big)\mathbf{B}_i(s+h_i)\Bigg\|\\
&\quad\times\sup_s\Bigg\|\Bigg(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\Bigg)^*\Bigg\|+(s_2-s_1)\sup_s\Bigg\|\sum_{i=1}^{N}\mathbf{\Phi}(s_2,s+h_i)\mathbf{B}_i(s+h_i)\Bigg\|\\
&\quad\times\sup_s\Bigg\|\Bigg(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{B}_i(s+h_i)\Bigg)^*\Bigg\|+\cdots\\
&\quad+(h_2-h_1)\sup_s\big\|\mathbf{\Phi}(s_1,s+h_1)-\mathbf{\Phi}(s_2,s+h_1)\big\|\big\|\mathbf{B}_1(s+h_1)\big\|\\
&\quad\times\sup_s\big\|\big(\mathbf{\Phi}(T,s+h_1)\mathbf{B}_1(s+h_1)\big)^*\big\|\\
&\quad+2(s_2-s_1)\sup_s\big\|\mathbf{\Phi}(s_2,s+h_1)\mathbf{B}_1(s+h_1)\big(\mathbf{\Phi}(T,s+h_1)\mathbf{B}_1(s+h_1)\big)^*\big\|\Bigg\}\\
&\quad\times\|\mathbf{W}^{-1}\|\Bigg\{\|\mathbf{x}_T\|_{\mathbb{R}^n}+M_1M_2+TM_1c_0+M_1\Bigg(T(a_0+b_0)+\sum_{k=1}^{M}(a_k+b_k)\Bigg)r_0\Bigg\}\\
&\quad+\Big\{s_1\sup_s\big\|\mathbf{\Phi}(s_1,s)-\mathbf{\Phi}(s_2,s)\big\|+M_1(s_2-s_1)\Big\}\big((a_0+b_0)r_0+c_0\big)\\
&\quad+\sum_{j=1}^{k}\|\mathbf{\Phi}(s_1,t_j)-\mathbf{\Phi}(s_2,t_j)\|(a_j+b_j)r_0,
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1}\\
&\quad\le M_1M_2+TM_1c_0+M_3\|\mathbf{W}^{-1}\|\big(M_1M_2+\|\mathbf{x}_T\|_{\mathbb{R}^n}+TM_1c_0\big)\\
&\qquad+M_1\big(1+M_3\|\mathbf{W}^{-1}\|\big)\Bigg[\Bigg(Ta_0+\sum_{k=1}^{M}a_k\Bigg)\|\mathbf{x}\|_{\mathcal{B}_1}+\Bigg(Tb_0+\sum_{k=1}^{M}b_k\Bigg)\|\mathbf{u}\|_{\mathcal{B}_2}\Bigg]\\
&\quad\le M_1M_2+TM_1c_0+M_3\|\mathbf{W}^{-1}\|(M_1M_2+\|\mathbf{x}_T\|_{\mathbb{R}^n}+TM_1c_0)\\
&\qquad+M_1\big(1+M_3\|\mathbf{W}^{-1}\|\big)\Bigg(T(a_0+b_0)+\sum_{k=1}^{M}(a_k+b_k)\Bigg)r_0.
\end{aligned}\tag{3.4.16}$$

Similarly, the equicontinuity of the set $\mathcal{K}_2(\mathcal{B})$ on $[t_0,T-h_N],(T-h_i,T-h_{i-1}]$ and $(T-h_1,T]$ and its uniform boundedness on $[t_0,T]$ are guaranteed by the following estimates:

$$\|\mathcal{K}_2(\mathbf{x},\mathbf{u})(s_1)-\mathcal{K}_2(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^m}$$

$$\leq \sum_i \left\| \Big(\mathbf{\Phi}(T, s_1 + h_i)\mathbf{B}_i(s_1 + h_i)\Big)^* - \Big(\mathbf{\Phi}(T, s_2 + h_i)\mathbf{B}_i(s_2 + h_i)\Big)^* \right\|$$

$$\times \|\mathbf{W}^{-1}\| \left\{ (\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_1 M_2 + T M_1 c_0) + M_1 \left(T(a_0 + b_0) + \sum_{k=1}^{M} (a_k + b_k) \right) \right\} r_0,$$

and

$$\begin{aligned}
&\|\mathcal{K}_2(\mathbf{x}, \mathbf{u})\|_{\mathcal{B}_2} \\
&\quad \leq M_4 \|\mathbf{W}^{-1}\| (M_1 M_2 + \|\mathbf{x}_T\|_{\mathbb{R}^n} + T M_1 c_0) \\
&\qquad + M_1 M_4 \|\mathbf{W}^{-1}\| \left[\left(T a_0 + \sum_{k=1}^{M} a_k \right) \|\mathbf{x}\|_{\mathcal{B}_1} + \left(T b_0 + \sum_{k=1}^{M} b_k \right) \|\mathbf{u}\|_{\mathcal{B}_2} \right] \\
&\quad \leq M_4 \|\mathbf{W}^{-1}\| (M_1 M_2 + \|\mathbf{x}_T\|_{\mathbb{R}^n} + T M_1 c_0) \\
&\qquad + M_1 M_4 \|\mathbf{W}^{-1}\| \left(T(a_0 + b_0) + \sum_{k=1}^{M} (a_k + b_k) \right) r_0.
\end{aligned} \tag{3.4.17}$$

Now, by the same argument as given in the proof of Theorem 3.4.2, we say that $\mathcal{K}(\mathcal{B})$ is compact set. Finally to show $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$, take an element $\mathcal{K}(\mathbf{x}, \mathbf{u}) \in \mathcal{K}(\mathcal{B})$ and use the estimates (3.4.16) and (3.4.17) to obtain

$$\begin{aligned}
&\|\mathcal{K}(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}} = \|\mathcal{K}_1(\mathbf{x}, \mathbf{u})\|_{\mathcal{B}_1} + \|\mathcal{K}_2(\mathbf{x}, \mathbf{u})\|_{\mathcal{B}_2} \\
&\quad \leq M_1 M_2 + T M_1 c_0 + \|\mathbf{W}^{-1}\| (M_3 + M_4)(M_1 M_2 + \|\mathbf{x}_T\|_{\mathbb{R}^n} + T M_1 c_0) \\
&\qquad + M_1 \big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big) \left[\left(T a_0 + \sum_{k=1}^{M} a_k \right) \|\mathbf{x}\|_{\mathcal{B}_1} + \left(T b_0 + \sum_{k=1}^{M} b_k \right) \|\mathbf{u}\|_{\mathcal{B}_2} \right] \\
&\quad \leq M_1 M_2 + T M_1 c_0 + \|\mathbf{W}^{-1}\| (M_3 + M_4)(M_1 M_2 + \|\mathbf{x}_T\|_{\mathbb{R}^n} + T M_1 c_0) \\
&\qquad + M_1 \big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big) \left(T(a_0 + b_0) + \sum_{k=1}^{M} (a_k + b_k) \right) \big(\|\mathbf{x}\|_{\mathcal{B}_1} + \|\mathbf{u}\|_{\mathcal{B}_2}\big).
\end{aligned}$$

Thus, we have

$$\lim_{\|(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}} \to \infty} \frac{\|\mathcal{K}(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}}}{\|(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}}} \leq M_1 \big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big) \left(T(a_0 + b_0) + \sum_{k=1}^{M} (a_k + b_k) \right) < 1.$$

Hence for some $\varepsilon \in (0, 1)$ with $M_1 \big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big) \left(T(a_0 + b_0) + \sum_{k=1}^{M} (a_k + b_k) \right) < \varepsilon$, we have $\|\mathcal{K}(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}} \leq \varepsilon \|(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}}$, for sufficiently large values of $\|(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}}$, say $r_0 > 0$. Hence $\|\mathcal{K}(\mathbf{x}, \mathbf{u})\|_{\mathcal{X}} \leq \varepsilon r_0 < r_0$. Thus, we have $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$. □

Remark 3.4.3. *According to the Theorem 3.4.4, if the semilinear impulsive delay system* (3.2.1) *possesses a unique solution on* $[t_0, T]$ *for any initial state* $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ *and* *for any control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$*, the linear delay system* (3.3.1) *is controllable on* $[t_0, T]$ *and the continuous functions* $\mathbf{f}(\cdot,\cdot,\cdot)$ *and each* $\mathbf{g}_k(\cdot,\cdot)$ *satisfies the linear growth condition as defined above, then the system* (3.2.1) *is also controllable on* $[t_0, T]$*, provided*

$$M_1\left(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\right)\left(T(a_0 + b_0) + \sum_{k=1}^{M}(a_k + b_k)\right) < 1.$$

3.5 Numerical examples

We consider the following two-dimensional nonautonomous impulsive semilinear system with two delays in the control and two impulses in the state,

$$\left.\begin{aligned}
\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}-2 & t\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\mathbf{u}(t-0.05) + \begin{bmatrix}1\\ -1.6\end{bmatrix}\mathbf{u}(t-0.1)\\
&\quad + t\sin(\mathbf{u}^2(t))\begin{bmatrix}\sin(x_1^2(t))\\ \cos(x_2^2(t))\end{bmatrix},\ t \in [0,3]\setminus\{1,2\},\\
\Delta(\mathbf{x}(1)) &= \begin{bmatrix}\sin(x_1^2(1)\mathbf{u}(1))\\ \cos(x_2^2(1)\mathbf{u}(1))\end{bmatrix},\ \ \Delta(\mathbf{x}(2)) = \begin{bmatrix}\cos(x_1^2(2)\mathbf{u}(2))\\ \sin(x_2^2(2)\mathbf{u}(2))\end{bmatrix},\\
\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} &= \begin{bmatrix}2\\ 1\end{bmatrix},\\
\mathbf{u}(t) &= t^3,\ t \in [-0.1, 0).
\end{aligned}\right\} \tag{3.5.1}$$

Comparing this equation with (3.2.1), we get

$$\begin{aligned}
&\mathbf{A}(t) = \begin{bmatrix}-2 & t\\ 0 & -1\end{bmatrix}, \mathbf{B}_1(t) = \begin{bmatrix}1\\ 0\end{bmatrix}, \mathbf{B}_2(t) = \begin{bmatrix}1\\ -1.6\end{bmatrix},\\
&h_1 = 0.05, h_2 = 0.1,\ t_0 = 0,\ t_1 = 1,\ t_2 = 2,\ T = 3,\\
&\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) = t\sin(\mathbf{u}^2(t))\begin{bmatrix}\sin(x_1^2(t))\\ \cos(x_2^2(t))\end{bmatrix}, \mathbf{g}_1(\mathbf{x}(t_1), \mathbf{u}(t_1)) = \begin{bmatrix}\sin(x_1^2(1)\mathbf{u}(1))\\ \cos(x_2^2(1)\mathbf{u}(1))\end{bmatrix},\\
&\mathbf{g}_2(\mathbf{x}(t_2), \mathbf{u}(t_2)) = \begin{bmatrix}\cos(x_1^2(2)\mathbf{u}(2))\\ \sin(x_2^2(2)\mathbf{u}(2))\end{bmatrix}.
\end{aligned}$$

We calculate the associated state-transition matrix as

$$\mathbf{\Phi}(t, s) = \begin{bmatrix}e^{-2(t-s)} & e^{-(t-s)}(t-1) - e^{-2(t-s)}(s-1)\\ 0 & e^{-(t-s)}\end{bmatrix}.$$

Further

$$\mathbf{W}_1 = \int_{2.9}^{2.95} \mathbf{\Phi}(3,\ s+0.05)\mathbf{B}_1\big(\mathbf{\Phi}(3,\ s+0.05)\mathbf{B}_1\big)^* \mathrm{d}s = \begin{bmatrix} 0.0453 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{W}_2 = \int_{0}^{2.9} \sum_{i=1}^{2} \mathbf{\Phi}(3, s+h_i)B_i \left(\sum_{i=1}^{2} \mathbf{\Phi}(3, s+h_i)\mathbf{B}_i \right)^* \mathrm{d}s = \begin{bmatrix} 0.914 & 0 \\ 0 & 1.265 \end{bmatrix}.$$

Clearly $\mathrm{rank}([\mathbf{W}_1,\ \mathbf{W}_2]) = 2$, and it follows from Theorem 3.3.1 that the linear part of system (3.5.1) without impulses, is controllable on $[0, 3]$. Furthermore, we calculate

$$\|\mathbf{W}^{-1}\| = \|(\mathbf{W}_1 + \mathbf{W}_2)^{-1}\| \approx 1.308,\ M_1 = \sup_{0\leq s\leq t\leq 3} \|\mathbf{\Phi}(t, s)\| = \sqrt{2},\ M_3 \approx 1.442,$$
$$M_4 \approx 2.487.$$

Since $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{C}\big([0, 3] \times \mathbb{R}^2 \times \mathbb{R};\ \mathbb{R}^2\big)$ and $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{C}\big(\mathbb{R}^2 \times \mathbb{R};\ \mathbb{R}^2\big)$ are bounded functions on their domain, i.e. $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathfrak{B}_1$ and $\mathbf{g}_k(\cdot,\cdot) \in \mathfrak{B}_2$, for $k = 1, 2$, so it follows from Theorem 3.4.2 that, the semilinear impulsive delay system (3.5.1) is also controllable on $[0, 3]$.
In (3.5.1), if we choose

$$\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) = \frac{1}{100} \begin{bmatrix} x_1(t) + \sin\big(x_1(t)\big) \\ x_2(t) + \sin\big(x_2(t)\big) \end{bmatrix},$$

$$\mathbf{g}_1(\mathbf{x}(t_1), \mathbf{u}(t_1)) = \frac{1}{100} \begin{bmatrix} x_1(1) \\ \cos\big(x_2(1)\big) \end{bmatrix},$$

$$\mathbf{g}_2(\mathbf{x}(t_2), \mathbf{u}(t_2)) = \frac{1}{100} \begin{bmatrix} \cos\big(x_1(2)\big) \\ x_2(2) \end{bmatrix},$$

then we see that $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{C}\big([0, 3] \times \mathbb{R}^2 \times \mathbb{R};\ \mathbb{R}^2\big)$ and $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{C}\big(\mathbb{R}^2 \times \mathbb{R};\ \mathbb{R}^2\big)$, $k = 1,\ 2$, are not bounded on their domain, so we cannot apply the Theorem 3.4.2 to check the controllability of (3.5.1). However, we see that $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{L}ip_1$ and $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{L}ip_2$, for $k = 1, 2$, such that $\alpha_0 = \frac{2\sqrt{2}}{100}$, $\beta_0 = 0$, $\alpha_1 = \frac{\sqrt{2}}{100}$, $\beta_1 = 0$, $\alpha_2 = \frac{\sqrt{2}}{100}$ and $\beta_2 = 0$. Hence

$$\gamma := max\left\{ \left(T\alpha_0 + \sum_{k=1}^{2} \alpha_k \right), \left(T\beta_0 + \sum_{k=1}^{2} \beta_k \right) \right\} = max\left\{ \left(3 \times \frac{2\sqrt{2}}{100} + \frac{\sqrt{2}}{100} + \frac{\sqrt{2}}{100} \right), 0 \right\}$$
$$\approx 0.11314.$$

After the calculation, we see that

$$M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\gamma = 0.982 < 1,$$

and hence by Theorem 3.4.3, the system (3.5.1) is controllable on $[0, 3]$.
Finally to apply the Theorem 3.4.4, choose the functions $\mathbf{f}(\cdot,\cdot,\cdot)$ and $\mathbf{g}_k(\cdot,\cdot)$ in the system (3.5.1) as

$$\begin{aligned}
\mathbf{f}(t,\mathbf{x}(t),\mathbf{u}(t)) &= c_0 \begin{bmatrix} x_1(t)\sin\left(\frac{1}{x_1^2(t)+1}\right) + t\mathbf{u}(t) \\ x_2(t)\sin\left(\frac{1}{x_2^2(t)+1}\right) \end{bmatrix}, \\
\mathbf{g}_1(\mathbf{x}(t_1),\mathbf{u}(t_1)) &= c_1 \begin{bmatrix} x_1(1) + \mathbf{u}(1)\sin(x_2^2(1)) \\ x_2(1) \end{bmatrix}, \\
\mathbf{g}_2(\mathbf{x}(t_2),\mathbf{u}(t_2)) &= c_2 \begin{bmatrix} x_1(2) \\ x_2(2) + \mathbf{u}(2)\cos(x_1^2(2)) \end{bmatrix},
\end{aligned}$$

where c_0, c_1, c_2 are positive constants. Note that $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{C}\big([0,3] \times \mathbb{R}^2 \times \mathbb{R};\, \mathbb{R}^2\big)$ and $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{C}\big(\mathbb{R}^2 \times \mathbb{R};\, \mathbb{R}^2\big)$ are not bounded and further $\mathbf{f}(\cdot,\cdot,\cdot) \notin \mathcal{L}ip_1$ and $\mathbf{g}_k(\cdot,\cdot) \notin \mathcal{L}ip_2$, for $k = 1, 2$. So, neither Theorem 3.4.2 nor Theorem 3.4.3 are applicable here to check the controllability of (3.5.1). However, we observe that $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{LG}_1$ and $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{LG}_2$ for $k = 1, 2$, with $a_0 = c_0$, $b_0 = 3c_0$, $a_1 = b_1 = c_1$ and $a_2 = b_2 = c_2$. Then for suitable choices of c_0, c_1 and c_2, we can get

$$M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\left(T(a_0 + b_0) + \sum_{k=1}^{2}(a_k + b_k)\right) = 8.6956\big(12c_0 + 2c_1 + 2c_2\big) < 1$$

and hence the semilinear impulsive delay system (3.5.1) is controllable on $[0, 3]$ by Theorem 3.4.4.

3.6 Conclusions

In this chapter, we considered an $n-$dimensional semilinear impulsive dynamical control system with multiple constant time-delays in control and derived the sufficient conditions to guarantee that this system is controllable on $[t_0, T]$ for certain classes of nonlinearities $\mathbf{f}(\cdot,\cdot,\cdot)$ and impulse functions $\mathbf{g}_k(\cdot,\cdot)$. The results are obtained by employing Schauder's fixed-point theorem and Banach contraction principle. By assuming that for a given initial state $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ and for a given $\mathbf{u}(\cdot) \in \mathcal{B}_2$, the semilinear impulsive delay system (3.2.1) admits a unique solution on $[t_0, T]$ and the linear delay system (3.3.1) is controllable on $[t_0, T]$, we have established that the semilinear impulsive delay system (3.2.1) is also controllable on $[t_0, T]$ under one of the following assumptions:

(i) $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathfrak{B}_1$ and each $\mathbf{g}_k(\cdot,\cdot) \in \mathfrak{B}_2$.

(ii) $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{L}ip_1$ and each $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{L}ip_2$ with

$$M_1\big(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\big)\gamma < 1,$$

where γ is given in eq (3.4.14).

(iii) $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{LG}_1$ and each $\mathbf{g}_k(\cdot,\cdot) \in \mathcal{LG}_2$ with

$$M_1\left(1 + (M_3 + M_4)\|\mathbf{W}^{-1}\|\right)\left(T(a_0 + b_0) + \sum_{k=1}^{M}(a_k + b_k)\right) < 1.$$

Numerical example is provided to demonstrate our theoretical results.

Furthermore, as every bounded function satisfy the linear growth condition, therefore $\mathfrak{B}_1 \subset \mathcal{LG}_1$ and $\mathfrak{B}_2 \subset \mathcal{LG}_2$. Also we know that, every Lipschitz function satisfy the linear growth condition, so $\mathcal{L}ip_1 \subset \mathcal{LG}_1$ and $\mathcal{L}ip_2 \subset \mathcal{LG}_2$. On the other hand, the functions satisfying a Lipschitz condition may not be bounded, for example, $\mathbf{f}(t,\mathbf{v},\mathbf{w}) = \mathbf{v} + \sin\mathbf{v}$ defined on $[0,1] \times \mathbb{R} \times \mathbb{R}$. Similarly, the bounded functions may not satisfy a Lipschitz condition, for example $\mathbf{f}(t,\mathbf{v},\mathbf{w}) = \sin(\mathbf{v}^2)$ defined on $[0,1] \times \mathbb{R} \times \mathbb{R}$. Therefore $\mathfrak{B}_1$ and $\mathcal{L}ip_1$ are not comparable. Similarly $\mathfrak{B}_2$ and $\mathcal{L}ip_2$. Further, the linear growth functions may not be bounded and may not satisfy a Lipschitz condition, for example, $\mathbf{f}(t,\mathbf{v},\mathbf{w}) = \mathbf{v}\sin\left(\mathbf{v^2}\right) + \mathbf{w}\cos\left(\mathbf{w}^2\right)$ defined on $[0,1] \times \mathbb{R} \times \mathbb{R}$. Thus from this work, we conclude that Theorem 3.4.4 gives the controllability conditions of the system (3.2.1) for much larger class of functions $\mathbf{f}(\cdot,\cdot,\cdot)$ and $\mathbf{g}_k(\cdot,\cdot)$ than that of Theorems 3.4.2 and 3.4.3.

Chapter 4

Controllability of a fractional-order semilinear system with multiple constant time-delays in control

4.1 Introduction

As we discussed earlier, many of the experiments show that some processes have a complex behaviour, due to which their dynamics have to be modelled with fractional-order derivatives. For a detailed study on fractional systems and their applications refer [52, 54] and references therein. The fractional-order derivatives frequently occurs in—modelling of viscoelastic materials, kinetics of anomalous diffusion, fractional wave equations, fractional Brownian motion, electrochemical process, feedback amplifiers, electrical circuits, biological systems etc. The study on the controllability of fractional-order dynamical systems gives important issues for many applied problems, as the fractional-order derivatives and integrals in control theory give better results than those of integer-order ones. The first research article on the controllability of linear autonomous fractional systems was published by Matignon and Novel [80]. Then many other people (see [13, 14, 19, 43] etc.) studied the controllability properties of different types of linear fractional systems, and the work on nonlinear fractional systems explored in [10, 12, 42, 44], [97]–[99] etc. The controllability of fractional systems which involves time-delays in control are obtained in [11, 103, 111] by using the time lead functions. In this chapter, without introducing the time lead functions we study the controllability issues and it makes easier for the investigation. Similar to Chapter 3, the controllability conditions are obtained for three classes of nonlinearities.

We organize this chapter as follows: in Section 4.2, the class of fractional-order semilinear delay system is defined and some preliminary results are given. In Section 4.3, a sufficient condition for controllability of the corresponding fractional-order linear delay system is obtained, and further discuss the controllability analysis of the linear fractional-order system with one delay in control in terms of controllability Grammian matrix. Section 4.4 deals with the controllability of the main system for three classes of nonlinearities by employing Schauder's fixed-point theorem and Banach contraction principle. Section 4.5 contains numerical examples to demonstrate our results. The chapter closes with Section 4.6 where some conclusions are given.

4.2 System description

We consider the dynamical control system modelled by the following $n-$dimensional semilinear autonomous differential equations in the sense of Caputo fractional derivative of order $\alpha \in (0,1)$ having multiple constant time-delays in the control function,

$$\left.\begin{aligned} \left({}^{c}D_t^{\alpha}\mathbf{x}\right)(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N}\mathbf{B}_i\mathbf{u}(t-h_i) + \mathbf{f}(t,\mathbf{x}(t),\mathbf{u}(t)), \quad t\in[0,T],\\ \mathbf{x}(0) &= \mathbf{x}_0,\\ \mathbf{u}(t) &= \mathbf{u}_0(t), \;\; t\in[-h_N,0), \end{aligned}\right\} \tag{4.2.1}$$

in which

(i) the state $\mathbf{x}(t) \in \mathbb{R}^n$ with a given initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$,

(ii) the control $\mathbf{u}(t) \in \mathbb{R}^m$,

(iii) $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_i \in \mathbb{R}^{n\times m}$ are the given constant matrices,

(iv) $0 \le h_1 \le h_2 \le \cdots \le h_N \le T$, h_i's are the known constant time-delays in the control function $\mathbf{u}(\cdot)$,

(v) $\mathbf{u}_0(\cdot) \in \mathcal{C}([-h_N,0);\mathbb{R}^m)$ is a given initial control function (and is assumed to be bounded on its domain) applied to the system (4.2.1),

(vi) the function $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{C}([0,T]\times\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^n)$ is nonlinear in its second argument.

The natural space to work on the solvability of fractional order $\alpha \in (0,1)$ semilinear control delay system (4.2.1) is the real Banach space given by

$$\mathcal{B}_1 := \Big\{\mathbf{x}(\cdot)\,\big|\,\mathbf{x}(\cdot):[0,T]\to\mathbb{R}^n \text{ such that } \left({}^{c}D_t^{\alpha}\mathbf{x}\right)(t) \text{ exists on } [0,T] \text{ and } \mathbf{x}(0) = \lim_{t\downarrow 0}\mathbf{x}(t)\Big\},$$

and the space for admissible controls is a real Banach space defined by

$$\mathcal{B}_2 := \Big\{\mathbf{u}(\cdot)\,\big|\,\mathbf{u}(\cdot):[0,T]\to\mathbb{R}^m, \mathbf{u}(\cdot) \text{ is continuous function a. e. on } [0,T] \text{and bounded on } [0,T]\Big\},$$

endowed with the norms

$$\|\mathbf{x}(\cdot)\|_{\mathcal{B}_1} := \sup_{t\in[0,T]} \|\mathbf{x}(t)\|_{\mathbb{R}^n} \text{ and } \|\mathbf{u}(\cdot)\|_{\mathcal{B}_2} := \sup_{t\in[0,T]} \|\mathbf{u}(t)\|_{\mathbb{R}^m},$$

respectively.

The following definition of controllability for the system (4.2.1) is adopted in this chapter.

Definition 4.2.1. *The system* (4.2.1) *is said to be controllable on* $[0,T]$ *over* $\mathbb{R}^n$, *if for every pair of vectors* $(\mathbf{x}_0, \mathbf{x}_T) \in \mathbb{R}^n \times \mathbb{R}^n$ *and for every bounded function* $\mathbf{u}_0(\cdot) \in \mathcal{C}\big([-h_N, 0)\,; \mathbb{R}^m\big)$ *there exists at least one control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$ *such that, with this control function on* $[0,T]$, *the corresponding solution to the system* (4.2.1) *with* $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{u}(t) = \mathbf{u}_0(t)$, $t \in [-h_N, 0)$, *satisfies the condition* $\mathbf{x}(T) = \mathbf{x}_T$.

4.3 Controllability of the fractional-order linear system with multiple constant time-delays in control

The associated linear system of (4.2.1) is given by,

$$\left.\begin{aligned} \big({}^cD_t^\alpha \mathbf{x}\big)(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_i \mathbf{u}(t-h_i), \quad t \in [0,T], \\ \mathbf{x}(0) &= \mathbf{x}_0, \\ \mathbf{u}(t) &= \mathbf{u}_0(t), \quad t \in [-h_N, 0). \end{aligned}\right\} \tag{4.3.1}$$

By using Mittag-Leffler function given in the Definition 2.1.6, the solution to this system (4.3.1) at any time $t \in [0,T]$ is given by,

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{E}_\alpha(t^\alpha \mathbf{A})\mathbf{x}_0 + \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s)^\alpha \mathbf{A}\big) \sum_{i=1}^{N} \mathbf{B}_i \mathbf{u}(s-h_i)\,\mathrm{d}s \\ &= \mathbf{E}_\alpha(t^\alpha \mathbf{A})\mathbf{x}_0 + \sum_{i=1}^{N} \int_{-h_i}^{0} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}_0(s)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{N} \int_0^{t-h_i} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s. \end{aligned}$$

Let us denote

$$\sum_{i=1}^{N} \int_{-h_i}^{0} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}_0(s)\,\mathrm{d}s = \mathbf{a}_0(t). \tag{4.3.2}$$

Therefore we have,

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{E}_\alpha(t^\alpha \mathbf{A})\mathbf{x}_0 + \mathbf{a}_0(t) \\ &\quad + \sum_{i=1}^{N} \int_0^{t-h_i} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s.\end{aligned} \tag{4.3.3}$$

Now let us simplify the summation given in equation (4.3.3) as

$$\begin{aligned}&\sum_{i=1}^{N} \int_0^{t-h_i} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s \\ &\quad = \int_0^{t-h_N} \sum_{i=1}^{N} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s \\ &\quad\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s.\end{aligned} \tag{4.3.4}$$

Using eq (4.3.4) in eq (4.3.3), the solution to the system (4.3.1) is given by,

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{E}_\alpha(t^\alpha \mathbf{A})\mathbf{x}_0 + \mathbf{a}_0(t) \\ &\quad + \int_0^{t-h_N} \sum_{i=1}^{N} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s.\end{aligned} \tag{4.3.5}$$

Let us now define

$$\begin{aligned}\mathbf{W}_l := \mathbf{W}_l(T) &= \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} (T-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad \times \left[\sum_{i=1}^{l} (T-s-h_i)^{1-\alpha} \mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right]^* \mathrm{d}s, \\ \mathbf{W}_N := \mathbf{W}_N(T) &= \int_0^{T-h_N} \left[\sum_{i=1}^{N} (T-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad \times \left[\sum_{i=1}^{N} (T-s-h_i)^{1-\alpha} \mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right]^* \mathrm{d}s,\end{aligned} \tag{4.3.6}$$

where $l = 1, 2, \ldots, (N-1)$ and denote $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_N$.

We now prove the sufficient condition of controllabilty of the system (4.3.1).

Theorem 4.3.1. *The linear delay system* (4.3.1) *is controllable over* $\mathbb{R}^n$ *on* $[0, T]$, *if* $\mathbf{W}$ *is non-singular.*

Proof. If $\mathbf{W}$ is non-singular matrix, then it is invertible. We define a control function $\mathbf{u}(t)$ as follows:

$$\mathbf{u}(t) := \begin{cases} \left[\sum_{i=1}^{N}(T-t-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-t-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathbf{W}^{-1} & \\ \times\,[\mathbf{x}_T-\mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0-\mathbf{a}_0(T)], & \text{for } t\in[0,T-h_N], \\ \left[\sum_{i=1}^{l}(T-t-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-t-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathbf{W}^{-1} & \\ \times\,[\mathbf{x}_T-\mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0-\mathbf{a}_0(T)], & \text{for } t\in(T-h_{l+1},T-h_l], \\ \mathbf{0}, & \text{for } t\in(T-h_1,T]. \end{cases} \tag{4.3.7}$$

The corresponding state of the system (4.3.1) given in eq (4.3.5) at $t=T$ becomes

$$\begin{aligned}\mathbf{x}(T) &= \mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0+\mathbf{a}_0(T)+\int_0^{T-h_N}\sum_{i=1}^{N}(T-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\mathbf{u}(s)\,\mathrm{d}s \\ &\quad+\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\sum_{i=1}^{l}(T-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}$$

Substitute $\mathbf{u}(s)$ from eq (4.3.7) in the above expression we have,

$$\begin{aligned}\mathbf{x}(T) &= \mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0+\mathbf{a}_0(T)+\Bigg\{\int_0^{T-h_N}\left[\sum_{i=1}^{N}(T-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad\times\left[\sum_{i=1}^{N}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathrm{d}s \\ &\quad+\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\left[\sum_{i=1}^{l}(T-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad\times\left[\sum_{i=1}^{l}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathrm{d}s\Bigg\}\mathbf{W}^{-1} \\ &\quad\times\big[\mathbf{x}_T-\mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0-\mathbf{a}_0(T)\big] \\ &= \mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0+\mathbf{a}_0(T)+\big\{\mathbf{W}_N+\cdots+\mathbf{W}_1\big\}\mathbf{W}^{-1}\big[\mathbf{x}_T-\mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0-\mathbf{a}_0(T)\big] \\ &= \mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0+\mathbf{a}_0(T)+\mathbf{W}\mathbf{W}^{-1}\big[\mathbf{x}_T-\mathbf{E}_{\alpha}(T^{\alpha}\mathbf{A})\mathbf{x}_0-\mathbf{a}_0(T)\big]=\mathbf{x}_T.\end{aligned}$$

Hence the linear delay system (4.3.1) is controllable over $\mathbb{R}^n$ on $[0,T]$. □

Corollary 4.3.1. *If the linear delay system* (4.3.1) *possesses only one delay in control, i.e.*

$N = 1$, *then the matrices in equations in*(4.3.6) *reduces to*

$$\mathbf{W} := \mathbf{W}_N(T) = \int_0^{T-h} \big[\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\big]\big[\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\big]^* \mathrm{d}s \tag{4.3.8}$$

and is called as controllability Grammian of the system (4.3.1) *for a single delay. Further for this system, the sufficient controllability condition given in Theorem 4.3.1 is also necessary which is shown in the following*

Proof. Let us prove this by contradiction. For this, let the system (4.3.1) be controllable on $[0, T]$, but assume that the matrix $\mathbf{W}$ is singular. Then there exists some non-zero vector, say $\mathbf{v} \in \mathbb{R}^n$ such that $\langle \mathbf{W}\mathbf{v},\ \mathbf{v}\rangle_{\mathbb{R}^n} = 0$. Therefore we have

$$\begin{aligned}
&\left\langle \int_0^{T-h} \big[\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\big]\big[\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\big]^* \mathrm{d}s\,\mathbf{v},\ \mathbf{v} \right\rangle_{\mathbb{R}^n} = 0\\
&\Longrightarrow \int_0^{T-h} \big[\mathbf{v}^*\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\big]\big[\mathbf{v}^*\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\big]^* \mathrm{d}s = 0\\
&\Longrightarrow \int_0^{T-h} \big\|\mathbf{B}^*\mathbf{E}^*_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{v}\big\|^2_{\mathbb{R}^m}\,\mathrm{d}s = 0.
\end{aligned}$$

Since $\mathbf{E}^*_{\alpha,\alpha}(\cdot)$ is a continuous function on $[0, T]$, therefore the above integral implies that

$$\mathbf{v}^*\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B} = \mathbf{0},\ \forall\, s \in [0, T-h]. \tag{4.3.9}$$

Since the system (4.3.1) is controllable on $[0, T]$, in particular it is null controllable. Now choose an initial state as $\mathbf{x}_0 = \mathbf{E}_\alpha(T^\alpha \mathbf{A})^{-1}(-\mathbf{a}_0(T) + \mathbf{v})$ and a final state as $\mathbf{x}_{_T} = \mathbf{0}$. Then with certain control function $\mathbf{u}(\cdot) \in \mathcal{B}_2$, the state of the system (4.3.1) given in eq (4.3.5) satisfies $\mathbf{x}(T) = \mathbf{0}$. That is,

$$\begin{aligned}
\mathbf{0} = \mathbf{x}(T) &= \mathbf{E}_\alpha(T^\alpha \mathbf{A})\mathbf{x}_0 + \mathbf{a}_0(T) + \int_0^{T-h} (T-s-h)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\mathbf{u}(s)\,\mathrm{d}s\\
&= \mathbf{E}_\alpha(T^\alpha \mathbf{A})\Big[\mathbf{E}_\alpha(T^\alpha \mathbf{A})^{-1}(-\mathbf{a}_0(T) + \mathbf{v})\Big] + \mathbf{a}_0(T)\\
&\quad + \int_0^{T-h} (T-s-h)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\mathbf{u}(s)\,\mathrm{d}s\\
&= \mathbf{v} + \int_0^{T-h} (T-s-h)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\mathbf{v} = -\int_0^{T-h} (T-s-h)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s-h)^\alpha \mathbf{A}\big)\mathbf{B}\mathbf{u}(s)\,\mathrm{d}s.$$

Now premultiply the above equation with $\mathbf{v}^*$ and use eq (4.3.9) to obtain $\mathbf{v}^*\mathbf{v} = 0$. Therefore we get $\|\mathbf{v}\|^2_{\mathbb{R}^n} = 0 \Longrightarrow \mathbf{v} = \mathbf{0}$. This is contradiction. Hence the matrix $\mathbf{W}$ is non-singular. □

Remark 4.3.1. *It should be mentioned that the linear delay system* (4.3.1) *with a single delay in control reduces to the first order linear control delay system for* $\alpha = 1$*, and is of the form*

$$\left.\begin{aligned}&\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-h), \ \textit{for all } t \in [0,T],\\&\mathbf{x}(0) = \mathbf{x}_0,\\&\mathbf{u}(t) = \mathbf{u}_0(t), \ \textit{for all } t \in [-h,0).\end{aligned}\right\} \tag{4.3.10}$$

Under this particular situation, the Mittag–Leffler function reduces to $\mathbf{E}_{1,1}(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} = e^{\mathbf{A}t}$*. Hence the controllability Grammian given in eq* (4.3.8) *becomes*

$$\mathbf{W} = \int_0^{T-h} e^{\mathbf{A}(T-s-h)}\mathbf{B}\mathbf{B}^* e^{\mathbf{A}^*(T-s-h)} \mathrm{d}s,$$

and a control function that steers the state of the system from $\mathbf{x}_0$ *to* $\mathbf{x}_T$ *given in eq* (4.3.7) *reduces to*

$$\mathbf{u}(t) = \begin{cases}\mathbf{B}^* e^{\mathbf{A}^*(T-t-h)}\mathbf{W}^{-1}\big[\mathbf{x}_T - e^{\mathbf{A}T}\mathbf{x}_0 - \mathbf{a}_0(T)\big], & t \in [0, T-h],\\ \mathbf{0}, & t \in (T-h, T],\end{cases}$$

where $\mathbf{a}_0(T) = \int_{-h}^{0} e^{\mathbf{A}(T-s-h)}\mathbf{B}\mathbf{u}_0(s)\,\mathrm{d}s$ *is obtained from eq* (4.3.2)*. This condition coincides with the results obtained in Theorem 3.3.1 of an autonomous case with* $N = 1$ *in Chapter 3.*

4.4 Main results

In this section, we obtain the controllability results of the semilinear delay system (4.2.1) for the classes of nonlinearities $\mathbf{f}(\cdot,\cdot,\cdot)$ given by

(i) the class of bounded functions:
$\mathfrak{B} := \Big\{\mathbf{f}(\cdot,\cdot,\cdot) \;\big|\; \mathbf{f}(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and bounded$\Big\}$,

(ii) the class of Lipschitz functions:
$\mathcal{L}ip := \Big\{\mathbf{f}(\cdot,\cdot,\cdot) \;\big|\; \mathbf{f}(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfying a Lipschitz condition with respect to second and third arguments$\Big\}$,

(iii) the class of linear growth functions:
$\mathcal{LG} := \Big\{\mathbf{f}(\cdot,\cdot,\cdot) \;\big|\; \mathbf{f}(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfying a linear growth condition with respect to second and third arguments$\Big\}$,

and to accomplish this goal, we adopt Schauder's fixed-point theorem and Banach contraction principle as in previous Chapter 3. Of course, first we assume that the fractional semilinear control delay system (4.2.1) admits only one solution for any initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ and for any control function $\mathbf{u}(\cdot) \in \mathcal{B}_2$, and the matrix $\mathbf{W}$ is non-singular. The solution to (4.2.1) satisfies the following equation:

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{E}_\alpha(t^\alpha \mathbf{A})\mathbf{x}_0 + \mathbf{a}_0(t) \\ &\quad + \int_0^{t-h_N} \sum_{i=1}^{N} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s)^\alpha \mathbf{A}\big)\mathbf{f}\big(s, \mathbf{x}(s), \mathbf{u}(s)\big)\,\mathrm{d}s, \text{ for all } t \in [0,T].\end{aligned} \tag{4.4.1}$$

Consider a real Banach space: $\mathcal{X} := \mathcal{B}_1 \times \mathcal{B}_2 = \big\{(\mathbf{x},\mathbf{u}) : \mathbf{x} \in \mathcal{B}_1,\ \mathbf{u} \in \mathcal{B}_2\big\}$ under the norm

$$\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} := \|\mathbf{x}\|_{\mathcal{B}_1} + \|\mathbf{u}\|_{\mathcal{B}_2}$$

and define an operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ by the formula

$$\mathcal{K}(\mathbf{x},\mathbf{u}) := \big(\mathcal{K}_1(\mathbf{x},\mathbf{u}), \mathcal{K}_2(\mathbf{x},\mathbf{u})\big) = (\boldsymbol{y}, \boldsymbol{v}), \tag{4.4.2}$$

where $\mathcal{K}_1 : \mathcal{X} \to \mathcal{B}_1$ is defined by

$$\begin{aligned}&\mathcal{K}_1(\mathbf{x},\mathbf{u})(t) = \boldsymbol{y}(t) \\ &:= \mathbf{E}_\alpha(t^\alpha \mathbf{A})\mathbf{x}_0 + \mathbf{a}_0(t) + \Bigg\{ \int_0^{t-h_N} \left[\sum_{i=1}^{N} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad \times \left[\sum_{i=1}^{N} (T-s-h_i)^{1-\alpha} \mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right]^* \mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} (t-s-h_i)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad \times \left[\sum_{i=1}^{l} (T-s-h_i)^{1-\alpha} \mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1} \mathcal{L}(\mathbf{x},\mathbf{u}) \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}\big((t-s)^\alpha \mathbf{A}\big)\mathbf{f}(s, \mathbf{x}(s), \mathbf{u}(s))\,\mathrm{d}s, \text{ for all } t \in [0,T],\end{aligned} \tag{4.4.3}$$

$\mathcal{K}_2 : \mathcal{X} \to \mathcal{B}_2$ is defined by

$$\mathcal{K}_2(\mathbf{x},\mathbf{u})(t) = \boldsymbol{v}(t)$$
$$:= \begin{cases} \left[\sum_{i=1}^{N} (T-t-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-t-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}), & \text{for all } t \in [0, T-h_N], \\ \left[\sum_{i=1}^{l} (T-t-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-t-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}), & \text{for all } t \in (T-h_{l+1}, T-h_l], \\ \mathbf{0}, & \text{for all } t \in (T-h_1, T], \end{cases} \tag{4.4.4}$$

and the operator $\mathcal{L} : \mathcal{X} \to \mathbb{R}^n$ is defined by

$$\begin{aligned} \mathcal{L}(\mathbf{x},\mathbf{u}) := \mathbf{x}_T - \mathbf{E}_\alpha(T^\alpha\mathbf{A})\mathbf{x}_0 - \mathbf{a}_0(T) \\ - \int_0^T (T-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((T-s)^\alpha\mathbf{A}\big)\mathbf{f}\big(s,\mathbf{x}(s),\mathbf{u}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{4.4.5}$$

We use the following theorem to establish the controllability of the system (4.2.1).

Theorem 4.4.1. *The system* (4.2.1) *is controllable on* $[0,T]$ *over* $\mathbb{R}^n$ *if and only if for every initial state* $\mathbf{x}_0 \in \mathbb{R}^n$ *and a final state* $\mathbf{x}_T \in \mathbb{R}^n$, *the operator* $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ *given in eqs* (4.4.2)–(4.4.5) *has a fixed-point, i.e.* $\mathcal{K}(\mathbf{x},\mathbf{u}) = (\mathbf{x},\mathbf{u})$ *for some* $(\mathbf{x},\mathbf{u}) \in \mathcal{X}$.

Proof. The proof of this is similar to Theorem 3.4.1 of Chapter 3. □

Now we introduce the following notations for the convenience, which will be used in the proof of next theorems:

$$\begin{aligned} M_1 &:= \sup_{0\le s\le t\le T} \|\mathbf{E}_{\alpha,\alpha}\big((t-s)^\alpha\mathbf{A}\big)\|, \\ M_2 &:= \sup_{t\in[0,T]} \|\mathbf{E}_\alpha(t^\alpha\mathbf{A})\|, \\ M_3 &:= \sup_{t\in[0,T]} \|\mathbf{a}_0(t)\|_{\mathbb{R}^n}, \\ M_4 &:= \sup_{t\in[0,T]} \Bigg\| \int_0^{t-h_N} \left[\sum_{i=1}^{N}(t-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^\alpha\mathbf{A}\big)\mathbf{B}_i\right] \\ &\quad\times \left[\sum_{i=1}^{N}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha\mathbf{A}\big)\mathbf{B}_i\right]^* \mathrm{d}s \\ &\quad+ \sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l}(t-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s-h_1)^\alpha\mathbf{A}\big)\mathbf{B}_i\right] \end{aligned}$$

$$\times \left[\sum_{i=1}^{l}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_1)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathrm{d}s\Bigg\|,$$

$$M_5 := \max_{l}\Bigg\{\sup_{[0,T-h_N]}\Bigg\|\left[\sum_{i=1}^{N}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\Bigg\|,$$
$$\sup_{(T-h_{l+1},T-h_l]}\Bigg\|\left[\sum_{i=1}^{l}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\Bigg\|\Bigg\}.$$

4.4.1 Controllability results for a class of bounded nonlinearities

For the class of bounded nonlinearities, the following theorem gives the controllability result.

Theorem 4.4.2. *If $\mathbf{f}(\cdot,\cdot,\cdot)$ is bounded on its domain, i.e. there exists some constant $K \geq 0$ such that $\|\mathbf{f}(t,\mathbf{v},\mathbf{w})\|_{\mathbb{R}^n} \leq K$, for all $(t,\mathbf{v},\mathbf{w}) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m$, then the fractional semilinear delay system* (4.2.1) *is controllable on* $[0,T]$.

Proof. Let $\mathcal{B} = \{(\mathbf{x},\mathbf{u}) \in \mathcal{X} : 0 \leq \|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq r_0\}$ be a nonempty, closed and convex subset of $\mathcal{X}$ for some $r_0 > 0$. To apply the Schauder's fixed-point theorem, we have to prove that $\mathcal{K}$ is a continuous operator from $\mathcal{B}$ into a compact subset of $\mathcal{B}$. Then the rest of the proof follows from Theorem 4.4.1.

Step 1: To show $\mathcal{K}$ is a continuous operator on $\mathcal{B}$.

As usual, first we show that $\mathcal{K}_1$ and $\mathcal{K}_2$ are continuous operators on $\mathcal{B}$. Let $(\mathbf{x}_1,\mathbf{u}_1), (\mathbf{x}_2,\mathbf{u}_2) \in \mathcal{B}$ be such that $\|(\mathbf{x}_1,\mathbf{u}_1)-(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} \to 0$. Since $\mathbf{f}(\cdot,\cdot,\cdot)$ is a continuous function on its domain, so in particular with respect to the second and third arguments, so

$$\sup_{t\in[0,T]}\|\mathbf{f}(t,\mathbf{x}_1(t),\mathbf{u}_1(t)) - \mathbf{f}(t,\mathbf{x}_2(t),\mathbf{u}_2(t))\|_{\mathbb{R}^n} \to 0,$$

as $\|(\mathbf{x}_1,\mathbf{u}_1)-(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}} \to 0$.

The continuity of $\mathcal{K}_1$ follows from the following estimation:

$$\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{B}_1}$$
$$= \sup_{t\in[0,T]}\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)(t)-\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)(t)\|_{\mathbb{R}^n}$$
$$= \sup_{t\in[0,T]}\Bigg\|\Bigg(\int_0^{t-h_N}\left[\sum_{i=1}^{N}(t-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]$$
$$\times\left[\sum_{i=1}^{N}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathrm{d}s$$
$$+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\left[\sum_{i=1}^{l}(t-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]$$

$$
\begin{aligned}
&\times \left[\sum_{i=1}^{l}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathrm{d}s\Bigg)\mathbf{W}^{-1}\\
&\times \big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)\\
&+\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s)^{\alpha}\mathbf{A}\big)\big[\mathbf{f}\big(s,\mathbf{x}_1(s),\mathbf{u}_1(s)\big)-\mathbf{f}\big(s,\mathbf{x}_2(s),\mathbf{u}_2(s)\big)\big]\mathrm{d}s\Bigg\|_{\mathbb{R}^n}\\
\leq{}& \big(1+M_4\|\mathbf{W}^{-1}\|\big)M_1T^{\alpha}\alpha^{-1}\sup_{s\in[0,T]}\big\|\mathbf{f}\big(s,\mathbf{x}_1(s),\mathbf{u}_1(s)\big)-\mathbf{f}\big(s,\mathbf{x}_2(s),\mathbf{u}_2(s)\big)\big\|_{\mathbb{R}^n}.
\end{aligned}
$$

The continuity of $\mathcal{K}_2$ follows from the following estimation:

$$
\begin{aligned}
&\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{B}_2}\\
&\quad=\sup_{[0,T]}\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)(t)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)(t)\|_{\mathbb{R}^m}\\
&\quad=\max_{l}\Bigg\{\sup_{[0,T-h_N]}\Bigg\|\left[\sum_{i=1}^{N}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathbf{W}^{-1}\\
&\qquad\qquad\times\big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)\Bigg\|_{\mathbb{R}^m},\\
&\qquad\sup_{(T-h_{l+1},T-h_l]}\Bigg\|\left[\sum_{i=1}^{l}(T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i\right]^{*}\mathbf{W}^{-1}\\
&\qquad\qquad\times\big(\mathcal{L}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{L}(\mathbf{x}_2,\mathbf{u}_2)\big)\Bigg\|_{\mathbb{R}^m}\Bigg\}\\
&\quad\leq M_5\|\mathbf{W}^{-1}\|\,M_1T^{\alpha}\alpha^{-1}\sup_{s\in[0,T]}\|\mathbf{f}(s,\mathbf{x}_1(s),\mathbf{u}_1(s))-\mathbf{f}(s,\mathbf{x}_2(s),\mathbf{u}_2(s))\|_{\mathbb{R}^n}.
\end{aligned}
$$

Finally, the continuity of $\mathcal{K}$ on $\mathcal{B}$ follows from following estimate:

$$
\begin{aligned}
\|\mathcal{K}(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}(\mathbf{x}_2,\mathbf{u}_2)\|_{\mathcal{X}}&=\big\|\big(\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1),\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)\big)-\big(\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2),\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\big)\big\|_{\mathcal{X}}\\
&=\big\|\big(\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2),\ \mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\big)\big\|_{\mathcal{X}}\\
&=\big\|\mathcal{K}_1(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_1(\mathbf{x}_2,\mathbf{u}_2)\big\|_{\mathcal{B}_1}+\big\|\mathcal{K}_2(\mathbf{x}_1,\mathbf{u}_1)-\mathcal{K}_2(\mathbf{x}_2,\mathbf{u}_2)\big\|_{\mathcal{B}_2}.
\end{aligned}
$$

Step 2: $\mathcal{K}(\mathcal{B})$ is a compact set.

For this we show that $\mathcal{K}_1(\mathcal{B})=\{\mathcal{K}_1(\mathbf{x},\mathbf{u}):\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}\leq r_0\}\subset\mathcal{B}_1$ is equicontinuous on $[0,T]$ and $\mathcal{K}_2(\mathcal{B})=\{\mathcal{K}_2(\mathbf{x},\mathbf{u}):\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}\leq r_0\}\subset\mathcal{B}_2$ is equicontinuous on $[0,T-h_N]$, $(T-h_i,T-h_{i-1}],\ldots,(T-h_1,T]$, $i=2,\ldots,N$, and both are uniformly bounded on $[0,T]$.

The following estimate obtained from eq (4.4.5) is used in this step.

$$
\|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n}\leq\|\mathbf{x}_T\|_{\mathbb{R}^n}+M_2\|\mathbf{x}_0\|_{\mathbb{R}^n}+M_3+M_1KT^{\alpha}\alpha^{-1}. \tag{4.4.6}
$$

For any s_1, s_2 $(s_1 < s_2)$ in $[0,T]$ and for any $\mathcal{K}_1(\mathbf{x},\mathbf{u}) \in \mathcal{K}_1(\mathcal{B})$, we have

$$
\begin{aligned}
&\|\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_1(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^n} \\
&\quad = \Bigg\| \big(\mathbf{E}_\alpha(s_1^\alpha \mathbf{A}) - \mathbf{E}_\alpha(s_2^\alpha \mathbf{A})\big)\mathbf{x}_0 + \mathbf{a}_0(s_1) - \mathbf{a}_0(s_2) \\
&\qquad + \Bigg[\Bigg\{ \int_0^{s_1-h_N} \Bigg(\sum_{i=1}^N (s_1-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \Bigg) \\
&\qquad \times \Bigg(\sum_{i=1}^N (T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \Bigg)^* \mathrm{d}s \\
&\qquad - \int_0^{s_2-h_N} \Bigg(\sum_{i=1}^N (s_2-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \Bigg) \\
&\qquad \times \Bigg(\sum_{i=1}^N (T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \Bigg)^* \mathrm{d}s \Bigg\} + \cdots \\
&\qquad + \Bigg\{ \int_{s_1-h_2}^{s_1-h_1} (s_1-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1 \\
&\qquad \times \big[(T-s-h_1)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1\big]^* \mathrm{d}s \\
&\qquad - \int_{s_2-h_2}^{s_2-h_1} (s_1-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1 \\
&\qquad \times \big[(T-s-h_1)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1\big]^* \mathrm{d}s \Bigg\}\Bigg] \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) \\
&\qquad + \int_0^{s_1} (s_1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s)^\alpha \mathbf{A}\big)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\,\mathrm{d}s \\
&\qquad - \int_0^{s_2} (s_1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s)^\alpha \mathbf{A}\big)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\,\mathrm{d}s \Bigg\|_{\mathbb{R}^n}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \leq \|\mathbf{x}_0\|_{\mathbb{R}^n} \|\mathbf{E}_\alpha(s_1^\alpha \mathbf{A}) - \mathbf{E}_\alpha(s_2^\alpha \mathbf{A})\| + \|\mathbf{a}_0(s_1) - \mathbf{a}_0(s_2)\|_{\mathbb{R}^n} \\
&\qquad + \Bigg\{ \sup_{s\in[0,T]} \Bigg\| \Bigg[\sum_{i=1}^N (T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \Bigg]^* \Bigg\| \\
&\qquad \times \Bigg(\Bigg\| \int_0^{s_1-h_N} \sum_{i=1}^N \big[(s_1-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_i)^\alpha \mathbf{A}\big) \\
&\qquad - (s_2-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_i)^\alpha \mathbf{A}\big)\big]\mathbf{B}_i\,\mathrm{d}s \Bigg\| \\
&\qquad + \Bigg\| \int_{s_1-h_N}^{s_2-h_N} \sum_{i=1}^N (s_2-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\,\mathrm{d}s \Bigg\| \Bigg) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{s\in[0,T]} \big\| \big[(T-s-h_1)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_1)^{\alpha}\mathbf{A}\big)\mathbf{B}_1\big]^*\big\| \\
&\times \Bigg(\Bigg\| \int_{s_1-h_2}^{s_1-h_1} \big[(s_1-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_1)^{\alpha}\mathbf{A}\big) \\
&- (s_2-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_1)^{\alpha}\mathbf{A}\big)\big]\mathbf{B}_1\,\mathrm{d}s \Bigg\| \\
&+ \Bigg\| \int_{s_2-h_2}^{s_1-h_2} (s_2-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_1)^{\alpha}\mathbf{A}\big)\mathbf{B}_1\,\mathrm{d}s \Bigg\| \\
&+ \Bigg\| \int_{s_1-h_1}^{s_2-h_1} (s_2-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_1)^{\alpha}\mathbf{A}\big)\mathbf{B}_1\,\mathrm{d}s \Bigg\| \Bigg) \Bigg\} \\
&\times \|\mathbf{W}^{-1}\| \left(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_1KT^{\alpha}\alpha^{-1}\right) \\
&+ K \left\| \int_0^{s_1} \big[(s_1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s)^{\alpha}\mathbf{A}\big) - (s_2-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s)^{\alpha}\mathbf{A}\big)\big]\mathrm{d}s \right\| \\
&+ M_1K\alpha^{-1}(s_2-s_1)^{\alpha}.
\end{aligned}
$$

Clearly, the right hand side is independent of $\mathbf{x}$ and $\mathbf{u}$, and if we take $s_1 \to s_2$, then we see that $\|\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_1(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^n} \to 0$, for all $\mathcal{K}_1(\mathbf{x},\mathbf{u}) \in \mathcal{K}_1(\mathcal{B})$. Therefore $\mathcal{K}_1(\mathcal{B})$ is equicontinuous on $[0,T]$.

The uniform boundedness of $\mathcal{K}_1(\mathcal{B})$ is confirmed from the following estimation:

$$
\begin{aligned}
\|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} &= \sup_{t\in[0,T]} \|\mathcal{K}_1(\mathbf{x},\mathbf{u})(t)\|_{\mathbb{R}^n} \\
&= \sup_{t\in[0,T]} \Bigg\| \mathbf{E}_{\alpha}(t^{\alpha}\mathbf{A})\mathbf{x}_0 + \mathbf{a}_0(t) \\
&\quad + \Bigg\{ \int_0^{t-h_N} \left[\sum_{i=1}^{N} (t-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i \right] \\
&\quad \times \left[\sum_{i=1}^{N} (T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i \right]^* \mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \left[\sum_{i=1}^{l} (t-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i) \right] \\
&\quad \times \left[\sum_{i=1}^{l} (T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^{\alpha}\mathbf{A}\big)\mathbf{B}_i \right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u}) \\
&\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((t-s)^{\alpha}\mathbf{A}\big)\mathbf{f}(s,\mathbf{x}(s),\mathbf{u}(s))\mathrm{d}s \Bigg\|_{\mathbb{R}^n} \\
&\le M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_4\|\mathbf{W}^{-1}\|\, \|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n} + M_1KT^{\alpha}\alpha^{-1} \\
&\le M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_4\|\mathbf{W}^{-1}\|\big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3
\end{aligned}
$$

$$+ M_1 K T^\alpha \alpha^{-1}\big) + M_1 K T^\alpha \alpha^{-1},$$

that is

$$\begin{aligned} \|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} \leq (1 + M_4\|\mathbf{W}^{-1}\|)\big(M_2\|\mathbf{x}_0\| + M_3 + M_1 K T^\alpha \alpha^{-1}\big) \\ + M_4\|\mathbf{W}^{-1}\|\|\mathbf{x}_T\|. \end{aligned} \tag{4.4.7}$$

For the equicontinuity of $\mathcal{K}_2(\mathcal{B})$, choose $s_1, s_2 \in [t_0, T - h_N]$ or $s_1, s_2 \in (T - h_i, T - h_{i-1}]$ or $s_1, s_2 \in (T - h_1, T]$ with $s_1 < s_2$, and $\mathcal{K}_2(\mathbf{x},\mathbf{u}) \in \mathcal{K}_2(\mathcal{B})$. And consider

$$\begin{aligned} \|\mathcal{K}_2(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_2(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^m} &\leq \sum_i \Big\| \Big(\big[(T - s_1 - h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T - s_1 - h_i)^\alpha \mathbf{A}\big) \\ &\quad - (T - s_2 - h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T - s_2 - h_i)^\alpha \mathbf{A}\big)\big]\mathbf{B}_i\Big)^* \Big\| \\ &\quad \times \|\mathbf{W}^{-1}\|\ \|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n} \\ &\leq \sum_i \Big\| \Big(\big[(T - s_1 - h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T - s_1 - h_i)^\alpha \mathbf{A}\big) \\ &\quad - (T - s_2 - h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T - s_2 - h_i)^\alpha \mathbf{A}\big)\big]\mathbf{B}_i\Big)^* \Big\| \\ &\quad \times \|\mathbf{W}^{-1}\|\ \Big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_1 K T^\alpha \alpha^{-1}\Big). \end{aligned}$$

For uniform boundedness of $\mathcal{K}_2(\mathcal{B})$, see the following estimation:

$$\begin{aligned} \|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} &= \sup_{[0,T]} \|\mathcal{K}_2(\mathbf{x},\mathbf{u})(t)\|_{\mathbb{R}^m} \\ &= \max_l \Bigg\{ \sup_{[0,T-h_N]} \Bigg\| \Bigg[\sum_{i=1}^N (T - t - h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T - t - h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\Bigg]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u})\Bigg\|_{\mathbb{R}^m}, \\ &\qquad \sup_{(T-h_{l+1},T-h_l]} \Bigg\| \Bigg[\sum_{i=1}^l (T - t - h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T - t - h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i\Bigg]^* \mathbf{W}^{-1}\mathcal{L}(\mathbf{x},\mathbf{u})\Bigg\|_{\mathbb{R}^m} \Bigg\}, \end{aligned}$$

that is

$$\|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} \leq M_5\ \|\mathbf{W}^{-1}\|\big(\|\mathbf{x}_T\|_{\mathbb{R}^n} + M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_1 K T^\alpha \alpha^{-1}\big). \tag{4.4.8}$$

Now $\mathcal{K}(\mathcal{B}) = \mathcal{K}_1(\mathcal{B}) \times \mathcal{K}_2(\mathcal{B}) = \big\{\big(\mathcal{K}_1(\mathbf{x},\mathbf{u}),\ \mathcal{K}_2(\mathbf{x},\mathbf{u})\big) : \|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq r_0\big\}$ is an equicontinuous set on each subintervals $[0, T - h_N]$, $(T - h_N, T - h_{N-1}], ..., (T - h_1, T]$ and uniformly bounded on $[0, T]$. Consequently, if we take a sequence $\big\{\mathcal{K}_1^n(\mathbf{x},\mathbf{u}), \mathcal{K}_2^n(\mathbf{x},\mathbf{u})\big\}$ in $\mathcal{K}(\mathcal{B})$, this sequence is uniformly bounded and equicontinuous on each interval, and in particular on $[0, T - h_N]$, so by Arzela–Ascoli theorem, there exists a subsequence $\big\{\mathcal{K}_1^{n_1}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_1}(\mathbf{x},\mathbf{u})\big\}$ of $\big\{\mathcal{K}_1^n(\mathbf{x},\mathbf{u}), \mathcal{K}_2^n(\mathbf{x},\mathbf{u})\big\}$ which is uniformly convergent on $[0, T - h_N]$.

Consider the sequence $\big\{\mathcal{K}_1^{n_1}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_1}(\mathbf{x},\mathbf{u})\big\}$ which is equicontinuous and uniformly bounded on each interval, in particular on $(T - h_N, T - h_{N-1}]$, and, for the same reason, there exists a subsequence $\{\mathcal{K}_1^{n_2}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_2}(\mathbf{x},\mathbf{u})\}$ of $\{\mathcal{K}_1^{n_1}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_1}(\mathbf{x},\mathbf{u})\}$ which is uniformly convergent on $[0, T - h_{N-1}]$. Continuing this process for the intervals $(T - h_{N-1}, T - h_{N-2}], ..., (T - h_1, T]$, we see that the sequence $\{\mathcal{K}_1^{n_{(N+1)}}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_{(N+1)}}(\mathbf{x},\mathbf{u})\}$ is uniformly convergent on $[0, T]$. Thus $\{\mathcal{K}_1^{n}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n}(\mathbf{x},\mathbf{u})\}$ being an arbitrary sequence in $\mathcal{K}(\mathcal{B})$, has a converging subsequence $\{\mathcal{K}_1^{n_{(N+1)}}(\mathbf{x},\mathbf{u}), \mathcal{K}_2^{n_{(N+1)}}(\mathbf{x},\mathbf{u})\}$ on $[0, T]$ in $\mathcal{K}(\mathcal{B})$. Therefore $\mathcal{K}(\mathcal{B})$ is compact in $\mathcal{X}$.

Step 3: $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$.

Let $\mathcal{K}(\mathbf{x},\mathbf{u}) \in \mathcal{K}(\mathcal{B})$ be any element. We use the estimates (4.4.7) and (4.4.8) to get

$$\begin{aligned}\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} &= \|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} + \|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} \\ &\leq \left(1 + (M_4 + M_5)\|\mathbf{W}^{-1}\|\right)\left(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_1 K T^{\alpha}\alpha^{-1}\right) \\ &\quad + \|\mathbf{x}_T\|_{\mathbb{R}^n}(M_4 + M_5)\,\|\mathbf{W}^{-1}\|.\end{aligned}$$

Then, we see that $\lim\limits_{\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}\to\infty} \frac{\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}}{\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}} = 0$. Therefore, for a fixed $\varepsilon \in (0,1)$, we have $\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq \varepsilon\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}$ for sufficiently large value of $\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}$, say $r_0 > 0$. Hence, we obtain $\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq \varepsilon r_0 < r_0$. Therefore, we have $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$. □

Remark 4.4.1. *Theorem 4.4.2 shows that, if the fractional-order* $\alpha \in (0,1)$ *semilinear control delay system* (4.2.1) *possesses a unique solution on* $[0,T]$ *for any initial state* $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ *and for any control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$*, the matrix* $\mathbf{W}$ *is non-singular and the continuous function* $\mathbf{f}(\cdot,\cdot,\cdot)$ *is bounded on its domain, then the system* (4.2.1) *is controllable on* $[0,T]$.

4.4.2 Controllability results for a class of Lipschitz nonlinearities

In this subsection, we prove controllability results of the system (4.2.1) for a class of Lipschitz nonlinearities.

Theorem 4.4.3. *In system* (4.2.1)*, if we assume that*

(i) the function $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{L}ip$ *with Lipschitz constants* $\alpha_0,\ \beta_0 \geq 0$*, i.e.*

$$\|\mathbf{f}(t,\mathbf{v}_1,\mathbf{w}_1) - f(t,\mathbf{v}_2,\mathbf{w}_2)\|_{\mathbb{R}^n} \leq \alpha_0\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{R}^n} + \beta_0\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbb{R}^m},$$

for all $(t,\mathbf{v}_1,\mathbf{w}_1),\ (t,\mathbf{v}_2,\mathbf{w}_2) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m$,

(ii) $\delta := M_1 T^{\alpha}\alpha^{-1}\gamma\left(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\right) < 1$*, where*

$$\gamma := \max\{\alpha_0,\ \beta_0\}, \tag{4.4.9}$$

then the fractional-order semilinear delay system (4.2.1) *is controllable on* $[0, T]$.

Proof. We apply Banach contraction principle to show that the operator $\mathcal{K}$ is contraction, and then the proof follows from Theorem 4.4.1. For this consider

$$\|\mathcal{K}(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{X}} = \|\mathcal{K}_1(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}_1(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_1} + \|K_2(\mathbf{x}_1, \mathbf{u}_1) - K_2(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_2}.$$

The estimations for $\|\mathcal{K}_1(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}_1(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_1}$ and $\|K_2(\mathbf{x}_1, \mathbf{u}_1) - K_2(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{B}_2}$ given in Step 1 in the proof of previous theorem forces us to write

$$\begin{aligned}
&\|\mathcal{K}(\mathbf{x}_1, \mathbf{u}_1) - \mathcal{K}(\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{X}} \\
&\qquad \le M_1 T^{\alpha} \alpha^{-1} \big(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\big) \\
&\qquad\quad \times \sup_{0 \le s \le T} \|\mathbf{f}(s, \mathbf{x}_1(s), \mathbf{u}_1(s)) - \mathbf{f}(s, \mathbf{x}_2(s), \mathbf{u}_2(s))\|_{\mathbb{R}^n} \\
&\qquad \le M_1 T^{\alpha} \alpha^{-1} \big(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\big) \gamma \big(\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathcal{B}_1} + \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{B}_2}\big) \\
&\qquad \le \delta \, \|(\mathbf{x}_1, \mathbf{u}_1) - (\mathbf{x}_2, \mathbf{u}_2)\|_{\mathcal{X}}.
\end{aligned}$$

Since $\delta < 1$, so $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a contraction map and hence from Banach contraction principle, $\mathcal{K}$ has a unique fixed-point in $\mathcal{X}$. Then by Theorem 4.4.1, the system (4.2.1) is controllable on $[0, T]$. □

Remark 4.4.2. *Theorem 4.4.3 shows that, if the fractional-order* $\alpha \in (0, 1)$ *semilinear delay system* (4.2.1) *possesses a unique solution on* $[0, T]$ *for any initial state* $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ *and for any control function* $\mathbf{u}(\cdot) \in \mathcal{B}_2$, *the matrix* $\mathbf{W}$ *is non-singular and the continuous function* $\mathbf{f}(\cdot, \cdot, \cdot)$ *satisfies a Lipschitz condition as defined above, then the system* (4.2.1) *is controllable on* $[0, T]$, *provided*

$$M_1 T^{\alpha} \alpha^{-1} \gamma \big(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\big) < 1,$$

where γ *is given in eq* (4.4.9).

4.4.3 Controllability results for a class of nonlinearities satisfying the linear growth condition

The controllability results of the system (4.2.1) for a class of nonlinearities satisfying the linear growth condition, is established in this subsection using Theorem 4.4.1.

Theorem 4.4.4. *In system* (4.2.1), *if we assume that*

(i) the function $\mathbf{f}(\cdot, \cdot, \cdot) \in \mathcal{LG}$ *with growth constants* $a_0, b_0, c_0 \ge 0$, *i.e.*

$$\|\mathbf{f}(t, \mathbf{v}, \mathbf{w})\|_{\mathbb{R}^n} \le a_0 \|\mathbf{v}\|_{\mathbb{R}^n} + b_0 \|\mathbf{w}\|_{\mathbb{R}^m} + c_0, \text{ for all } (t, \mathbf{v}, \mathbf{w}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

(ii) $M_1 T^{\alpha} \alpha^{-1} (a_0 + b_0) \big(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\big) < 1$,

then the fractional semilinear delay system (4.2.1) *is controllable on* $[0, T]$.

Proof. The proof is again based on Schauder's fixed-point theorem and is similar to the proof of Theorem 4.4.2. Let $\mathcal{B}$ be a nonempty, closed and convex subset of $\mathcal{X}$ as defined earlier. The continuity of $\mathcal{K}$ on $\mathcal{B}$ is already established in the proof of Theorem 4.4.2. Now to show $\mathcal{K}(\mathcal{B})$ is compact in $\mathcal{B}$, first recall the following inequality to be derived from eq (4.4.5):

$$\begin{aligned} \|\mathcal{L}(\mathbf{x},\mathbf{u})\|_{\mathbb{R}^n} \leq \big(&M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + \|\mathbf{x}_T\|_{\mathbb{R}^n} + M_3 + M_1 T^\alpha \alpha^{-1}\big) \\ &+ M_1 T^\alpha \alpha^{-1}\big(a_0\|\mathbf{x}\|_{\mathcal{B}_1} + b_0\|\mathbf{u}\|_{\mathcal{B}_2}\big). \end{aligned} \tag{4.4.10}$$

The equicontinuity of the set $\mathcal{K}_1(\mathcal{B})$ on $[0, T]$ and its uniform boundedness on $[0, T]$ follows from following estimations:

$$\begin{aligned} &\|\mathcal{K}_1(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_1(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^n} \\ &\quad\leq \|\mathbf{x}_0\|_{\mathbb{R}^n}\|\mathbf{E}_\alpha(s_1^\alpha \mathbf{A}) - \mathbf{E}_\alpha(s_2^\alpha \mathbf{A})\| + \|\mathbf{a}_0(s_1) - \mathbf{a}_0(s_2)\|_{\mathbb{R}^n} \\ &\qquad + \Bigg\{ \sup_{s\in[0,T]} \Bigg\| \Bigg[\sum_{i=1}^{N} (T-s-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \Bigg]^* \Bigg\| \\ &\qquad \times \Bigg(\Bigg\| \int_0^{s_1-h_N} \sum_{i=1}^{N} \big[(s_1-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_i)^\alpha \mathbf{A}\big) \\ &\qquad - (s_2-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_i)^\alpha \mathbf{A}\big)\big]\mathbf{B}_i \mathrm{d}s \Bigg\| \\ &\qquad + \Bigg\| \int_{s_1-h_N}^{s_2-h_N} \sum_{i=1}^{N} (s_2-s-h_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_i)^\alpha \mathbf{A}\big)\mathbf{B}_i \,\mathrm{d}s \Bigg\| \Bigg) + \cdots \\ &\qquad + \sup_{s\in[0,T]} \big\| \big[(T-s-h_1)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1\big]^* \big\| \\ &\qquad \times \Bigg(\Bigg\| \int_{s_1-h_2}^{s_1-h_1} \big[(s_1-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s-h_1)^\alpha \mathbf{A}\big) \\ &\qquad - (s_2-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_1)^\alpha \mathbf{A}\big)\big]\mathbf{B}_1 \,\mathrm{d}s \Bigg\| \\ &\qquad + \Bigg\| \int_{s_2-h_2}^{s_1-h_2} (s_2-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1 \,\mathrm{d}s \Bigg\| \\ &\qquad + \Bigg\| \int_{s_1-h_1}^{s_2-h_1} (s_2-s-h_1)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s-h_1)^\alpha \mathbf{A}\big)\mathbf{B}_1 \,\mathrm{d}s \Bigg\| \Bigg) \Bigg\} \\ &\qquad \times \|\mathbf{W}^{-1}\| \Big(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + \|\mathbf{x}_T\|_{\mathbb{R}^n} + M_3 + M_1 T^\alpha \alpha^{-1}\big((a_0+b_0)r_0 + c_0\big) \Big) \\ &\qquad + \Bigg(\Bigg\| \int_0^{s_1} \big[(s_1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_1-s)^\alpha \mathbf{A}\big) - (s_2-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big((s_2-s)^\alpha \mathbf{A}\big)\big]\mathrm{d}s \Bigg\| \end{aligned}$$

$$+ M_1\alpha^{-1}(s_2 - s_1)^{\alpha}\Big)\big((a_0+b_0)r_0 + c_0\big).$$

and

$$\begin{aligned}\|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} &\leq M_4\|\mathbf{W}^{-1}\|\|\mathbf{x}_T\|_{\mathbb{R}^n} + \big(1+M_4\|\mathbf{W}^{-1}\|\big)\\ &\quad\times\Big(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_1T^{\alpha}\alpha^{-1}\big(a_0\|\mathbf{x}\|_{\mathcal{B}_1} + b_0\|\mathbf{u}\|_{\mathcal{B}_2} + c_0\big)\Big)\\ &\leq M_4\|\mathbf{W}^{-1}\|\|\mathbf{x}_T\|_{\mathbb{R}^n} + \big(1+M_4\|\mathbf{W}^{-1}\|\big)\\ &\quad\times\Big(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3 + M_1T^{\alpha}\alpha^{-1}\big((a_0+b_0)r_0 + c_0\big)\Big).\end{aligned} \tag{4.4.11}$$

Similarly, the equicontinuity of the set $\mathcal{K}_2(\mathcal{B})$ on $[0, T-h_N]$, $(T-h_i, T-h_{i-1}]$ and $(T-h_1, T]$ and its uniform boundedness on $[0, T]$ are guaranteed by the following estimations:

$$\begin{aligned}\|\mathcal{K}_2(\mathbf{x},\mathbf{u})(s_1) - \mathcal{K}_2(\mathbf{x},\mathbf{u})(s_2)\|_{\mathbb{R}^m} &\leq \sum_i \bigg\|\Big(\big[(T-s_1-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s_1-h_i)^{\alpha}\mathbf{A}\big)\\ &\quad - (T-s_2-h_i)^{1-\alpha}\mathbf{E}_{\alpha,\alpha}\big((T-s_2-h_i)^{\alpha}\mathbf{A}\big)\big]\mathbf{B}_i\Big)^*\bigg\|\\ &\quad\times\|\mathbf{W}^{-1}\|\Big(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + \|\mathbf{x}_T\|_{\mathbb{R}^n} + M_3 + M_1T^{\alpha}\alpha^{-1}\\ &\quad\times\big((a_0+b_0)r_0 + c_0\big)\Big)\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2} &\leq M_5\|\mathbf{W}^{-1}\|\Big(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + \|\mathbf{x}_T\|_{\mathbb{R}^n} + M_3 + M_1T^{\alpha}\alpha^{-1}\\ &\quad\times\big(a_0\|\mathbf{x}\|_{\mathcal{B}_1} + b_0\|\mathbf{u}\|_{\mathcal{B}_2} + c_0\big)\Big)\\ &\leq M_5\|\mathbf{W}^{-1}\|\Big(M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + \|\mathbf{x}_T\|_{\mathbb{R}^n} + M_3 + M_1T^{\alpha}\alpha^{-1}\\ &\quad\times\big((a_0+b_0)r_0 + c_0\big)\Big).\end{aligned} \tag{4.4.12}$$

Now by the same argument as given in the proof of Theorem 4.4.2, we say that $\mathcal{K}(\mathcal{B})$ is compact set. Finally to show $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$, take an element $\mathcal{K}(\mathbf{x},\mathbf{u}) \in \mathcal{K}(\mathcal{B})$ and use the estimates (4.4.11) and (4.4.12) to obtain

$$\begin{aligned}\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} &= \|\mathcal{K}_1(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_1} + \|\mathcal{K}_2(\mathbf{x},\mathbf{u})\|_{\mathcal{B}_2}\\ &\leq \|\mathbf{W}^{-1}\|\|\mathbf{x}_T\|_{\mathbb{R}^n}(M_4+M_5) + \big(1+\|\mathbf{W}^{-1}\|(M_4+M_5)\big)\Big(M_1T^{\alpha}\alpha^{-1}\\ &\quad\times\big((a_0+b_0)\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} + c_0\big) + M_2\|\mathbf{x}_0\|_{\mathbb{R}^n} + M_3\Big).\end{aligned}$$

Then we have,

$$\lim_{\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}\to\infty}\frac{\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}}{\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}} \leq M_1T^{\alpha}\alpha^{-1}(a_0+b_0)\big(1+\|\mathbf{W}^{-1}\|(M_4+M_5)\big) < 1.$$

Hence for some $\varepsilon \in (0,1)$ with $M_1 T^{\alpha}\alpha^{-1}(a_0+b_0)\big(1+\|\mathbf{W}^{-1}\|(M_4+M_5)\big) < \varepsilon$, we have $\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq \varepsilon\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}$, for sufficiently large values of $\|(\mathbf{x},\mathbf{u})\|_{\mathcal{X}}$, say $r_0 > 0$. Hence $\|\mathcal{K}(\mathbf{x},\mathbf{u})\|_{\mathcal{X}} \leq \varepsilon r_0 < r_0$. Thus we finally have $\mathcal{K}(\mathcal{B}) \subset \mathcal{B}$. □

Remark 4.4.3. *According to Theorem 4.4.4, if the fractional-order $\alpha \in (0,1)$ semilinear delay system (4.2.1) possesses a unique solution on $[0,T]$ for any initial state $\mathbf{x}(0)=\mathbf{x}_0 \in \mathbb{R}^n$ and for any control function $\mathbf{u}(\cdot) \in \mathcal{B}_2$, the matrix $\mathbf{W}$ is non-singular and the continuous function $\mathbf{f}(\cdot,\cdot,\cdot)$ satisfies the linear growth condition as defined above, then the system (4.2.1) is controllable on $[0,T]$, provided*

$$M_1 T^{\alpha}\alpha^{-1}(a_0+b_0)\big(1+\|\mathbf{W}^{-1}\|(M_4+M_5)\big) < 1.$$

4.5 Numerical examples

Consider the following semilinear autonomous system of fractional-order $\alpha = 0.5$ with two delays in control function:

$$\left.\begin{aligned}
\begin{bmatrix} {}^cD_t^{0.5}x_1(t) \\ {}^cD_t^{0.5}x_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbf{u}(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}(t-2) \\
&\quad + t\,\sin\left(\frac{1}{1+\mathbf{u}^2(t)}\right)\begin{bmatrix} \sin(x_1(t)) \\ \cos(x_2(t)) \end{bmatrix}, \text{ for } t \in [0,3], \\
\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
\mathbf{u}(t) &= t^2, \ \ t \in [-2,0).
\end{aligned}\right\} \tag{4.5.1}$$

On comparing this equation with (4.2.1), we get

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ h_1 = 1,\ h_2 = 2,\ T = 3,$$
$$\mathbf{f}\big(t,\mathbf{x}(t),\mathbf{u}(t)\big) = t\,\sin\left(\frac{1}{1+\mathbf{u}^2(t)}\right)\begin{bmatrix} \sin(x_1(t)) \\ \cos(x_2(t)) \end{bmatrix}.$$

Further $\mathbf{W}_1$ and $\mathbf{W}_2$ are given by

$$\mathbf{W}_1 = \int_1^2 \mathbf{E}_{0.5,0.5}\big((2-s)^{0.5}\mathbf{A}\big)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{E}_{0.5,0.5}\big((2-s)^{0.5}\mathbf{A}^*\big)\,\mathrm{d}s,$$
$$\mathbf{W}_2 = \int_0^1 \mathbf{E}_{0.5,0.5}\big((2-s)^{0.5}\mathbf{A}\big)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{E}_{0.5,0.5}\big((2-s)^{0.5}\mathbf{A}^*\big)\,\mathrm{d}s$$

$$
\begin{aligned}
&+\int_0^1 (2-s)^{-0.5}(1-s)^{0.5}\mathbf{E}_{0.5,0.5}\big((2-s)^{0.5}\mathbf{A}\big)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\mathbf{E}_{0.5,0.5}\big((1-s)^{0.5}\mathbf{A}^*\big)\,\mathrm{d}s\\
&+\int_0^1 (2-s)^{0.5}(1-s)^{-0.5}\mathbf{E}_{0.5,0.5}\big((1-s)^{0.5}\mathbf{A}\big)\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\mathbf{E}_{0.5,0.5}\big((2-s)^{0.5}\mathbf{A}^*\big)\,\mathrm{d}s\\
&+\int_0^1 \mathbf{E}_{0.5,0.5}\big((1-s)^{0.5}\mathbf{A}\big)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\mathbf{E}_{0.5,0.5}\big((1-s)^{0.5}\mathbf{A}^*\big)\,\mathrm{d}s,
\end{aligned}
$$

where $\mathbf{E}_{0.5,0.5}((2-s)^{0.5}\mathbf{A}) = \begin{bmatrix}P_1 & P_2\\ -P_2 & P_1\end{bmatrix}$, $\mathbf{E}_{0.5,0.5}((1-s)^{0.5}\mathbf{A}) = \begin{bmatrix}Q_1 & Q_2\\ -Q_2 & Q_1\end{bmatrix}$;

$$
\begin{aligned}
P_1 &= 0.5\big[\mathbf{E}_{0.5}(i(2-s)^{0.5}) + \mathbf{E}_{0.5}(-i(2-s)^{0.5})\big],\\
P_2 &= 0.5i\big[\mathbf{E}_{0.5}(-i(2-s)^{0.5}) - \mathbf{E}_{0.5}(i(2-s)^{0.5})\big],\\
Q_1 &= 0.5\big[\mathbf{E}_{0.5}(i(1-s)^{0.5}) + \mathbf{E}_{0.5}(-i(1-s)^{0.5})\big],\\
Q_2 &= 0.5i\big[\mathbf{E}_{0.5}(-i(1-s)^{0.5}) - \mathbf{E}_{0.5}(i(1-s)^{0.5})\big].
\end{aligned}
$$

By calculation we found that

$$
\mathbf{W}_1 = \int_1^2 \begin{bmatrix}P_1^2 & -P_1P_2\\ -P_2P_1 & P_2^2\end{bmatrix}\mathrm{d}s = \begin{bmatrix}0.43233 & 0.61934+0.864665i\\ 0.61934+0.864665i & -0.58364+2.477364i\end{bmatrix} \text{ and}
$$

$$
\begin{aligned}
\mathbf{W}_2 &= \int_0^1 \begin{bmatrix}P_1^2 & -P_1P_2\\ -P_2P_1 & P_2^2\end{bmatrix}\mathrm{d}s + \int_0^1 (2-s)^{-0.5}(1-s)^{0.5}\begin{bmatrix}P_1Q_2 & P_1Q_1\\ -P_2Q_2 & -P_2Q_1\end{bmatrix}\mathrm{d}s\\
&\quad+\int_0^1 (2-s)^{0.5}(1-s)^{-0.5}\begin{bmatrix}Q_2P_1 & -Q_2P_2\\ Q_1P_1 & -Q_1P_2\end{bmatrix}\mathrm{d}s + \int_0^1 \begin{bmatrix}Q_2^2 & Q_2Q_1\\ Q_1Q_2 & Q_1^2\end{bmatrix}\mathrm{d}s\\
&= \begin{bmatrix}0.058510 & 0.266053+0.11702i\\ 0.266053+0.11702i & 1.041312+1.064212i\end{bmatrix}\\
&\quad+\begin{bmatrix}-0.121814-0.142785i & 0.0713927\\ -0.368482-0.943107i & 0.349739+0.142785i\end{bmatrix}\\
&\quad+\begin{bmatrix}-0.475277-0.957679i & -0.347954-4.83987i\\ 0.478839 & 1.94466+0.957679i\end{bmatrix}\\
&\quad+\begin{bmatrix}-0.58364+2.477364i & -0.61934-0.864665i\\ -0.61934-0.864665i & 0.43233\end{bmatrix}\\
&= \begin{bmatrix}-1.122214+1.3769i & -0.6298483-5.5875i\\ -0.24293-1.690737i & 3.768041+2.164676i\end{bmatrix}.
\end{aligned}
$$

$$
\therefore \mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 = \begin{bmatrix}-0.689884+1.3769i & -0.0105083-4.722835i\\ 0.37641-0.826072i & 3.184401+4.64204i\end{bmatrix}.
$$

Since $\mathbf{W}$ is non-singular matrix and $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{C}([0,3] \times \mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$ is bounded on its domain, hence by Theorem 4.4.2, the system (4.5.1) is controllable on $[0,3]$.

4.6 Conclusions

In this chapter, we considered an $n-$dimensional fractional-order $\alpha \in (0,1)$ semilinear control system with multiple time-delays in control function and derived the sufficient conditions to guarantee that this system (4.2.1) is controllable on $[0,T]$ for certain classes of nonlinearities $\mathbf{f}(\cdot,\cdot,\cdot)$. The results are obtained by employing Schauder's fixed-point theorem and Banach contraction principle. By assuming that, for a given initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ and for a given control function $\mathbf{u}(\cdot) \in \mathcal{B}_2$, the fractional-order semilinear delay system (4.2.1) admits a unique solution on $[0,T]$ and the matrix $\mathbf{W}$ is non-singular, we have established that the system (4.2.1) is controllable on $[0,T]$ under one of the following assumptions:

(i) $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathfrak{B}$.

(ii) $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{L}ip$ with

$$M_1 T^{\alpha} \alpha^{-1} \gamma \big(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\big) < 1,$$

where γ is given in eq (4.4.9).

(iii) $\mathbf{f}(\cdot,\cdot,\cdot) \in \mathcal{LG}$ with

$$M_1 T^{\alpha} \alpha^{-1} (a_0 + b_0) \big(1 + \|\mathbf{W}^{-1}\|(M_4 + M_5)\big) < 1.$$

Numerical example is provided to demonstrate our theoretical results.

Here we note that Theorem 4.4.4 gives the controllability conditions of the system (4.2.1) for much larger class of functions $\mathbf{f}(\cdot,\cdot,\cdot)$ than that of Theorems 4.4.2 and 4.4.3 as the class of linear growth functions contains the class of bounded and Lipschitz functions.

Chapter 5

Controllability of linear impulsive matrix Lyapunov ordinary differential systems with multiple constant time-delays in control

5.1 Introduction

The matrix differential equations occurs naturally in many areas of control theory and game theory [39]. Recently the research on controllability of matrix Lyapunov differential systems have attracted many researchers, as these systems have numerous applications in control theory, such as stability analysis and optimal control. For the linear matrix Lyapunov systems, the controllability was examined in [83]. Later the work has been extended to the nonlinear systems in [36]. More recently, in [37], authors investigated the controllability of both linear and nonlinear finite-dimensional matrix Lyapunov impulsive ordinary differential systems. Further, if such systems involves time-delays in control, then characterizing their controllability properties is important, and so far now, there are no results are available in the literature in this direction and it motivates to study this problem.

In this chapter, we establish the controllability results of linear impulsive matrix Lyapunov systems having multiple constant time-delays in control, in terms of a matrix rank condition for two classes of admissible control functions. Further, under each class of admissible control functions, the corresponding steering control has been computed. The controllability conditions are further reduced for the corresponding system without impulses and with delays; system with impulses and without delays; system without impulses and without delays. The numerical examples given in the last section of this chapter, shows how the controlled trajectory and steering control behaves under different classes of admissible control functions.

This chapter is organized as follows: in Section 5.2, some preliminaries are given and the controllability problem for a class of linear impulsive matrix Lyapunov ordinary differential system with multiple constant time delays in control is formulated. In Section 5.3, the matrix Lyapunov ordinary differential system is converted into vector differential system, by applying vector operator. Section 5.4 contains the main results of the chapter, where we establish controllability results for the system for two classes of admissible control functions. In Section 5.5, we give illustrative examples by considering the autonomous systems to demonstrate the theoretical results. Further, the control function and controlled trajectory are plotted for the given classes of admissible control functions. Conclusions are given in Section 5.6.

5.2 System description

The following dynamical control system modelled by an $(n\times n)-$dimensional linear impulsive matrix Lyapunov ordinary differential equations with multiple constant time-delays in control as

$$\left.\begin{aligned} \dot{\mathbf{X}}(t) &= \mathbf{A}_1(t)\mathbf{X}(t)+\mathbf{X}(t)\mathbf{A}_2(t)+\sum_{i=1}^{N}\mathbf{B}_i(t)\mathbf{U}(t-h_i),\\ &\qquad\qquad t\in[t_0,T]\setminus\{t_k : k=1,\dots,M\},\\ \mathbf{X}(t_0) &= \mathbf{X}_0,\\ \Delta\mathbf{X}(t_k) &:= \mathbf{X}(t_k^+)-\mathbf{X}(t_k) = D^k\mathbf{U}(t_k)\mathbf{X}(t_k),\\ \mathbf{U}(t) &= \mathbf{U}_0(t),\ \ t\in[t_0-h_N,t_0), \end{aligned}\right\} \tag{5.2.1}$$

where,

(i) the state $\mathbf{X}(t)\in\mathbb{R}^{n\times n}$ with a given initial state $\mathbf{X}(t_0)=\mathbf{X}_0\in\mathbb{R}^{n\times n}$,

(ii) the control $\mathbf{U}(t)\in\mathbb{R}^{m\times n}$,

(iii) $\mathbf{A}_1(\cdot)$, $\mathbf{A_2}(\cdot)\in\mathcal{C}\big([t_0,T];\,\mathbb{R}^{n\times n}\big)$ and $\mathbf{B}_i(\cdot)\in\mathcal{C}\big([t_0,T];\,\mathbb{R}^{n\times m}\big)$,

(iv) $t_0\le t_1\le\cdots\le t_M<T$, t_k's are the fixed times at which the state function $\mathbf{X}(\cdot)$ experiences impulses and are state independent,

(v) $0\le h_1\le h_2\le\cdots\le h_N\le\min\big\{(t_1-t_0),(t_2-t_1),\dots,(t_M-t_{M-1}),(T-t_M)\big\}$, h_i's are the known constant time delays in the control function $\mathbf{U}(\cdot)$,

(vi) $\Delta(\mathbf{X}(t_k))$ is an impulse in the state function $\mathbf{X}(\cdot)$ at the time t_k,

(vii) $D^k\mathbf{U}(t_k):=\sum_{r=1}^{m}\sum_{j=1}^{n}d_{rj}^kU_{rj}(t_k)\mathbf{I}_n$, d_{rj}^k are the known constant real numbers, $\mathbf{I}_n$ is an $n\times n$ identity matrix, and the matrix $\mathbf{U}(t_k)=\big(U_{rj}(t_k)\big)$,

(viii) $\mathbf{U}_0(\cdot)\in\mathcal{C}([t_0-h_N,t_0);\mathbb{R}^{m\times n})$ denotes the given initial control function (and is assumed to be bounded on its domain) applied to the system (5.2.1), the subscripts $i=1,\dots,N$ and $k=1,\dots,M$.

The natural space to work on the solvability of system (5.2.1) is the real Banach space given by

$$\mathcal{B}_1 := \Big\{\mathbf{X}(\cdot) \,\Big|\, \mathbf{X}(\cdot) : [t_0, T] \to \mathbb{R}^{n\times n},\ \mathbf{X}(\cdot) \text{ is a continuous function on } [t_0, T] \setminus \{t_k : k = 1, \ldots, M\} \text{ and differentiable a.e. on } [t_0, T] \text{ such that there exists } \mathbf{X}(t_k^-) := \lim_{t\uparrow t_k} \mathbf{X}(t) \text{ and } \mathbf{X}(t_k^+) := \lim_{t\downarrow t_k} \mathbf{X}(t) \text{ with } \mathbf{X}(t_k^-) = \mathbf{X}(t_k),\ \forall\, k \text{ and } \mathbf{X}(t_0) = \lim_{t\downarrow t_0} \mathbf{X}(t)\Big\},$$

endowed with the norm

$$\|\mathbf{X}(\cdot)\|_{\mathcal{B}_1} := \sup_{t\in[t_0,T]} \|\mathbf{X}(t)\|,$$

and we choose the space of admissible controls as

$$\mathcal{B}_2 := \Big\{\mathbf{U}(\cdot) \,\Big|\, \mathbf{U}(\cdot) : [t_0, T] \to \mathbb{R}^{m\times n},\ \mathbf{U}(\cdot) \text{ is continuous and bounded function with finite number of discontinuity points on } [t_0, T]\Big\},$$

endowed with the norm

$$\|\mathbf{U}(\cdot)\|_{\mathcal{B}_2} := \sup_{t\in[t_0,T]} \|\mathbf{U}(t)\|.$$

Definition 5.2.1 (Controllability)**.** *The system* (5.2.1) *is said to be controllable over* $\mathbb{R}^{n\times n}$ *on* $[t_0, T]$*, if for all* $\mathbf{X}_0, \mathbf{X}_T \in \mathbb{R}^{n\times n}$ *and for every bounded function* $\mathbf{U}_0(\cdot) \in \mathcal{C}\big([t_0 - h_N, t_0)\,; \mathbb{R}^{m\times n}\big)$ *there exists at least one control function* $\mathbf{U}(\cdot) \in \mathcal{B}_2$ *such that, with this control function on* $[t_0, T]$*, the corresponding solution to the system* (5.2.1) *with* $\mathbf{X}(t_0) = \mathbf{X}_0$, $\mathbf{U}(t) = \mathbf{U}_0(t),\ t \in [t_0 - h_N, t_0)$*, satisfies the condition* $\mathbf{X}(T) = \mathbf{X}_T$*.*

5.3 Conversion of matrix Lyapunov ordinary differential system into vector differential system

In this section, the matrix Lyapunov differential system (5.2.1) is converted into vector differential system by applying the vector operator defined in Definition 2.1.2.

$$\begin{aligned}
vec\,\dot{\mathbf{X}}(t) &= vec\left(\mathbf{A}_1(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{A}_2(t) + \sum_{i=1}^{N} \mathbf{B}_i(t)\mathbf{U}(t - h_i)\right), \\
&\qquad\qquad \text{for } t \in [t_0, T] \setminus \{t_k : k = 1, \ldots, M\}, \\
vec\,\mathbf{X}(t_0) &= vec\,\mathbf{X}_0, \\
vec\,\Delta(\mathbf{X}(t_k)) &= vec\,\big(D^k\mathbf{U}(t_k)\mathbf{X}(t_k)\big), \\
vec\,\mathbf{U}(t) &= vec\,\mathbf{U}_0(t),\ t \in [t_0 - h_N, t_0).
\end{aligned}$$

That is,

$$\left.\begin{aligned} vec\,\dot{\mathbf{X}}(t) &= \big(\mathbf{I}_n \otimes \mathbf{A}_1(t)+\mathbf{A}_2^*(t)\otimes\mathbf{I}_n\big)\,vec\,\mathbf{X}(t)+\sum_{i=1}^{N}\big(\mathbf{I}_n\otimes\mathbf{B}_i(t)\big)\,vec\,\mathbf{U}(t-h_i),\\ &\qquad\qquad\text{for } t\in[t_0,T]\setminus\{t_k : k=1,\ldots,M\},\\ vec\,\mathbf{X}(t_0) &= vec\,\mathbf{X}_0,\\ vec\,\Delta(\mathbf{X}(t_k)) &= \big(\mathbf{I}_n\otimes D^k\mathbf{U}(t_k)\big)\,vec\,\mathbf{X}(t_k),\\ vec\,\mathbf{U}(t) &= vec\,\mathbf{U}_0(t),\ t\in[t_0-h_N,t_0). \end{aligned}\right\} \tag{5.3.1}$$

Now we introduce the following notations:

$$\begin{aligned} \mathbf{x}(t) &:= vec\,\mathbf{X}(t)\in\mathbb{R}^{n^2},\\ \mathbf{u}(t) &:= vec\,\mathbf{U}(t)\in\mathbb{R}^{mn},\\ \mathbf{x}_0 &:= vec\,\mathbf{X}_0\in\mathbb{R}^{n^2},\\ \mathbf{A}(t) &:= \big(\mathbf{I}_n\otimes\mathbf{A}_1(t)+\mathbf{A}_2^*(t)\otimes\mathbf{I}_n\big)_{n^2\times n^2},\\ \mathbf{C}_i(t) &:= \big(\mathbf{I}_n\otimes\mathbf{B}_i(t)\big)_{n^2\times mn},\\ d^k\mathbf{u}(t_k) &:= \big(\mathbf{I}_n\otimes D^k\mathbf{U}(t_k)\big)_{n^2\times n^2} = \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k)\mathbf{I}_{n^2} = \alpha_k\mathbf{I}_{n^2},\ \text{where}\\ \alpha_k &:= \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k)\in\mathbb{R}. \end{aligned}$$

With the above notations, the system (5.3.1) becomes,

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t)+\sum_{i=1}^{N}\mathbf{C}_i(t)\mathbf{u}(t-h_i),\ \ t\in[t_0,T]\setminus\{t_k : k=1,\ldots,M\},\\ \mathbf{x}(t_0) &= \mathbf{x}_0,\\ \Delta\mathbf{x}(t_k) &= d^k\mathbf{u}(t_k)\mathbf{x}(t_k) = \alpha_k\mathbf{x}(t_k),\\ \mathbf{u}(t) &= \mathbf{u}_0(t),\ t\in[t_0-h_N,t_0). \end{aligned}\right\} \tag{5.3.2}$$

Proposition 5.3.1. *The matrix Lyapunov ordinary differential system* (5.2.1) *is controllable over* $\mathbb{R}^{n\times n}$ *on the time interval* $[t_0,T]$ *if and only if the vector differential system* (5.3.2) *is controllable over* $\mathbb{R}^{n^2}$ *on the same time interval* $[t_0,T]$.

The proof of this proposition is trivial, as the systems (5.2.1) and (5.3.2) are identical with each other under the vector operator.

5.4 Main results

In this section, the controllability results of the system (5.2.1) for certain classes of admissible control functions are obtained. First, we recall the following lemma from [36].

Lemma 5.4.1. *Let* $\mathbf{\Phi}_1(t,t_0)$ *and* $\mathbf{\Phi}_2(t,t_0)$ *be the state-transition matrices for* $\mathbf{A}_1(t)$ *and* $\mathbf{A}_2^*(t)$, *respectively. Then the state-transition matrix for* $\mathbf{A}(t)$ *is given by* $\mathbf{\Phi}(t,t_0) = \mathbf{\Phi}_2(t,t_0) \otimes \mathbf{\Phi}_1(t,t_0)$, *where* $\mathbf{A}(t) = \big(\mathbf{I}_n \otimes \mathbf{A}_1(t) + \mathbf{A}_2^*(t) \otimes \mathbf{I}_n\big)_{n^2 \times n^2}$.

The solution to the linear impulsive vector differential delay system (5.3.2) is obtained in the following theorem.

Theorem 5.4.1. *The solution to the system* (5.3.2) *in the time interval* $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, M$, *with* $t_{M+1} = T$ *is given by,*

$$
\begin{aligned}
\mathbf{x}(t) = {} & \prod_{j=1}^{k}(1+\alpha_j)\mathbf{\Phi}(t,t_0)(\mathbf{x_0}+\mathbf{a_0}) \\
& + \int_{t_0}^{t_1-h_N} \prod_{j=1}^{k}(1+\alpha_j)\sum_{i=1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l} \Bigg\{ \prod_{j=1}^{k}(1+\alpha_j)\sum_{i=1}^{l} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \\
& + \prod_{j=2}^{k}(1+\alpha_j)\sum_{i=l+1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \Bigg\} \mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{q=1}^{k-1}\Bigg\{ \int_{t_q-h_1}^{t_{q+1}-h_N} \prod_{j=q+1}^{k}(1+\alpha_j)\sum_{i=1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l} \Bigg(\prod_{j=q+1}^{k}(1+\alpha_j)\sum_{i=1}^{l} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \\
& + \prod_{j=q+2}^{k}(1+\alpha_j)\sum_{i=l+1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \Bigg) \mathbf{u}(s)\,\mathrm{d}s \Bigg\} \\
& + \int_{t_k-h_1}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s,
\end{aligned}
\tag{5.4.1}
$$

where for convention it is assumed that $\prod_{j=k+1}^{k}(1+\alpha_j) = 1$, $\prod_{j=k+2}^{k}(1+\alpha_j) = 0$, *for all*

$k = 1, 2, \ldots, M$ *and*

$$\mathbf{a}_0 := \sum_{i=1}^{N} \int_{t_0-h_i}^{t_0} \mathbf{\Phi}(t_0, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}_0(s)\,\mathrm{d}s.$$

Proof. Let $\mathbf{\Phi}(t)$ be the fundamental matrix solution of the homogeneous system $\dot{\mathbf{x}}(\mathbf{t})(t) = \mathbf{A}(t)\mathbf{x}(t)$ and hence $\mathbf{\Phi}(t,s) := \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$ is the corresponding state-transition matrix associated with the matrix $\mathbf{A}(t)$. The solution to the system (5.3.2) on $[t_0, t_1]$ is given by,

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t,s) \sum_{i=1}^{N} \mathbf{C}_i(s)\mathbf{u}(s-h_i)\,\mathrm{d}s \\ &= \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \mathbf{\Phi}(t,t_0) \sum_{i=1}^{N} \int_{t_0-h_i}^{t_0} \mathbf{\Phi}(t_0, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}_0(s)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{N} \int_{t_0}^{t-h_i} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}$$

Since

$$\sum_{i=1}^{N} \int_{t_0-h_i}^{t_0} \mathbf{\Phi}(t_0, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}_0(s)\,\mathrm{d}s = \mathbf{a}_0 \in \mathbb{R}^{n^2}, \tag{5.4.2}$$

therefore

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \sum_{i=1}^{N} \int_{t_0}^{t-h_i} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s. \tag{5.4.3}$$

Now simplify the summation given in eq (5.4.3) as

$$\begin{aligned}&\sum_{i=1}^{N} \int_{t_0}^{t-h_i} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\qquad = \int_{t_0}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \qquad (5.4.4) \\ &\qquad\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}$$

Using eq (5.4.4) in eq (5.4.3), the solution to the system (5.3.2) on $[t_0, t_1]$ is given by,

$$\begin{aligned}\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned} \tag{5.4.5}$$

Now as $\mathbf{x}(t_1^+) = \mathbf{x}(t_1) + \Delta\mathbf{x}(t_1) = (1+\alpha_1)\mathbf{x}(t_1)$, the solution to the system (5.3.2) on $(t_1, t_2]$ is given by

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t, t_1)\mathbf{x}(t_1^+) + \int_{t_1}^{t} \mathbf{\Phi}(t, s) \sum_{i=1}^{N} \mathbf{C}_i(s)\mathbf{u}(s-h_i)\,\mathrm{d}s \\ &= (1+\alpha_1)\mathbf{\Phi}(t, t_1)\Bigg\{\mathbf{\Phi}(t_1, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{t_1-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t_1, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{t_1-h_{l+1}}^{t_1-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t_1, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\Bigg\} \\ &\quad + \sum_{i=1}^{N} \int_{t_1-h_i}^{t-h_i} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &= (1+\alpha_1)\Bigg\{\mathbf{\Phi}(t, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{t_1-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{t_1-h_{l+1}}^{t_1-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\Bigg\} \\ &\quad + \sum_{l=1}^{N-1} \int_{t_1-h_{l+1}}^{t_1-h_l} \sum_{i=l+1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \int_{t_1-h_1}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathbf{x}(t) = &\prod_{j=1}^{1}(1+\alpha_j)\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\
&+ \int_{t_0}^{t_1-h_N}\prod_{j=1}^{1}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\Bigg\{\prod_{j=1}^{1}(1+\alpha_1)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \\
&+ \prod_{j=2}^{1}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\Bigg\}\mathbf{u}(s)\,\mathrm{d}s \\
&+ \int_{t_1-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}
\tag{5.4.6}
$$

Again, as $\mathbf{x}(t_2^+) = \mathbf{x}(t_2) + \Delta\mathbf{x}(t_2) = (1+\alpha_2)\mathbf{x}(t_2)$, the solution to the system (5.3.2) on $(t_2, t_3]$ is given by

$$
\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Phi}(t,t_2)\mathbf{x}(t_2^+) + \int_{t_2}^{t}\mathbf{\Phi}(t,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\mathbf{u}(s-h_i)\,\mathrm{d}s \\
&= (1+\alpha_2)\mathbf{\Phi}(t,t_2)\Bigg\{(1+\alpha_1)\mathbf{\Phi}(t_2,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\
&\quad + \int_{t_0}^{t_1-h_N}(1+\alpha_1)\sum_{i=1}^{N}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\Bigg[(1+\alpha_1)\sum_{i=1}^{l}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i) \\
&\quad + \sum_{i=l+1}^{N}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\Bigg]\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \int_{t_1-h_1}^{t_2-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1}\int_{t_2-h_{l+1}}^{t_2-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\Bigg\} \\
&\quad + \sum_{i=1}^{N}\int_{t_2-h_i}^{t-h_i}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= (1+\alpha_1)(1+\alpha_2)\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\
&\quad + \int_{t_0}^{t_1-h_N} (1+\alpha_1)(1+\alpha_2)\sum_{i=1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l} \Bigg[(1+\alpha_1)(1+\alpha_2)\sum_{i=1}^{l} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \\
&\quad + (1+\alpha_2)\sum_{i=l+1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\Bigg]\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \int_{t_1-h_1}^{t_2-h_N} (1+\alpha_2)\sum_{i=1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1}\int_{t_2-h_{l+1}}^{t_2-h_l} (1+\alpha_2)\sum_{i=1}^{l} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1}\int_{t_2-h_{l+1}}^{t_2-h_l} \sum_{i=l+1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \int_{t_2-h_1}^{t-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&\quad + \sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{x}(t) &= \prod_{j=1}^{2}(1+\alpha_j)\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\
&+ \int_{t_0}^{t_1-h_N} \prod_{j=1}^{2}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\left\{\prod_{j=1}^{2}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&+ \left.\prod_{j=2}^{2}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right\}\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{q=1}^{1}\left\{\int_{t_q-h_1}^{t_{q+1}-h_N}\prod_{j=q+1}^{2}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\right. \qquad (5.4.7) \\
&+ \sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l}\left(\prod_{j=q+1}^{2}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&+ \left.\left.\prod_{j=q+2}^{2}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right)\mathbf{u}(s)\mathrm{d}s\right\} \\
&+ \int_{t_2-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}
$$

Continuing this process on subintervals $(t_3,t_4],\ldots,(t_M,T]$, the solution to system (5.3.2) on subinterval $(t_k,t_{k+1}]$ is given by eq (5.4.1). □

Let us now define the following matrices which will be used throughout the chapter:

$$
\begin{aligned}
\mathbf{W}_{1} :=& \int_{t_0}^{t_1-h_N} \left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right] \\
&\times \left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s, \\
\mathbf{W}_{kN+1} :=& \int_{t_k-h_1}^{t_{k+1}-h_N} \left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right] \\
&\times \left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s, \\
\mathbf{W}_{qN+1-l} :=& \int_{t_q-h_{l+1}}^{t_q-h_l} \left[\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\left.+\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right] \\
&\times \left[\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\left.+\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s,
\end{aligned}
\tag{5.4.8}
$$

where, for convention it is assumed that $\prod_{j=k+1}^{k}(1+\alpha_j)=1,\ \prod_{j=k+2}^{k}(1+\alpha_j)=0;\ k=1,2,\dots,M;\ l=1,2,\dots,(N-1)$ and $q=1,2,\dots,(M+1)$.

Lemma 5.4.2. *Each* $\mathbf{W}_p,\ p=1,2,\dots,(M+1)N$, *given in* (5.4.8) *is symmetric positive semidefinite* $(n^2\times n^2)-$*matrix and*

$$\operatorname{rank}\big([\mathbf{W}_{1},\ \mathbf{W}_{2},\dots,\mathbf{W}_{(M+1)N}]\big)=\operatorname{rank}\big(\mathbf{W}_{1}+\mathbf{W}_{2}+\cdots+\mathbf{W}_{(M+1)N}\big).$$

Proof. The proof is similar to the Lemma 3.3.1 of Chapter 3. □

The following two classes of admissible control functions are considered in this chapter, for which the controllability results of the system (5.2.1) are obtained:

(i) $\mathcal{U}_1 := \left\{\mathbf{U}(\cdot)\,\middle|\,\mathbf{U}(\cdot)\in\mathcal{B}_2,\ \alpha_k := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^{k}U_{rj}(t_k)\neq -1,\ \forall\, k=1,2,\dots,M\right\}$

(ii) $\mathcal{U}_2 := \left\{\mathbf{U}(\cdot)\,\middle|\,\mathbf{U}(\cdot)\in\mathcal{B}_2,\ \alpha_M := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^{M}U_{rj}(t_M) = -1\right\}.$

5.4.1 Controllability under the class $\mathcal{U}_1$ controllers

Now we will establish the controllability for the system (5.2.1) with the class $\mathcal{U}_1$ controllers.

Theorem 5.4.2. *In system* (5.2.1)*, if the control function* $\mathbf{U}(\cdot) \in \mathcal{U}_1$*, then a necessary and sufficient condition for the controllability of the system* (5.2.1) *on* $[t_0, T]$ *is*

$$\operatorname{rank}\big([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}]\big) = n^2.$$

Proof. To prove this theorem, it is enough to show that the necessary and sufficient condition for the controllability of the system (5.3.2) is $\operatorname{rank}\big([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}]\big) = n^2$ for this class of controllers. Then the proof follows by Proposition 5.3.1.

First let us show that, the condition is sufficient. For this, assume

$$\operatorname{rank}\big([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}]\big) = n^2,$$

so that $\mathbf{W} := \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}$ is a positive definite matrix by Lemma 5.4.2, and hence $\mathbf{W}$ is invertible. Let us define a control function $\mathbf{u}(\cdot) = vec\, \mathbf{U}(\cdot) \in \mathcal{U}_1$ as follows:

$$\mathbf{u}(t) := \begin{cases} \left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)\right. \\ \left. \times\, (\mathbf{x}_0 + \mathbf{a}_0)\right], \quad \text{for all } t \in [t_0, t_1 - h_N], \\ \left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)\right. \\ \left. \times\, (\mathbf{x}_0 + \mathbf{a}_0)\right], \quad \text{for all } t \in (t_k - h_1, t_{k+1} - h_N] \setminus \{t_k\}, \\ \left[\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i) + \prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T, t+h_i)\right. \\ \left. \times\, \mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0)\right], \\ \qquad\qquad \text{for all } t \in (t_q - h_{l+1}, t_q - h_l], \\ \mathbf{v}_k, \qquad\qquad \text{at } t = t_k, \\ \mathbf{0}, \qquad\qquad \text{for all } t \in (T - h_1, T], \end{cases} \tag{5.4.9}$$

where $k = 1, \ldots, M;\ l = 1, \ldots, (N-1);\ q = 1, 2, \ldots, (M+1)$ and $\mathbf{v}_k = \big(v_{rj}^k\big) \in \mathbb{R}^{mn}$ is any vector such that $\sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k v_{rj}^k \neq -1$.

The state $\mathbf{x}(t)$ of the system (5.3.2) given in eq (5.4.1) at $t = T$ satisfies

$$\begin{aligned}
\mathbf{x}(T) = &\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\
&+ \int_{t_0}^{t_1-h_N} \prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\left\{\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\left. + \prod_{j=2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right\}\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{q=1}^{M-1}\left\{\int_{t_q-h_1}^{t_{q+1}-h_N}\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\right. \\
&+ \sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l}\left(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\left.\left. + \prod_{j=q+2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)\mathbf{u}(s)\,\mathrm{d}s\right\} \\
&+ \int_{t_M-h_1}^{T-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}$$

Substituting $\mathbf{u}(t)$ from eq (5.4.9) in the above expression we get,

$$\begin{aligned}
\mathbf{x}(T) = &\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\
&+ \left[\int_{t_0}^{t_1-h_N}\left(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)\right. \\
&\times\left(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)^{*}\mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\left\{\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\left. + \prod_{j=2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right. \\
&\left. + \prod_{j=2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right\}^{*} \mathrm{d}s \\
&+ \sum_{q=1}^{M-1}\left\{ \int_{t_q-h_1}^{t_{q+1}-h_N} \left(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right) \right. \\
&\times \left(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right)^{*} \mathrm{d}s \\
&+ \sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l} \left(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right. \\
&\left. + \prod_{j=q+2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right) \\
&\times \left(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right. \\
&\left.\left. + \prod_{j=q+2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right)^{*} \mathrm{d}s \right\} \\
&+ \int_{t_M-h_1}^{T-h_N} \left(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right) \left(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right)^{*} \mathrm{d}s \\
&+ \left. \sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l} \left(\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right) \left(\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \right)^{*} \mathrm{d}s \right] \\
&\times \mathbf{W}^{-1}\left(\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) \right) \\
= &\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) + \big(\mathbf{W}_1 + \ldots + \mathbf{W}_{(M+1)N}\big)\mathbf{W}^{-1} \\
&\times \left(\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) \right) \\
= &\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) + \mathbf{W}\mathbf{W}^{-1}\left(\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) \right) = \mathbf{x}_T.
\end{aligned}
$$

Therefore the system (5.3.2) is controllable on $[t_0, T]$, and hence by Proposition 5.3.1, the system (5.2.1) also is controllable on $[t_0, T]$.

The necessary condition can be proved by contradiction. For this, let the system (5.2.1) be controllable on $[t_0, T]$, but assume that $0 \le \mathrm{rank}\big([\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}]\big) < n^2$. Then by Lemma 5.4.2, $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}$ is singular matrix. Hence there exists at least one non-zero vector, say $\mathbf{v} \in \mathbb{R}^{n^2}$ such that $\mathbf{W}\mathbf{v} = \mathbf{0}$, i.e.

$$\big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)\mathbf{v} = \mathbf{0} \Longrightarrow \mathbf{W}_1\mathbf{v} + \mathbf{W}_2\mathbf{v} + \cdots + \mathbf{W}_{(M+1)N}\mathbf{v} = \mathbf{0}.$$

Hence $\mathbf{W}_p\mathbf{v} = \mathbf{0}$ for all p (since each $\mathbf{W}_p$ is positive semidefinite matrix). This shows that each $\mathbf{W}_p$ is a singular matrix and $\big\langle \mathbf{W}_p\mathbf{v}, \mathbf{v}\big\rangle_{\mathbb{R}^{n^2}} = 0$, for all p, i. e.,

$$\begin{cases} \Bigg\langle \displaystyle\int_{t_0}^{t_1-h_N} \left(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right) \\ \times \left(\displaystyle\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)^{*} \mathrm{d}s\,\mathbf{v},\ \mathbf{v}\Bigg\rangle_{\mathbb{R}^{n^2}} = 0, \\ \Bigg\langle \displaystyle\int_{t_k-h_1}^{t_{k+1}-h_N} \left(\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right) \\ \times \left(\displaystyle\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)^{*} \mathrm{d}s\,\mathbf{v},\ \mathbf{v}\Bigg\rangle_{\mathbb{R}^{n^2}} = 0, \\ \Bigg\langle \displaystyle\int_{t_q-h_{l+1}}^{t_q-h_l} \left(\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\ \left. +\displaystyle\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right) \\ \times \left(\displaystyle\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\ \left. +\displaystyle\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)^{*} \mathrm{d}s\,\mathbf{v},\ \mathbf{v}\Bigg\rangle_{\mathbb{R}^{n^2}} = 0. \end{cases}$$

$$\Longrightarrow \begin{cases} \displaystyle\int_{t_0}^{t_1-h_N} \left\|\left(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)^{*}\mathbf{v}\right\|_{\mathbb{R}^{mn}}^{2} \mathrm{d}s = 0, \\ \displaystyle\int_{t_k-h_1}^{t_{k+1}-h_N} \left\|\left(\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)^{*}\mathbf{v}\right\|_{\mathbb{R}^{mn}}^{2} \mathrm{d}s = 0, \\ \displaystyle\int_{t_q-h_{l+1}}^{t_q-h_l} \left\|\left\{\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right.\right. \\ \quad \left.\left. +\displaystyle\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right\}^{*}\mathbf{v}\right\|_{\mathbb{R}^{mn}}^{2} \mathrm{d}s = 0. \end{cases}$$

Since each $\mathbf{C}_i^*(\cdot)$ and $\mathbf{\Phi}^*(\cdot,\cdot)$ are continuous functions, so the above integrals implies that

$$\begin{cases} \mathbf{v}^*\left(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)=\mathbf{0}, \\ \mathbf{v}^*\left(\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)=\mathbf{0}, \\ \mathbf{v}^*\left(\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\ \quad \left.+\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)=\mathbf{0}, \end{cases} \tag{5.4.10}$$

for all $k=1,\ldots,M;\ l=1,\ldots,(N-1);\ q=1,\ldots,(M+1)$, and some $\mathbf{v}\neq 0\in\mathbb{R}^{n^2}$.

We assumed that the system (5.2.1) is controllable on $[t_0,T]$, so the system (5.3.2). In particular, this system (5.3.2) is null controllable on $[t_0,T]$. Now, let us choose an initial state $\mathbf{x}_0=-\mathbf{a}_0+\left(\prod_{j=1}^{M}(1+\alpha_j)\right)^{-1}\mathbf{\Phi}^{-1}(T,t_0)\mathbf{v}$ and a final state $\mathbf{x}(T)=\mathbf{0}$. Then with some control $\mathbf{u}(\cdot)$, the state of the system (5.3.2) given in eq (5.4.1) satisfies $\mathbf{x}(T)=\mathbf{0}$. That is,

$$\begin{aligned} \mathbf{0}=\mathbf{x}(T) &= \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0) \\ &+\int_{t_0}^{t_1-h_N}\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\left\{\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\ &\left.+\prod_{j=2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right\}\mathbf{u}(s)\,\mathrm{d}s \\ &+\sum_{q=1}^{M-1}\left\{\int_{t_q-h_1}^{t_{q+1}-h_N}\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\right. \\ &+\sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l}\left(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\ &\left.\left.+\prod_{j=q+2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right)\mathbf{u}(s)\,\mathrm{d}s\right\} \\ &+\int_{t_M-h_1}^{T-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \end{aligned}$$

$$+\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.$$

In the above expression, substitute $\mathbf{x}_0 = -\mathbf{a}_0 + \Big(\prod_{j=1}^{M}(1+\alpha_j)\Big)^{-1}\mathbf{\Phi}^{-1}(T,t_0)\mathbf{v}$, then premultiply with $\mathbf{v}^*$ and use the results given in (5.4.10) to get, $0 = \mathbf{v}^*\mathbf{v}$. Thus, $\mathbf{v} = \mathbf{0}$. This is a contradiction. Hence our assumption that $0 \le \mathrm{rank}\big([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}]\big) < n^2$ is wrong. Thus, finally we have $\mathrm{rank}\big([\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}]\big) = n^2$. □

Corollary 5.4.1. *In system* (5.2.1)*, if delays are absent in the control function* $\mathbf{U}(\cdot) \in \mathcal{U}_1$*, i.e.* $h_i = 0,\ \forall\, i$*, then the necessary and sufficient condition of controllability of the system* (5.2.1) *given in Theorem 5.4.2 reduces to*

$$\mathrm{rank}\big([\mathbf{W}_1,\, \mathbf{W}_{N+1},\, \mathbf{W}_{2N+1}, \ldots, \mathbf{W}_{MN+1}]\big) = n^2,$$

where $\mathbf{W}_1,\ \mathbf{W}_{N+1},\ \mathbf{W}_{2N+1}, \ldots, \mathbf{W}_{MN+1}$ *are obtained from* (5.4.8) *by taking* $h_i = 0$*, for all* $i = 1, 2, \ldots, N$*, i.e.*

$$\mathbf{W}_{kN+1} := \int_{t_k}^{t_{k+1}}\left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s)\mathbf{C}_i(s)\right]\left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s)\mathbf{C}_i(s)\right]^*\mathrm{d}s,$$

$\forall\, k = 0, 1, \ldots, M$*. Note that in this case, the other* $\mathbf{W}_p$*'s are zero matrices. Further, the steering controller defined in eq* (5.4.9) *reduces to*

$$\mathbf{u}(t) := \begin{cases}\left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,t)\mathbf{C}_i(t)\right]^*\mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\right], & \\ \qquad\qquad \textit{for all } t \in [t_0, t_1], & \\ \left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,t)\mathbf{C}_i(t)\right]^*\mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\right], & \\ \qquad\qquad \textit{for all } t \in (t_k, t_{k+1}], & \end{cases}$$

where $\mathbf{W} := \mathbf{W}_1 + \mathbf{W}_{N+1} + \mathbf{W}_{2N+1} + \cdots + \mathbf{W}_{MN+1}$ *and* $k = 1, \ldots, M$*.*

An interesting special case is the situation when the system (5.2.1) does not have the impulses, for, in that case $\alpha_k := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^{k} U_{rj}(t_k) = 0,\ \forall\ k = 1, 2, \ldots, M$, then the first $(MN+1)-$matrices can be combined to form a single matrix. Thus as a consequence we have the following

Corollary 5.4.2. *If in the system* (5.2.1)*, the control function* $\mathbf{U}(\cdot) \in \mathcal{U}_1$ *such that* $\alpha_k := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^{k} U_{rj}(t_k) = 0$*, for all* $k = 1, 2, \ldots, M$*, then the necessary and sufficient* *condition for the controllability of the system* (5.2.1) *is given by*

$$\mathrm{rank}\big([\mathbf{V},\ \mathbf{W}_{MN+2},\dots,\mathbf{W}_{(M+1)N}]\big)=n^2,$$

where

$$\mathbf{V}:=\int_{t_0}^{T-h_N}\Big[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\Big]\Big[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\Big]^*\mathrm{d}s. \quad (5.4.11)$$

and $\mathbf{W}_{MN+2},\dots,\mathbf{W}_{(M+1)N}$ *are obtained from* (5.4.8) *by taking* $h_i=0,\ \forall\, i=1,\dots,N$, *i.e.*

$$\mathbf{W}_{(M+1)N+1-l}:=\int_{T-h_{l+1}}^{T-h_l}\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^*\mathrm{d}s,$$

where $l=1,\dots,(N-1)$.

Proof. In this scenario, there are no impulses in the system (5.2.1) as $\sum_{r=1}^{m}\sum_{j=1}^{n}d_{rj}^{k}U_{rj}(t_k) = \alpha_k = 0$, so that $D^k\mathbf{U}(t_k)=\mathbf{O}$ and hence $\Delta\mathbf{X}(t_k)=\mathbf{O}$, for all $k=1,\dots,M$. As a result, the matrices $\mathbf{W}_1,\mathbf{W}_2,\dots,\mathbf{W}_{MN+1}$ can be combined to form a single matrix $\mathbf{V}$, i.e. $\mathbf{V}=\mathbf{W}_1+\mathbf{W}_2+\cdots+\mathbf{W}_{MN+1}$. The proof of this corollary lies along the same line as that of Theorem 5.4.2. The steering control function given in eq (5.4.9) reduces to

$$\mathbf{u}(t):=\begin{cases}\left[\displaystyle\sum_{i=1}^{N}\mathbf{\Phi}(T,t+h_i)\mathbf{C}_i(t+h_i)\right]^*\mathbf{W}^{-1}\big[\mathbf{x}_T-\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\big], & \\ \qquad\qquad \text{for all } t\in[t_0,T-h_N], & \\ \left[\displaystyle\sum_{i=1}^{l}\mathbf{\Phi}(T,t+h_i)\mathbf{C}_i(t+h_i)\right]^*\mathbf{W}^{-1}\big[\mathbf{x}_T-\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\big], & \\ \qquad\qquad \text{for all } t\in(T-h_{l+1},T-h_l], & \\ \mathbf{0}, \qquad \text{for all } t\in(T-h_1,T], & \end{cases} \quad (5.4.12)$$

where $\mathbf{W}:=\mathbf{V}+\mathbf{W}_{MN+2}+\cdots+\mathbf{W}_{(M+1)N}$ and $l=1,2,\dots,(N-1)$. □

Corollary 5.4.3. *In system* (5.2.1)*, if delays are absent in the control function* $\mathbf{U}(\cdot)\in\mathcal{U}_1$ *and* $\alpha_k:=\sum_{r=1}^{m}\sum_{j=1}^{n}d_{rj}^{k}U_{rj}(t_k)=0$, *for all* $k=1,2,\dots,M$ *then the necessary and sufficient condition for the controllability of the system* (5.2.1) *obtained in Corollary 5.4.2 reduces to the controllability condition given in Remark 3.4, p.3 of Dubey and George* [36] *and is given by*

$$\mathrm{rank}(\mathbf{V})=n^2,$$

where $\mathbf{V}$ *is obtained from eq* (5.4.11) *by substituting* $h_i = 0$, *for all* $i = 1, 2, \ldots, N$ *i. e.*

$$\mathbf{V} = \int_{t_0}^{T} \left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\right]\left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\right]^* \mathrm{d}s.$$

The matrix $\mathbf{V}$ *given above is called the controllability Grammian of the system* (5.2.1) *having no impulses and no delays. Further the steering controller defined in eq* (5.4.12) *reduces to*

$$\mathbf{u}(t) := \left[\mathbf{\Phi}(T,t)\sum_{i=1}^{N}\mathbf{C}_i(t)\right]^* \mathbf{V}^{-1}\big[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0 + \mathbf{a}_0)\big], \textit{ for all } t \in [t_0, T].$$

5.4.2 Controllability under the class $\mathcal{U}_2$ controllers

In this subsection, a necessary and sufficient condition for the controllability of the system (5.2.1) for the class $\mathcal{U}_2$ of control functions is derived. Further, if the control function $\mathbf{U}(\cdot) \in \mathcal{U}_2$, then $\mathbf{W}_p = \mathbf{O}$, for $p = 1, 2, \ldots, (M-1)N$, $(M-1)N+1$.

Theorem 5.4.3. *In system* (5.2.1), *if the control function belongs to the class* $\mathcal{U}_2$, *then a necessary and sufficient condition for the controllability of the system* (5.2.1) *on* $[t_0, T]$ *is given by*

$$\mathrm{rank}\big([\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}]\big) = n^2.$$

Here $\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}$ *are defined in* (5.4.8) *and are given by*

$$\begin{aligned}
\mathbf{W}_{MN+1-l} &:= \int_{t_M - h_{l+1}}^{t_M - h_l} \left[\sum_{i=l+1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=l+1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s,\\
\mathbf{W}_{MN+1} &:= \int_{t_M - h_1}^{T - h_N} \left[\sum_{i=1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s,\\
\mathbf{W}_{(M+1)N+1-l} &:= \int_{T - h_{l+1}}^{T - h_l} \left[\sum_{i=1}^{l}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{l}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s,
\end{aligned}$$

where $l = 1, \ldots, (N-1)$.

Proof. To prove the sufficiency, we show that if

$$\mathrm{rank}\big([\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}]\big) = n^2,$$

then the system (5.3.2) is controllable under the class $\mathcal{U}_2$ controllers. Then the proof follows by Proposition 5.3.1. Let us begin by considering

$$\operatorname{rank}\big([\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \dots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \dots, \mathbf{W}_{(M+1)N}]\big) = n^2.$$

Then $\mathbf{W} := \mathbf{W}_{(M-1)N+2} + \mathbf{W}_{(M-1)N+3} + \dots + \mathbf{W}_{MN} + \mathbf{W}_{MN+1} + \dots + \mathbf{W}_{(M+1)N}$ is a positive definite matrix. Now define a control function $\mathbf{u}(\cdot) = vec\,\mathbf{U}(\cdot) \in \mathcal{U}_2$ as follows:

$$\mathbf{u}(t) := \begin{cases} \left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\mathbf{x}_T, \text{ for all } t \in (t_M - h_{l+1}, t_M - h_l], \\ \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\mathbf{x}_T, \text{ for all } t \in (t_M - h_1, T - h_N] \setminus \{t_M\}, \\ \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\mathbf{x}_T, \text{ for all } t \in (T - h_{l+1}, T - h_l], \\ \mathbf{v}_M, \quad \text{at } t = t_M, \\ \mathbf{0}, \quad \text{for all } t \in [t_0, t_M - h_N] \cup (T - h_1, T], \end{cases} \tag{5.4.13}$$

where $l = 1, \dots, (N-1)$ and $\mathbf{v}_M = \big(v_{rj}^M\big) \in \mathbb{R}^{mn}$ is an arbitrary vector such that $\sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^M v_{rj}^M = -1$. Now the state $\mathbf{x}(t)$ of the system (5.3.2) given in eq (5.4.1) at $t = T$ satisfies,

$$\begin{aligned} \mathbf{x}(T) = &\sum_{l=1}^{N-1} \int_{t_M - h_{l+1}}^{t_M - h_l} \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &+ \int_{t_M - h_1}^{T - h_N} \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &+ \sum_{l=1}^{N-1} \int_{T - h_{l+1}}^{T - h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s. \end{aligned}$$

Substitute $\mathbf{u}(t)$ from eq (5.4.13) in the above expression to get,

$$\begin{aligned} \mathbf{x}(T) = &\Bigg\{ \sum_{l=1}^{N-1} \int_{t_M - h_{l+1}}^{t_M - h_l} \left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s \\ &+ \int_{t_M - h_1}^{T - h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s \\ &+ \sum_{l=1}^{N-1} \int_{T - h_{l+1}}^{T - h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s \Bigg\} \mathbf{W}^{-1}\mathbf{x}_T \\ = &\big\{\mathbf{W}_{(M-1)N+2} + \mathbf{W}_{(M-1)N+3} + \dots + \mathbf{W}_{MN} + \mathbf{W}_{MN+1} + \dots + \mathbf{W}_{(M+1)N}\big\}\mathbf{W}^{-1}\mathbf{x}_T \\ = &\mathbf{W}\mathbf{W}^{-1}\mathbf{x}_T = \mathbf{x}_T. \end{aligned}$$

Hence the system (5.3.2) is controllable over $\mathbb{R}^{n^2}$ on $[t_0, T]$. Then by Proposition 5.3.1, the system (5.2.1) also is controllable over $\mathbb{R}^{n\times n}$ on $[t_0, T]$.

Now the necessity of the condition can be proved by contradiction. Let the system (5.2.1) be controllable on $[t_0, T]$ for the class $\mathcal{U}_2$ controllers, but assume that

$$0 \le \operatorname{rank}\big([\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \dots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \dots, \mathbf{W}_{(M+1)N}]\big) < n^2.$$

Then the matrix $\mathbf{W} = \mathbf{W}_{(M-1)N+2} + \mathbf{W}_{(M-1)N+3} + \cdots + \mathbf{W}_{MN} + \mathbf{W}_{MN+1} + \cdots + \mathbf{W}_{(M+1)N}$ is singular and hence there exists a non-zero vector, say $\mathbf{v} \in \mathbb{R}^{n^2}$ such that $\mathbf{W}\mathbf{v} = \mathbf{0}$, i.e.

$$\big(\mathbf{W}_{(M-1)N+2} + \mathbf{W}_{(M-1)N+3} + \cdots + \mathbf{W}_{MN} + \mathbf{W}_{MN+1} + \cdots + \mathbf{W}_{(M+1)N}\big)\mathbf{v} = \mathbf{0},$$

and hence each $\mathbf{W}_{(M-1)N+2}$, $\mathbf{W}_{(M-1)N+3}$, $\mathbf{W}_{MN}$, $\mathbf{W}_{MN+1}$, $\mathbf{W}_{(M+1)N}$ is a singular matrix and $\big\langle \mathbf{W}_p\mathbf{v}, \mathbf{v}\big\rangle_{\mathbb{R}^{n^2}} = 0$ for $p = (M-1)N+2,\ (M-1)N+3, \dots, MN,\ (MN+1), \dots, (M+1)N$. Proceeding similar to the Theorem 5.4.2, we get

$$\begin{cases} \mathbf{v}^*\left(\displaystyle\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right) = \mathbf{0}, \\ \mathbf{v}^*\left(\displaystyle\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right) = \mathbf{0}, \\ \mathbf{v}^*\left(\displaystyle\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right) = \mathbf{0}, \end{cases} \tag{5.4.14}$$

for all $l = 1, 2, \dots, (N-1)$.

Since the system (5.2.1) is controllable, so the system (5.3.2) on $[t_0, T]$, and hence any initial state $\mathbf{x}_0$ can be steered to the final state $\mathbf{x}(T) = \mathbf{v}$ with certain control function $\mathbf{u}(\cdot) = vec\,\mathbf{U}(\cdot)$, where $\mathbf{U}(\cdot) \in \mathcal{U}_2$. That is,

$$\begin{aligned} \mathbf{v} = \mathbf{x}(T) &= \sum_{l=1}^{N-1} \int_{t_M-h_{l+1}}^{t_M-h_l} \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \int_{t_M-h_1}^{T-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &\quad + \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s. \end{aligned}$$

Premultiply the above expression with $\mathbf{v}^*$ and use the estimates (5.4.14), to get $\mathbf{v}^*\mathbf{v} = 0$, and hence $\mathbf{v} = \mathbf{0}$, a contradiction. Hence

$$\operatorname{rank}\big([\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \dots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \dots, \mathbf{W}_{(M+1)N}]\big) = n^2.$$

$\square$

Remark 5.4.1. *The control function given in eq* (5.4.13) *is independent of an initial state of the system* (5.2.1) *and depends only on the final state* $\mathbf{X}(T)$ *(where* $\mathbf{x}_T = vec\,\mathbf{X}(T)$*). Therefore this control function steers any initial state of the system* (5.2.1) *to* $\mathbf{X}(T)$.

Corollary 5.4.4. *Suppose the control function* $\mathbf{U}(\cdot) \in \mathcal{U}_2$ *in the system* (5.2.1) *does not have delays, then the necessary and sufficient condition of this system obtained in the Theorem 5.4.3 reduces to the controllability condition given in Theorem 3.1, p.330 of Dubey and George* [37] *and is given by*

$$\text{rank}(\mathbf{W}_{MN+1}) = n^2,$$

where $\mathbf{W}_{MN+1}$ *is obtained from* (5.4.8) *by taking* $h_i = 0$, *for all* $i = 1, 2, \ldots, N$, *i.e.*

$$\mathbf{W}_{MN+1} := \int_{t_M}^{T} \left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\right]\left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\right]^* \mathrm{d}s.$$

Further the steering control function given in eq (5.4.13) *reduces to*

$$\mathbf{u}(t) := \begin{cases} \mathbf{0}, & \text{for all } t \in [t_0, t_M], \\ \left[\mathbf{\Phi}(T,t)\sum_{i=1}^{N}\mathbf{C}_i(t)\right]^* \mathbf{W}_{MN+1}^{-1}\mathbf{x}_T, & \text{for all } t \in (t_M, T]. \end{cases}$$

5.5 Numerical examples

1. Consider the following $(2 \times 2)-$dimensional linear impulsive matrix Lyapunov autonomous ordinary differential system with one impulse and two delays in the control

function:

$$
\left.
\begin{aligned}
\begin{bmatrix} \dot{x}_{11}(t) & \dot{x}_{12}(t) \\ \dot{x}_{21}(t) & \dot{x}_{22}(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} + \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} U_{11}(t-0.2) & U_{12}(t-0.2) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} U_{11}(t-0.4) & U_{12}(t-0.4) \end{bmatrix}, \ t \in [0,1] \setminus \{0.5\}, \\
\begin{bmatrix} x_{11}(0) & x_{12}(0) \\ x_{21}(0) & x_{22}(0) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
\begin{bmatrix} \Delta x_{11}(0.5) & \Delta x_{12}(0.5) \\ \Delta x_{21}(0.5) & \Delta x_{22}(0.5) \end{bmatrix} &= \big(U_{11}(0.5) + U_{12}(0.5)\big) \begin{bmatrix} x_{11}(0.5) & x_{12}(0.5) \\ x_{21}(0.5) & x_{22}(0.5) \end{bmatrix}, \\
\begin{bmatrix} U_{11}(t) & U_{12}(t) \end{bmatrix} &= \begin{bmatrix} 1 & t \end{bmatrix}, \quad t \in [-0.4, 0).
\end{aligned}
\right\} \tag{5.5.1}
$$

After applying the vector operator, the system (5.5.1) becomes

$$
\begin{bmatrix} \dot{x}_{11}(t) \\ \dot{x}_{21}(t) \\ \dot{x}_{12}(t) \\ \dot{x}_{22}(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ x_{12}(t) \\ x_{22}(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{11}(t-0.2) \\ U_{12}(t-0.2) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{11}(t-0.4) \\ U_{12}(t-0.4) \end{bmatrix},
$$

$$
t \in [0,1] \setminus \{0.5\},
$$

$$
\begin{bmatrix} x_{11}(0) \\ x_{21}(0) \\ x_{12}(0) \\ x_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
$$

$$
\begin{bmatrix} \Delta x_{11}(0.5) \\ \Delta x_{21}(0.5) \\ \Delta x_{12}(0.5) \\ \Delta x_{22}(0.5) \end{bmatrix} = \big(U_{11}(0.5) + U_{12}(0.5)\big) \begin{bmatrix} x_{11}(0.5) \\ x_{21}(0.5) \\ x_{12}(0.5) \\ x_{22}(0.5) \end{bmatrix},
$$

$$
\begin{bmatrix} U_{11}(t) \\ U_{12}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad t \in [-0.4, 0).
$$

On comparing the above system with (5.3.2), we get

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, t_0 = 0, h_1 = 0.2, h_2 = 0.4, t_1 = 0.5,$$

$T = 1,\ \alpha = U_{11}(0.5) + U_{12}(0.5).$

By calculation, we get

$$\mathbf{\Phi}(t,s) = \begin{bmatrix} e^{2(t-s)} & 0 & 0 & 0 \\ 0 & e^{4(t-s)} & 0 & 0 \\ 0 & 0 & e^{(t-s)} & 0 \\ 0 & 0 & 0 & e^{3(t-s)} \end{bmatrix}, \quad \mathbf{a}_0 = \begin{bmatrix} 0.1648 \\ 0.1995 \\ -0.0187 \\ -0.0557 \end{bmatrix},$$

$$\mathbf{W}_1 = (1+\alpha)^2 \begin{bmatrix} 2.0217 & 4.1054 & 0 & 0 \\ 4.1054 & 8.3637 & 0 & 0 \\ 0 & 0 & 0.4487 & 1.1094 \\ 0 & 0 & 1.1094 & 2.7521 \end{bmatrix},$$

$$\mathbf{W}_2 = \begin{bmatrix} 2.2636(1+\alpha)^2 & 3.4896(1+\alpha) & 0 & 0 \\ 3.4896(1+\alpha) & 5.4467 & 0 & 0 \\ 0 & 0 & 0.6684(1+\alpha)^2 & 1.2424(1+\alpha) \\ 0 & 0 & 1.2424(1+\alpha) & 2.3393 \end{bmatrix},$$

$$\mathbf{W}_3 = \begin{bmatrix} 1.2907 & 1.2555 & 0 & 0 \\ 1.2555 & 1.2529 & 0 & 0 \\ 0 & 0 & 0.6131 & 0.7084 \\ 0 & 0 & 0.7084 & 0.8416 \end{bmatrix} \text{ and } \mathbf{W}_4 = \begin{bmatrix} 0.3064 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2459 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let the desired final state of the system (5.5.1) be $\mathbf{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Now we compute the steering controller and controlled trajectory in different cases.

Case (i): If we choose the control function from the class $\mathcal{U}_1$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = 1$, then $\text{rank}\big([\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4]\big) = 4$, hence the system (5.5.1) is controllable on $[0,1]$ by Theorem 5.4.2.

Further, $\mathbf{W}^{-1} = \begin{bmatrix} 0.2779 & -0.1706 & 0 & 0 \\ -0.1706 & 0.1297 & 0 & 0 \\ 0 & 0 & 0.8172 & -0.4395 \\ 0 & 0 & -0.4395 & 0.3068 \end{bmatrix}$ and one of the control function that steers the state from $\mathbf{X}_0$ to $\mathbf{X}_1$ is given by

$$\mathbf{U}(t) = \begin{cases} \begin{bmatrix}(31.2e^{-2t} - 54.0563e^{-4t}) & (-2.326e^{-t} + 6.1598e^{-3t})\end{bmatrix}, \text{ for all } t \in [0, 0.1], \\ \begin{bmatrix}(31.2e^{-2t} - 27.0283e^{-4t}) & (-2.326e^{-t} + 3.08e^{-3t})\end{bmatrix}, \text{ for all } t \in (0.1, 0.3], \\ \begin{bmatrix}(15.6e^{-2t} - 27.028e^{-4t}) & (-1.163e^{-t} + 3.08e^{-3t})\end{bmatrix}, \text{ for all } t \in (0.3, 0.6] \setminus \{0.5\}, \\ \begin{bmatrix}15.6e^{-2t} & -1.163e^{-t}\end{bmatrix}, \text{ for all } t \in (0.6, 0.8], \\ \begin{bmatrix}0 & 0\end{bmatrix}, \text{ for all } t \in (0.8, 1], \end{cases}$$

and the controlled trajectory is given by

$$\mathbf{X}(t) = \begin{cases} \begin{bmatrix} 0.5(e^{2t} - 1) & 0.8(e^{t} - 1.25t - 1) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.0135t - 1) \end{bmatrix}, \quad \text{for all } t \in [0, 0.2], \\ \begin{bmatrix} (20.05e^{-4t} - 11.629e^{-2t} - 0.6456e^{2t}) & (-2.806e^{-3t} + 1.4198e^{-t} + 0.289e^{t}) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.136t - 1) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.2, 0.3], \\ \begin{bmatrix} (10.025e^{-4t} - 11.634e^{-2t} + 1.0125e^{2t}) & (-1.403e^{-3t} + 1.4198e^{-t} - 1.3258e^{t}) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.136t - 1) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.3, 0.4], \\ \begin{bmatrix} (10.025e^{-4t} - 11.689e^{-2t} + 1.0406e^{2t}) & (-1.403e^{-3t} + 1.4198e^{-t} - 0.1326e^{t}) \\ (33.469e^{-4t} - 11.4464e^{-2t} - 0.1272e^{4t}) & (-3.409e^{-3t} + 0.8693e^{-t} + 0.0784e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.4, 0.5], \\ \begin{bmatrix} (10.025e^{-4t} - 5.8175e^{-2t} + 0.1584e^{2t}) & (-1.403e^{-3t} + 0.7102e^{-t} + 0.3286e^{t}) \\ (16.734e^{-4t} - 11.5716e^{-2t} + 0.1138e^{4t}) & (-1.7042e^{-3t} + 0.8674e^{-t} + 0.0194e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.5, 0.7], \\ \begin{bmatrix} (10.025e^{-4t} - 5.8175e^{-2t} + 0.1584e^{2t}) & (-1.403e^{-3t} + 0.7102e^{-t} + 0.3286e^{t}) \\ (16.734e^{-4t} - 5.785e^{-2t} + 0.0268e^{4t}) & (-1.7042e^{-3t} + 0.4334e^{-t} + 0.046e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.7, 0.8], \\ \begin{bmatrix} -5.8178e^{-2t} + 0.2415e^{2t} & 0.7102e^{-t} + 0.2715e^{t} \\ (16.734e^{-4t} - 5.785e^{-2t} + 0.0268e^{4t}) & (-1.7042e^{-3t} + 0.4334e^{-t} + 0.046e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.8, 1], \end{cases}$$

and these are shown in the Figures 5.1 and 5.2 respectively.

Case (ii): If we choose the control function from the class $\mathcal{U}_1$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = 0$, then there are no impulses in the system (5.5.1) and hence

the matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ can be combined to get a matrix $\mathbf{V} = \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = \begin{bmatrix} 5.576 & 8.8505 & 0 & 0 \\ 8.8505 & 15.0633 & 0 & 0 \\ 0 & 0 & 1.7302 & 3.0602 \\ 0 & 0 & 3.0602 & 5.933 \end{bmatrix}$. Then we see that $\text{rank}([\mathbf{V},\ \mathbf{W}_4]) = 4$, and hence by Corollary 5.4.2, the system (5.5.1) is controllable on $[0, 1]$. Further,
$\mathbf{W}^{-1} = \begin{bmatrix} 1.4606 & -0.8578 & 0 & 0 \\ -0.8578 & 0.5701 & 0 & 0 \\ 0 & 0 & 2.5147 & -1.2970 \\ 0 & 0 & -1.2970 & 0.8375 \end{bmatrix}$ and one of the control function that steers the state from $\mathbf{X}_0$ to $\mathbf{X}_1$ is given by

$$\mathbf{U}(t) = \begin{cases} \begin{bmatrix} (-60.11e^{-4t} + 40.4515e^{-2t}) & (2.4901e^{-3t} - 0.2351e^{-t}) \end{bmatrix}, \ \forall\, t \in [0, 0.6] \setminus \{0.5\}, \\ \begin{bmatrix} 40.4515e^{-2t} & -0.2351e^{-t} \end{bmatrix}, \ \forall\, t \in (0.6, 0.8], \\ \begin{bmatrix} 0 & 0 \end{bmatrix}, \ \forall\, t \in (0.8, 1], \end{cases}$$

and the controlled trajectory is given by

$$\mathbf{X}(t) = \begin{cases} \begin{bmatrix} 0.5(e^{2t} - 1) & 0.8(e^{t} - 1.25t - 1) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.0135t - 1) \end{bmatrix}, \quad \text{for all } t \in [0, 0.2], \\ \begin{bmatrix} (22.2946e^{-4t} - 15.086e^{-2t} + 0.2285e^{2t}) & (-1.1342e^{-3t} + 0.1436e^{-t} + 0.3947e^{t}) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.0135t - 1) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.2, 0.4], \\ \begin{bmatrix} (22.2954e^{-4t} - 15.086e^{-2t} + 0.22825e^{2t}) & (-1.1343e^{-3t} + 0.1436e^{-t} + 0.3947e^{t}) \\ (37.216e^{-4t} - 15.005e^{-2t} + 0.0437e^{4t}) & (-1.378e^{-3t} + 0.0877e^{-t} + 0.0516e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.4, 0.8], \\ \begin{bmatrix} -15.086e^{-2t} + 0.4117e^{2t} & 0.1436e^{-t} + 0.3484e^{t} \\ (37.216e^{-4t} - 15.005e^{-2t} + 0.0437e^{4t}) & (-1.378e^{-3t} + 0.0877e^{-t} + 0.0516e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.8, 1], \end{cases}$$

and these are shown in Figures 5.3 and 5.4 respectively.

Case (iii): If we choose the control function from the class $\mathcal{U}_2$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = -1$, then $\mathbf{W}_1 = \mathbf{O}$ and $\text{rank}\big([\mathbf{W}_2,\ \mathbf{W}_3,\ \mathbf{W}_4]\big) = 4$, hence by Theorem 5.4.3, the system (5.5.1) is controllable on $[0,1]$. Now,

$$\mathbf{W}^{-1} = \begin{bmatrix} 0.7342 & -0.1374 & 0 & 0 \\ -0.1374 & 0.1748 & 0 & 0 \\ 0 & 0 & 1.4260 & -0.3176 \\ 0 & 0 & -0.3176 & 0.3851 \end{bmatrix}$$

and one of the control function that steers the state from $\mathbf{X}_0$ to $\mathbf{X}_1$ is given by

$$\mathbf{U}(t) = \begin{cases} \begin{bmatrix} 0.4122e^{-4t} & 0.40846e^{-3t} \end{bmatrix}, \ \forall\, t \in (0.1, 0.3], \\ \begin{bmatrix} (2.9556e^{-2t} + 0.4122e^{-4t}) & (2.4668e^{-t} + 0.40846e^{-3t}) \end{bmatrix}, \ \forall\, t \in (0.3, 0.6] \setminus \{0.5\}, \\ \begin{bmatrix} 2.9556e^{-2t} & 2.4668e^{-t} \end{bmatrix}, \ \forall\, t \in (0.6, 0.8], \\ \begin{bmatrix} 0 & 0 \end{bmatrix}, \ \forall\, t \in [0, 0.1] \cup (0.8, 1], \end{cases}$$

and the controlled trajectory is given by

$$\mathbf{X}(t) = \begin{cases} \begin{bmatrix} 0.5(e^{2t} - 1) & 0.8(e^{t} - 1.25t - 1) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.0135t - 1) \end{bmatrix}, \ \text{ for all } t \in [0, 0.2], \\ \begin{bmatrix} 0.1648e^{2t} & -0.0187e^{t} \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.136t - 1) \end{bmatrix}, \ \text{ for all } t \in (0.2, 0.3], \\ \begin{bmatrix} 0.19e^{2t} - 0.153e^{4t} & -0.186e^{-3t} + 0.0373e^{t} \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.136t - 1) \end{bmatrix}, \ \text{ for all } t \in (0.3, 0.4], \\ \begin{bmatrix} 0.19e^{2t} - 0.153e^{4t} & -0.186e^{-3t} + 0.0373e^{t} \\ 0.1995e^{4t} & -0.0557e^{3t} \end{bmatrix}, \ \text{ for all } t \in (0.4, 0.5], \\ \begin{bmatrix} (-0.153e^{-4t} - 1.103e^{-2t} + 0.1568e^{2t}) & (-0.186e^{-3t} - 1.5064e^{-t} + 0.5794e^{t}) \\ -0.255e^{-4t} + 0.0047e^{4t} & -0.226e^{-3t} + 0.0113e^{3t} \end{bmatrix}, \\ \qquad \text{for all } t \in (0.5, 0.7], \\ \begin{bmatrix} (-0.153e^{-4t} - 1.103e^{-2t} + 0.1568e^{2t}) & (-0.186e^{-3t} - 1.5064e^{-t} + 0.5794e^{t}) \\ (-0.255e^{-4t} - 1.096e^{-2t} + 0.0211e^{4t}) & (-0.226e^{-3t} - 0.9204e^{-t} + 0.067e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.7, 0.8], \\ \begin{bmatrix} -1.1023e^{-2t} + 0.1555e^{2t} & -1.5064e^{-t} + 0.5718e^{t} \\ (-0.255e^{-4t} - 1.096e^{-2t} + 0.0211e^{4t}) & (-0.226e^{-3t} - 0.9204e^{-t} + 0.067e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.8, 1], \end{cases}$$

and these are shown in Figures 5.5 and 5.6 respectively.

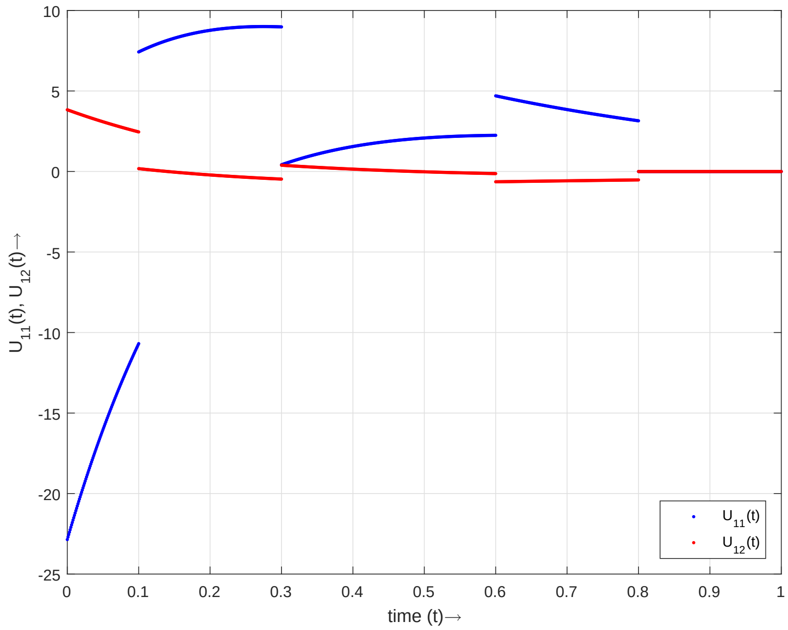

Figure 5.1: Plot of control function in case (i) of example (5.5.1).

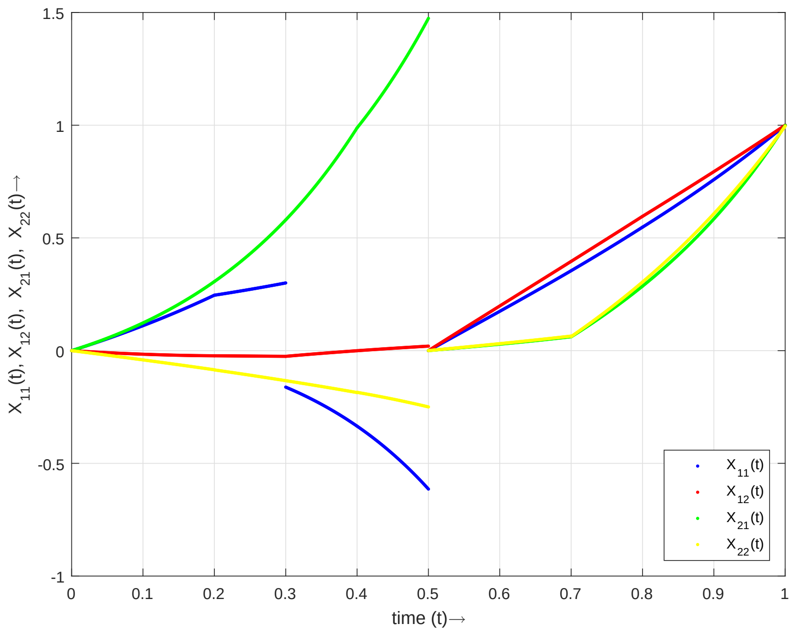

Figure 5.2: Plot of controlled trajectory in case (i) of example (5.5.1).

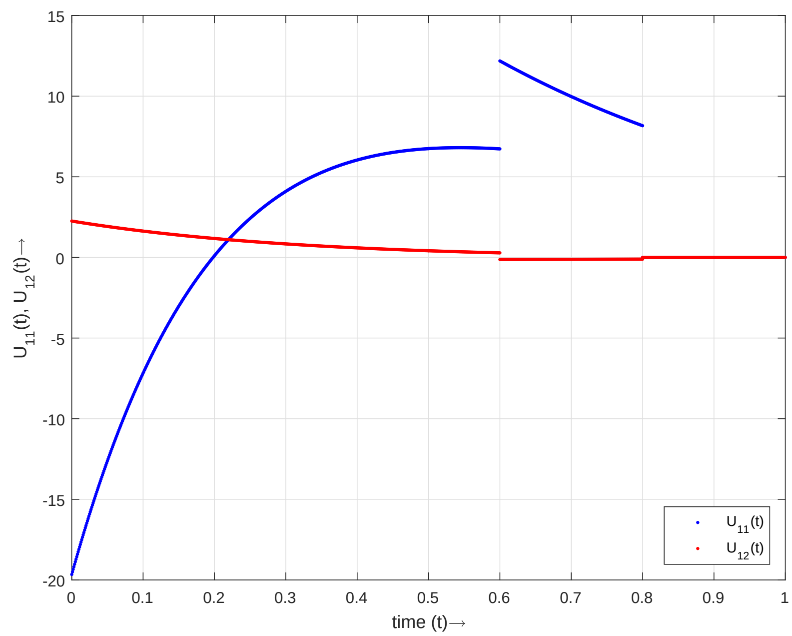

Figure 5.3: Plot of control function in case (ii) of example (5.5.1).

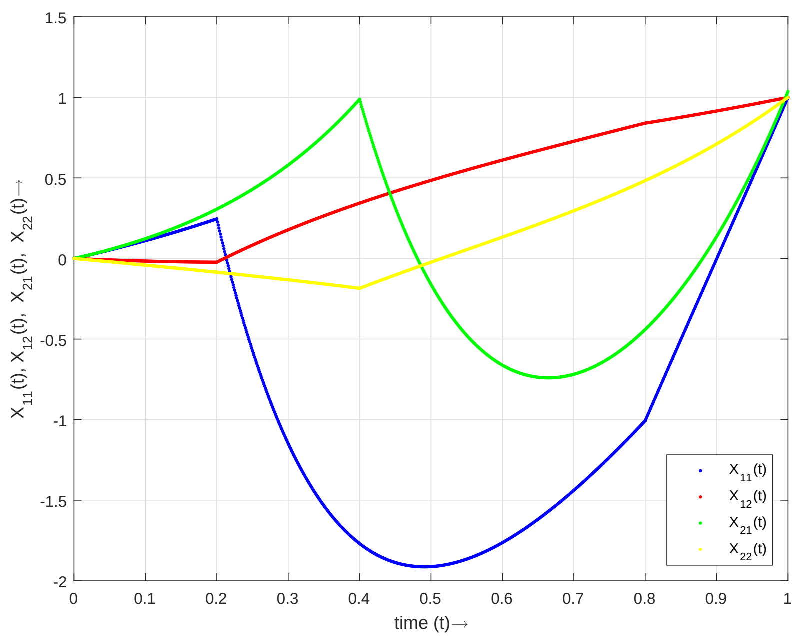

Figure 5.4: Plot of controlled trajectory in case (ii) of example (5.5.1).

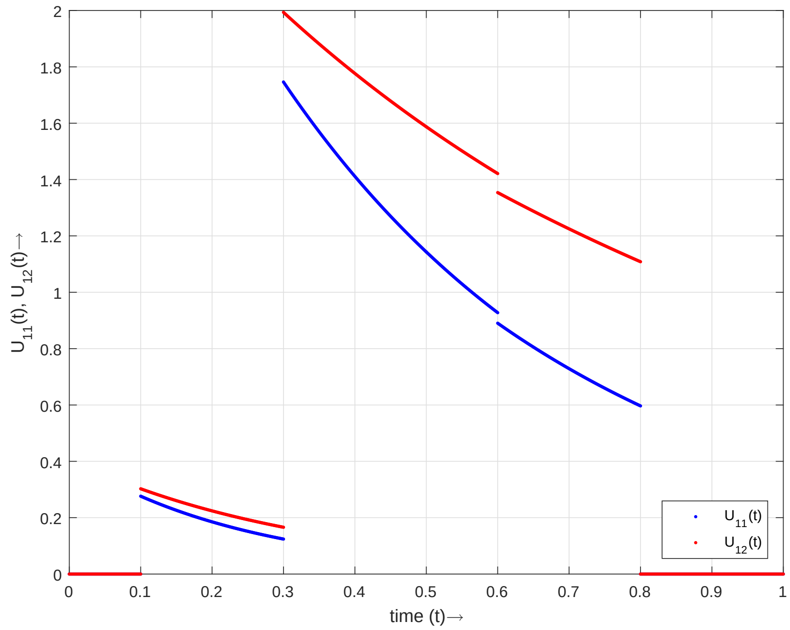

Figure 5.5: Plot of control function in case (iii) of example (5.5.1).

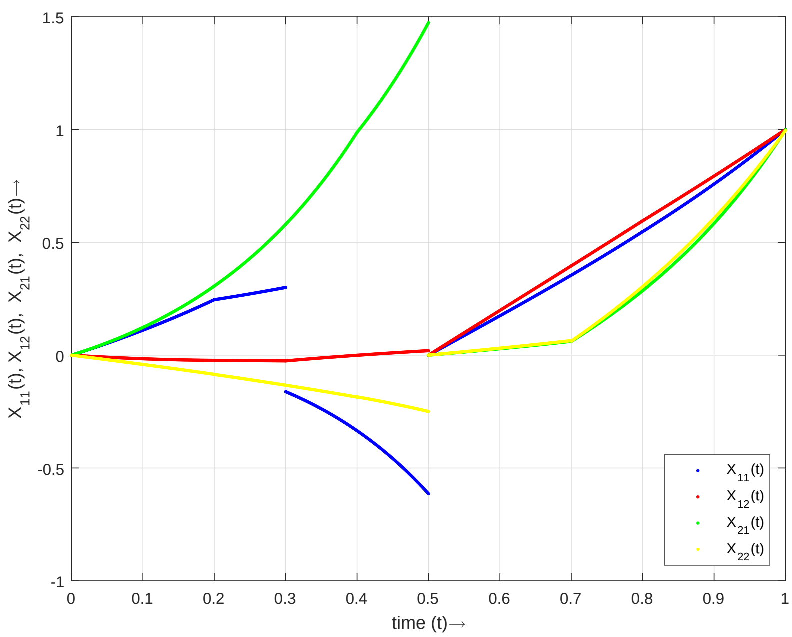

Figure 5.6: Plot of controlled trajectory in case (iii) of example (5.5.1).

2. Consider another $(2\times 2)-$dimensional linear impulsive matrix Lyapunov autonomous ordinary differential system with one impulse and two delays in the control function:

$$\left.\begin{aligned}
\begin{bmatrix}\dot{x}_{11}(t) & \dot{x}_{12}(t)\\ \dot{x}_{21}(t) & \dot{x}_{22}(t)\end{bmatrix} &= \begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_{11}(t) & x_{12}(t)\\ x_{21}(t) & x_{22}(t)\end{bmatrix} + \begin{bmatrix}x_{11}(t) & x_{12}(t)\\ x_{21}(t) & x_{22}(t)\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\\
&\quad + \begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}U_{11}(t-0.2) & U_{12}(t-0.2)\end{bmatrix}\\
&\quad + \begin{bmatrix}0\\ 2\end{bmatrix}\begin{bmatrix}U_{11}(t-0.4) & U_{12}(t-0.4)\end{bmatrix},\ t\in[0,1]\setminus\{0.5\},\\
\begin{bmatrix}x_{11}(0) & x_{12}(0)\\ x_{21}(0) & x_{22}(0)\end{bmatrix} &= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\\
\begin{bmatrix}\Delta x_{11}(0.5) & \Delta x_{12}(0.5)\\ \Delta x_{21}(0.5) & \Delta x_{22}(0.5)\end{bmatrix} &= \big(U_{11}(0.5)+U_{12}(0.5)\big)\begin{bmatrix}x_{11}(0.5) & x_{12}(0.5)\\ x_{21}(0.5) & x_{22}(0.5)\end{bmatrix},\\
\begin{bmatrix}U_{11}(t) & U_{12}(t)\end{bmatrix} &= \begin{bmatrix}1 & t\end{bmatrix},\qquad t\in[-0.4,0).
\end{aligned}\right\}\tag{5.5.2}$$

After applying the vector operator, the system (5.5.2) becomes

$$\begin{aligned}
\begin{bmatrix}\dot{x}_{11}(t)\\ \dot{x}_{21}(t)\\ \dot{x}_{12}(t)\\ \dot{x}_{22}(t)\end{bmatrix} &= \begin{bmatrix}1 & 2 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 2\\ 0 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_{11}(t)\\ x_{21}(t)\\ x_{12}(t)\\ x_{22}(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}U_{11}(t-0.2)\\ U_{12}(t-0.2)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 2 & 0\\ 0 & 0\\ 0 & 2\end{bmatrix}\\
&\quad\times\begin{bmatrix}U_{11}(t-0.4)\\ U_{12}(t-0.4)\end{bmatrix},\qquad t\in[0,1]\setminus\{0.5\},\\
\begin{bmatrix}x_{11}(0)\\ x_{21}(0)\\ x_{12}(0)\\ x_{22}(0)\end{bmatrix} &= \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix},\\
\begin{bmatrix}\Delta x_{11}(0.5)\\ \Delta x_{21}(0.5)\\ \Delta x_{12}(0.5)\\ \Delta x_{22}(0.5)\end{bmatrix} &= \big(U_{11}(0.5)+U_{12}(0.5)\big)\begin{bmatrix}x_{11}(0.5)\\ x_{21}(0.5)\\ x_{12}(0.5)\\ x_{22}(0.5)\end{bmatrix},\\
\begin{bmatrix}U_{11}(t)\\ U_{12}(t)\end{bmatrix} &= \begin{bmatrix}1\\ t\end{bmatrix},\qquad t\in[-0.4,0).
\end{aligned}$$

On comparing the above system with (5.3.2), we get

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \ \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{C}_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix},$$

$$t_0 = 0,\ h_1 = 0.2,\ h_2 = 0.4,\ t_1 = 0.5,\ T = 1,\ \ \alpha = U_{11}(0.5) + U_{12}(0.5).$$

By calculation, we get

$$\mathbf{\Phi}(t,s) = \frac{1}{6}\begin{bmatrix} \begin{pmatrix} e^{-2(t-s)}+1 \\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}-2 \\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -e^{-2(t-s)}+1 \\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}-2 \\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix} \\ \begin{pmatrix} -e^{-2(t-s)}-1 \\ +e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}+2 \\ +e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} e^{-2(t-s)}-1 \\ -e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}+2 \\ -e^{t-s}+e^{3(t-s)} \end{pmatrix} \\ \begin{pmatrix} -e^{-2(t-s)}+1 \\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}-2 \\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} e^{-2(t-s)}+1 \\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}-2 \\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix} \\ \begin{pmatrix} e^{-2(t-s)}-1 \\ -e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}+2 \\ -e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -e^{-2(t-s)}-1 \\ +e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}+2 \\ +e^{t-s}+e^{3(t-s)} \end{pmatrix} \end{bmatrix},$$

$$\mathbf{a}_0 = \begin{bmatrix} -0.1602 \\ 1.0713 \\ 0.0883 \\ 0.5817 \end{bmatrix}, \ \mathbf{W}_1 = (1+\alpha)^2 \begin{bmatrix} 17.4115 & 11.135 & 15.277 & 9.709 \\ 11.1350 & 7.125 & 9.709 & 6.172 \\ 15.2770 & 9.709 & 17.4115 & 11.1349 \\ 9.709 & 6.172 & 11.1349 & 7.125 \end{bmatrix},$$

$$\mathbf{W}_2 = \begin{bmatrix} 6.915(1+\alpha)^2 & 3.8854(1+\alpha)^2 & 5.8225(1+\alpha)^2 & 3.2689(1+\alpha)^2 \\ +5.8373(1+\alpha) & +4.965(1+\alpha) & +4.5299(1+\alpha) & +3.7637(1+\alpha) \\ +1.2553 & +1.3835 & +0.8625 & +0.9396 \\ & & & \\ 3.8854(1+\alpha)^2 & 2.1839(1+\alpha)^2 & 3.269(1+\alpha)^2 & 1.8359(1+\alpha)^2 \\ +4.965(1+\alpha) & +3.646(1+\alpha) & +3.7636(1+\alpha) & 2.8035(1+\alpha) \\ +0.8625 & +1.5526 & +0.9395 & 1.041 \\ & & & \\ 5.8225(1+\alpha)^2 & 3.269(1+\alpha)^2 & 6.9152(1+\alpha)^2 & 3.8854(1+\alpha)^2 \\ +4.5299(1+\alpha) & +3.7636(1+\alpha) & +5.8373(1+\alpha) & +4.8776(1+\alpha) \\ +0.8625 & +0.9395 & +1.2553 & +1.3834 \\ & & & \\ 3.2689(1+\alpha)^2 & 1.8359(1+\alpha)^2 & 3.8854(1+\alpha)^2 & 2.1839(1+\alpha)^2 \\ +3.7637(1+\alpha) & +2.8035(1+\alpha) & +4.8776(1+\alpha) & +3.646(1+\alpha) \\ +0.9396 & +1.041 & +1.3834 & +1.5525 \end{bmatrix},$$

$$\mathbf{W}_3 = \begin{bmatrix} 4.143 & 4.3136 & 2.5038 & 2.3366 \\ 4.3136 & 4.6915 & 2.3366 & 2.1719 \\ 2.5038 & 2.3366 & 4.143 & 4.3136 \\ 2.3366 & 2.1719 & 4.3136 & 4.6915 \end{bmatrix}, \quad \mathbf{W}_4 = \begin{bmatrix} 0.3787 & 0.313 & 0.0874 & 0.0694 \\ 0.313 & 0.2605 & 0.0694 & 0.0554 \\ 0.0874 & 0.0694 & 0.3787 & 0.313 \\ 0.0694 & 0.0554 & 0.313 & 0.2605 \end{bmatrix}.$$

Let the desired final state of the system (5.5.2) be $\mathbf{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Now we discuss the controllability of the system (5.5.2) in different cases.

Case (i): If we choose the control function $\mathbf{U}(\cdot) \in \mathcal{U}_1$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = 1$, then $\text{rank}\big([\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4]\big) = 4$, and hence by Theorem 5.4.2, the system (5.5.2) is controllable on $[0, 1]$.

Case (ii): If the control function $\mathbf{U}(\cdot) \in \mathcal{U}_1$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = 0$, then there are no impulses in the system (5.5.2) and hence the matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ can be combined to get a matrix

$$\mathbf{V} = \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = \begin{bmatrix} 35.5621 & 25.6825 & 28.9957 & 20.0177 \\ 25.6825 & 19.199 & 20.0177 & 14.0243 \\ 28.9957 & 20.0177 & 35.5623 & 25.5949 \\ 20.0177 & 14.0243 & 25.5949 & 19.1989 \end{bmatrix}.$$

Then we see that $\text{rank}([\mathbf{V}, \mathbf{W}_4]) = 4$, and hence by Corollary 5.4.2, the system (5.5.2) is controllable on $[0, 1]$.

Case (iii): If we choose the control function $\mathbf{U}(\cdot) \in \mathcal{U}_2$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = -1$, then $\mathbf{W}_1 = \mathbf{O}$ and $\text{rank}\big([\mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4]\big) = 4$. Therefore by Theorem 5.4.3, the system (5.5.2) is controllable on $[0, 1]$.

In all the above three cases, the computation of the control function and corresponding controlled trajectory are similar to that of system (5.5.1).

5.6 Conclusions

In this chapter, a dynamical control system modelled by an $(n \times n)-$dimensional linear impulsive matrix Lyapunov ordinary differential equations having multiple constant time-delays in its control function is considered. The controllability conditions of this system for certain classes of admissible control functions are derived. Further, these controllability conditions are reduced to the special cases, namely, system without impulses and with delays; with impulses and without delays; and without impulses and without delays. In each of such case, the controllability results coincides with the results available in the existing works in the literature. Numerical examples are given to substantiate our results and plots of steering controllers and controlled trajectory are also provided.

Chapter 6

Controllability of a linear impulsive system—an eigenvalue approach

6.1 Introduction

The studies on the controllability of impulsive control systems was initiated by Leela et al [70] in 1993 by explaining with some simple results that how the impulsive control affects the controllability of the system. Later the research on the controllability of impulsive systems has grown more rapidly as many other control theorists started investigating the controllability properties for different types of such systems. Some of the remarkable contributions can be seen in [21, 41, 45, 46, 49, 112, 114, 115] etc. In [21], a homogeneous linear impulsive system is considered and its global null controllability is established. In [41], the authors investigated the controllability of impulsive systems with nonlinear perturbations. In [45, 46], various necessary and sufficient controllability conditions are obtained for the linear impulsive systems of both autonomous and nonautonomous cases. In [112], authors established the controllability of linear piecewise constant impulsive systems and the obtained results are further extended in [114] to the corresponding time-varying case. But it is worth pointing out that, in all these papers the authors investigated the null controllability (i.e. controllable to the origin from any initial state) of impulsive systems, which is not equivalent to the classical controllability unlike the systems without impulses.

As we know that for the time-invariant linear systems without impulses, a Popov–Bilewitch–Hautus (PBH) rank condition which adopts the eigenvalues of the system matrix is one of the easily verifiable and a powerful tool in the analysis of the controllability in addition to Kalman's matrix rank condition [106]. A literature survey shows that, in none of the articles on impulsive control systems, a PBH-type rank condition for the controllability is established.

Based on the above discussions, in this chapter we establish the various necessary as well as sufficient criteria for controllability of the linear impulsive systems. The derived results are further reduced to the corresponding time-invariant case and subsequently obtained a

Kalman's type matrix rank condition and a PBH-type rank condition under some specific conditions satisfying by the system parameters. When we specialize these conditions of controllability to that of null controllability, the results coincides with the results of [45].

This chapter consists of four sections: in Section 6.2, some of the preliminaries required for the establishment of the results and a class of linear impulsive control systems is introduced. The main results begins with Section 6.3, where we discuss various necessary as well as sufficient conditions for controllability of the system. Also some examples are provided in this section to support the theory—the null controllability need not imply controllability for the impulsive systems. In Section 6.4, the controllability results of the Section 6.3 are applied to the corresponding time-invariant system, and obtained the conditions in terms of the system matrices and eigenvalues of the system matrices. Conclusions of this chapter are made in Section 6.5.

6.2 System description

We consider the dynamical control system modeled by the following $n-$dimensional linear impulsive ordinary differential equations:

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad t \in [t_0, \infty) \setminus \{t_k : k = 1, 2, \ldots\}, \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \\ \Delta\mathbf{x}(t_k) &= \mathbf{E}_k\mathbf{x}(t_k) + \mathbf{F}_k\mathbf{u}(t_k), \end{aligned}\right\} \tag{6.2.1}$$

where,

(i) the state $\mathbf{x}(t) \in \mathbb{R}^n$ with a known initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, the control $\mathbf{u}(t) \in \mathbb{R}^m$,

(ii) $\mathbf{A}(\cdot) \in \mathcal{C}\big([t_0, \infty); \mathbb{R}^{n\times n}\big)$ and $\mathbf{B}(\cdot) \in \mathcal{C}\big([t_0, \infty); \mathbb{R}^{n\times m}\big)$ are the known matrix valued functions; if these are constant functions, then system (6.2.1) is called autonomous, otherwise (6.2.1) is nonautonomous system,

(iii) $\Delta\mathbf{x}(t_k) := \mathbf{x}(t_k^+) - \mathbf{x}(t_k)$ is an impulse in the state function at the time t_k,

(iv) $\mathbf{E}_k \in \mathbb{R}^{n\times n}$ and $\mathbf{F}_k \in \mathbb{R}^{n\times m}$ are the known constant matrices.

The trajectory of this system belongs to the set

$$\begin{aligned} \mathcal{B} := \Big\{ &\mathbf{x}(\cdot) \,\big|\, \mathbf{x}(\cdot) : [t_0, \infty) \to \mathbb{R}^n \text{ is a continuous and bounded on } [t_0, \infty) \setminus \{t_k : k = 1, 2, \ldots\} \\ &\text{and differentiable a.e on } [t_0, \infty) \text{ such that } \exists\, \mathbf{x}(t_k^-) := \lim_{t\uparrow t_k} \mathbf{x}(t) \text{ and } \mathbf{x}(t_k^+) := \lim_{t\downarrow t_k} \mathbf{x}(t) \\ &\text{with } \mathbf{x}(t_k^-) = \mathbf{x}(t_k) \text{ and } \mathbf{x}(t_0) = \lim_{t\downarrow t_0} \mathbf{x}(t) \Big\}, \end{aligned}$$

and an admissible control function belongs to the set

$$\mathcal{PC} := \Big\{ \mathbf{u}(\cdot) \,\big|\, \mathbf{u}(\cdot) : [t_0, \infty) \to \mathbb{R}^m \text{ is a bounded piecewise continuous function on } [t_0, \infty) \Big\}.$$

Note that, if we define a sup-norm

$$\big\|\mathbf{x}(\cdot)\big\|_{\mathcal{B}} := \sup_{t\in[t_0,\infty)} \big\|\mathbf{x}(t)\big\|_{\mathbb{R}^n} \quad \text{and} \quad \big\|\mathbf{u}(\cdot)\big\|_{\mathcal{PC}} := \sup_{t\in[t_0,\infty)} \big\|\mathbf{u}(t)\big\|_{\mathbb{R}^m},$$

then $\mathcal{B}$ and $\mathcal{PC}$ are the real Banach spaces.

Let us recall the definition of controllability of the system (6.2.1).

Definition 6.2.1. *The system* (6.2.1) *is said to be controllable on* $[t_0, t_f]$, $t_0 < t_f < \infty$, *over* $\mathbb{R}^n$, *if for all vectors* $\mathbf{x}_0$, $\mathbf{x}_f \in \mathbb{R}^n$, *there exists at least one control function* $\mathbf{u}(\cdot) \in \mathcal{PC}$ *such that the corresponding solution of the system* (6.2.1) *with an initial state* $\mathbf{x}(t_0) = \mathbf{x}_0$ *also satisfies the condition:* $\mathbf{x}(t_f) = \mathbf{x}_f$.

The following lemma gives the solution to the system (6.2.1).

Lemma 6.2.1. *By assuming there are* $M-$*impulses,* $M \in \mathbb{N}$, *in the time interval* $[t_0, t_f]$, $t_f = t_{M+1}$, *the solution to the system* (6.2.1) *in the time-duration* $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, M$, *is given by*

$$\begin{aligned}\mathbf{x}(t) = \mathbf{\Phi}(t, t_k)\Bigg\{ &\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0 \\ &+ \sum_{i=1}^{k} \left(\prod_{j=k}^{i+1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)(\mathbf{I}_n + \mathbf{E}_i) \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{i=2}^{k} \prod_{j=k}^{i} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{F}_k\mathbf{u}(t_k) \Bigg\} \\ &+ \int_{t_k}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,\end{aligned} \tag{6.2.2}$$

where $\mathbf{\Phi}(t)$ *is the fundamental solution matrix of the homogeneous system* $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ *and hence* $\mathbf{\Phi}(t, s) := \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$ *is the state-transition matrix associated with* $\mathbf{A}(t)$; *and for a convention it is assumed that* $\prod_{j=k-1}^{k} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) = \mathbf{I}_n$.

Proof. The solution to the system (6.2.1) in $[t_0, t_1]$ is given by using the method of variation of parameters as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s. \tag{6.2.3}$$

Then

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,$$

so that

$$\begin{aligned}\mathbf{x}(t_1^+) &= (\mathbf{I}_n + \mathbf{E}_1)\mathbf{x}(t_1) + \mathbf{F}_1\mathbf{u}(t_1) \\ &= (\mathbf{I}_n + \mathbf{E}_1)\left\{\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\right\} + \mathbf{F}_1\mathbf{u}(t_1).\end{aligned} \tag{6.2.4}$$

Using eq (6.2.4), the solution to the system (6.2.1) in $(t_1, t_2]$ is given by

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t, t_1)\mathbf{x}(t_1^+) + \int_{t_1}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &= \mathbf{\Phi}(t, t_1)\Bigg\{(\mathbf{I}_n + \mathbf{E}_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + (\mathbf{I}_n + \mathbf{E}_1)\int_{t_0}^{t_1} \mathbf{\Phi}(t_1, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &\quad + \mathbf{F}_1\mathbf{u}(t_1)\Bigg\} + \int_{t_1}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &= \mathbf{\Phi}(t, t_1)\Bigg\{\prod_{j=1}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0 \\ &\quad + \sum_{i=1}^{1}\left(\prod_{j=1}^{i+1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)(\mathbf{I}_n + \mathbf{E}_i)\int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &\quad + \sum_{i=2}^{1}\prod_{j=1}^{i}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{F}_1\mathbf{u}(t_1)\Bigg\} \\ &\quad + \int_{t_1}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\end{aligned} \tag{6.2.5}$$

Then

$$\begin{aligned}\mathbf{x}(t_2) = \mathbf{\Phi}(t_2, t_1)&\left\{(\mathbf{I}_n + \mathbf{E}_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + (\mathbf{I}_n + \mathbf{E}_1)\int_{t_0}^{t_1} \mathbf{\Phi}(t_1, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s + \mathbf{F}_1\mathbf{u}(t_1)\right\} \\ &+ \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,\end{aligned}$$

so that

$$
\begin{aligned}
\mathbf{x}(t_2^+) &= (\mathbf{I}_n + \mathbf{E}_2)\mathbf{x}(t_2) + \mathbf{F}_2\mathbf{u}(t_2) \\
&= (\mathbf{I}_n + \mathbf{E}_2)\mathbf{\Phi}(t_2, t_1)(\mathbf{I}_n + \mathbf{E}_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \\
&\quad + (\mathbf{I}_n + \mathbf{E}_2)\mathbf{\Phi}(t_2, t_1)(\mathbf{I}_n + \mathbf{E}_1)\int_{t_0}^{t_1}\mathbf{\Phi}(t_1, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&\quad + (\mathbf{I}_n + \mathbf{E}_2)\mathbf{\Phi}(t_2, t_1)\mathbf{F}_1\mathbf{u}(t_1) + (\mathbf{I}_n + \mathbf{E}_2)\int_{t_1}^{t_2}\mathbf{\Phi}(t_2, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&\quad + \mathbf{F}_2\mathbf{u}(t_2).
\end{aligned}
\tag{6.2.6}
$$

Using eq (6.2.6), the solution to the system (6.2.1) in $(t_2, t_3]$ is given by

$$
\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Phi}(t, t_2)\mathbf{x}(t_2^+) + \int_{t_2}^{t}\mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&= \mathbf{\Phi}(t, t_2)\Big\{(\mathbf{I}_n + \mathbf{E}_2)\mathbf{\Phi}(t_2, t_1)(\mathbf{I}_n + \mathbf{E}_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \\
&\quad + (\mathbf{I}_n + \mathbf{E}_2)\mathbf{\Phi}(t_2, t_1)(\mathbf{I}_n + \mathbf{E}_1)\int_{t_0}^{t_1}\mathbf{\Phi}(t_1, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&\quad + (\mathbf{I}_n + \mathbf{E}_2)\mathbf{\Phi}(t_2, t_1)\mathbf{F}_1\mathbf{u}(t_1) + (\mathbf{I}_n + \mathbf{E}_2)\int_{t_1}^{t_2}\mathbf{\Phi}(t_2, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&\quad + \mathbf{F}_2\mathbf{u}(t_2)\Big\} + \int_{t_2}^{t}\mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&= \mathbf{\Phi}(t, t_2)\Big\{\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0 \\
&\quad + \sum_{i=1}^{2}\left(\prod_{j=2}^{i+1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)(\mathbf{I}_n + \mathbf{E}_i)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t_i, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&\quad + \sum_{i=2}^{2}\prod_{j=2}^{i}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{F}_2\mathbf{u}(t_2)\Big\} \\
&\quad + \int_{t_2}^{t}\mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,
\end{aligned}
\tag{6.2.7}
$$

Observing eq (6.2.5) and eq (6.2.7), in general, the solution to the system (6.2.1) in $(t_k, t_{k+1}]$ is given by eq (6.2.2). □

Lemma 6.2.2. *If all* $\mathbf{E}_k$ *commutes with the state transition matrix* $\mathbf{\Phi}(t, s)$*, i.e.* $\mathbf{E}_k\mathbf{\Phi}(t, s) = \mathbf{\Phi}(t, s)\mathbf{E}_k$*,* $\forall\, k = 1, 2, \ldots, M$*, then the solution to the system* (6.2.1) *given in eq* (6.2.2) *reduces to*

$$
\begin{aligned}
\mathbf{x}(t) = &\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \sum_{i=1}^{k}\prod_{j=k}^{i}(\mathbf{I}_n + \mathbf{E}_j)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
&+ \sum_{i=2}^{k}\prod_{j=k}^{i}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t,t_{i-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{\Phi}(t,t_k)\mathbf{F}_k\mathbf{u}(t_k) \\
&+ \int_{t_k}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s, \ \text{ for } t \in (t_k, t_{k+1}].
\end{aligned}
\tag{6.2.8}
$$

Remark 6.2.1. *For an autonomous case of the system* (6.2.1), *the state-transition matrix is given by* $\mathbf{\Phi}(t,s) = e^{\mathbf{A}(t-s)}$. *Then the condition* $\mathbf{E}_k e^{\mathbf{A}(t-s)} = e^{\mathbf{A}(t-s)}\mathbf{E}_k$ *satisfies if* $\mathbf{A}$ *commutes with* $\mathbf{E}_k$, *i.e.* $\mathbf{A}\mathbf{E}_k = \mathbf{E}_k\mathbf{A}, \ \ k = 1, 2, \ldots, M,$

6.3 Controllability results for a time-varying system

In this section, we obtain several sufficient as well as necessary conditions associated with the controllability of system (6.2.1) under various assumptions on the system components.

Theorem 6.3.1 (Sufficient conditions)**.** *If one of the following conditions holds true, then the impulsive system* (6.2.1) *is controllable on* $[t_0, t_f]$.

(i) There exists at least one $l \in \{1, 2, \ldots, (M-1)\}$ *and a* $(m \times n)-$*matrix* $\mathbf{F}'_l$ *such that* $\mathbf{F}_l\mathbf{F}'_l = \mathbf{I}_n$ *and* $(\mathbf{I}_n + \mathbf{E}_{l+1})$, $(\mathbf{I}_n + \mathbf{E}_{l+2}), \ldots, (\mathbf{I}_n + \mathbf{E}_M)$ *are invertible.*

(ii) There exists a $(m \times n)-$*matrix* $\mathbf{F}'_M$ *such that* $\mathbf{F}_M\mathbf{F}'_M = \mathbf{I}_n$.

(iii) There exists at least one $k \in \{1, 2, \ldots, M\}$ *such that* $(\mathbf{I}_n + \mathbf{E}_k), \ldots, (\mathbf{I}_n + \mathbf{E}_M)$ *are invertible and* $\int_{t_{k-1}}^{t_k}\big[\mathbf{\Phi}(t_k,s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_k,s)\mathbf{B}(s)\big]^*\mathrm{d}s$ *is positive definite matrix.*

(iv) $\int_{t_M}^{t_f}\big[\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big]^*\mathrm{d}s$ *is positive definite matrix.*

Proof. (a) First we consider case (i). Without loss of generality, suppose there exists a $l \in \{1, 2, \ldots, (M-1)\}$ and a $(m \times n)-$matrix $\mathbf{F}'_l$ such that $\mathbf{F}_l\mathbf{F}'_l = \mathbf{I}_n$ and $(\mathbf{I}_n + \mathbf{E}_{l+1}), \ldots, (\mathbf{I}_n + \mathbf{E}_M)$ are invertible. Then, given an initial state $\mathbf{x}_0 \in \mathbb{R}^n$ and a desired final state $\mathbf{x}_f \in \mathbb{R}^n$, by using a control function

$$
\mathbf{u}(t) := \begin{cases}
\mathbf{F}'_l\left(\displaystyle\prod_{j=M}^{l+1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\right)^{-1}\mathbf{\Phi}(t_M,t_f) \\
\quad \times \left\{\mathbf{x}_f - \mathbf{\Phi}(t_f,t_M)\displaystyle\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0\right\}, \ \text{ at } t = t_l, \\
\mathbf{0}, \qquad \text{for } t \in [t_0, t_f] \setminus \{t_l\},
\end{cases}
\tag{6.3.1}
$$

the solution to the system (6.2.1) given in eq (6.2.2) satisfies $\mathbf{x}(t_f) = \mathbf{x}_f$.

(b) Now we consider case (ii). Let there exists a $(m \times n)-$matrix $\mathbf{F}'_M$ such that $\mathbf{F}_M\mathbf{F}'_M = \mathbf{I}_n$. Then for a given initial state $\mathbf{x}_0 \in \mathbb{R}^n$ and a desired final state $\mathbf{x}_f \in \mathbb{R}^n$, we consider a control function

$$\mathbf{u}(t) := \begin{cases} \mathbf{F}'_M \mathbf{\Phi}(t_M, t_f)\left\{\mathbf{x}_f - \mathbf{\Phi}(t_f, t_M) \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0\right\}, & \text{at } t = t_M, \\ \mathbf{0}, & \text{for } t \in [t_0, t_f] \setminus \{t_M\}. \end{cases} \tag{6.3.2}$$

One can easily verify that, this control function steers the state of system (6.2.1) given in eq (6.2.2) from $\mathbf{x}_0$ to $\mathbf{x}_f$.

(c) To prove the case (iii), let there exists a $k \in \{1, 2, \dots, M\}$ such that $(\mathbf{I}_n + \mathbf{E}_k)$, $(\mathbf{I}_n + \mathbf{E}_{k+1}), \dots, (\mathbf{I}_n + \mathbf{E}_M)$ are all invertible and $\mathbf{W} = \int_{t_{k-1}}^{t_k} \big[\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\big]^* \mathrm{d}s$ is positive definite matrix. Then we are giving the following control function which makes the state of the system (6.2.1) given in eq (6.2.2) to steer from $\mathbf{x}_0$ to $\mathbf{x}_f$.

$$\mathbf{u}(t) := \begin{cases} \big[\mathbf{\Phi}(t_k, t)\mathbf{B}(t)\big]^* \mathbf{W}^{-1}(\mathbf{I}_n + \mathbf{E}_k)^{-1}\left(\prod_{j=M}^{k+1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)^{-1} \mathbf{\Phi}(t_M, t_f) & \\ \quad \times \left\{\mathbf{x}_f - \mathbf{\Phi}(t_f, t_M) \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0\right\}, & \text{for } t \in (t_{k-1}, t_k), \\ \mathbf{0}, & \text{for } t \in [t_0, t_f] \setminus (t_{k-1}, t_k), \end{cases} \tag{6.3.3}$$

(d) Finally to prove case (iv), let $\mathbf{W} = \int_{t_M}^{t_f} \big[\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\big]^* \mathrm{d}s$ be positive definite matrix and consider the following control function in order to steer the state of the system (6.2.1) given in eq (6.2.2) from $\mathbf{x}_0$ to $\mathbf{x}_f$.

$$\mathbf{u}(t) := \begin{cases} \big[\mathbf{\Phi}(t_f, t)\mathbf{B}(t)\big]^* \mathbf{W}^{-1}\left\{\mathbf{x}_f - \mathbf{\Phi}(t_f, t_M) \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0\right\}, & t \in (t_M, t_f], \\ \mathbf{0}, & \text{for } t \in [t_0, t_f] \setminus (t_M, t_f], \end{cases} \tag{6.3.4}$$

□

Theorem 6.3.2 (Sufficient conditions)**.** *If one of the following conditions holds, then the impulsive system* (6.2.1) *is null controllable on* $[t_0, t_f]$.

(i) There exists at least one $k \in \{1, 2, \dots, M\}$ *and a* $(m \times n)-$*matrix* $\mathbf{F}'_k$ *such that* $\mathbf{F}_k\mathbf{F}'_k = \mathbf{I}_n$.

(ii) There exists a positive definite matrix $\mathbf{W} = \int_{t_{k-1}}^{t_k} \big[\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\big]^* \mathrm{d}s$*, for*

some $k \in \{1, \ldots, (M+1)\}$.

Proof. (a) First we consider case (i). Without loss of generality, suppose there exists a $l \in \{1, 2, \ldots, M\}$ and a $(m \times n)-$matrix $\mathbf{F}'_l$ such that $\mathbf{F}_l \mathbf{F}'_l = \mathbf{I}_n$. Then, given an initial state $\mathbf{x}_0 \in \mathbb{R}^n$, the control function

$$\mathbf{u}(t) := \begin{cases} -\mathbf{F}'_l \prod_{j=l}^{1} (\mathbf{I}_n + \mathbf{E}_j) \mathbf{\Phi}(t_j, t_{j-1}) \mathbf{x}_0, & \text{at } t = t_l, \\ \mathbf{0}, & \text{for } t \in [t_0, t_f] \setminus \{t_l\}, \end{cases} \tag{6.3.5}$$

steers the state of the system (6.2.1) given in eq (6.2.2) from $\mathbf{x}_0$ to $\mathbf{0}$.

(b) To prove the case (ii), let $\mathbf{W} = \int_{t_{k-1}}^{t_k} \big[\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\big]^* \mathrm{d}s$ is positive definite matrix for some $k \in \{1, \ldots, (M+1)\}$. Then the following control function steers the system (6.2.1) given in eq (6.2.2) from $\mathbf{x}_0$ to $\mathbf{0}$.

$$\mathbf{u}(t) := \begin{cases} -\big[\mathbf{\Phi}(t_k, t)\mathbf{B}(t)\big]^* \mathbf{W}^{-1} \mathbf{\Phi}(t_k, t_{k-1}) \prod_{j=k-1}^{1} (\mathbf{I}_n + \mathbf{E}_j) \mathbf{\Phi}(t_j, t_{j-1}) \mathbf{x}_0, & t \in (t_{k-1}, t_k), \\ \mathbf{0}, & \text{for } t \in [t_0, t_f] \setminus (t_{k-1}, t_k). \end{cases} \tag{6.3.6}$$

□

The following example shows that, a system is null controllable without being controllable.

Example: Consider a 2-D system with two impulses as

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{u}(t), \qquad t \in [0, 3] \setminus \{1, 2\}, \\ \mathbf{x}(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \Delta \mathbf{x}(1) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \mathbf{u}(1), \\ \Delta \mathbf{x}(2) &= \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \mathbf{x}(2) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{u}(2). \end{aligned}\right\} \tag{6.3.7}$$

In this system, $n = 2,\ m = 3$ and $\mathbf{F}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. We observe that there exists a matrix

$\mathbf{F}_1' = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \\ -2 & 0 \end{bmatrix}$ such that $\mathbf{F}_1\mathbf{F}_1' = \mathbf{I}_2$, therefore by condition (i) given in Theorem 6.3.2, system (6.3.7) is null controllable on $[0, 3]$. But there exists no control function that steers the state of system (6.3.7) from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, showing that this system is not controllable on $[0, 3]$.

The special cases for the controllability and null controllability are the situation when all $\mathbf{E}_k$ commutes with the state-transition matrix $\mathbf{\Phi}(t, s)$. These are shown in the following two corollaries.

Corollary 6.3.1 (Sufficient conditions)**.** *If in system* (6.2.1)*,* $\mathbf{E}_k$ *commutes with the state-transition matrix, i.e.* $\mathbf{E}_k\mathbf{\Phi}(t, s) = \mathbf{\Phi}(t, s)\mathbf{E}_k$, $\forall\, k = 1, 2, \ldots, M$, *then the sufficient conditions of controllability on* $[t_0, t_f]$ *for the system* (6.2.1) *given in Theorem 6.3.1 reduces to*

(i) There exists at least one $l \in \{1, 2, \ldots, (M-1)\}$ *and a* $(m \times n)-$*matrix* $\mathbf{F}_l'$*, such that* $\mathbf{F}_l\mathbf{F}_l' = \mathbf{I}_n$ *and* $(\mathbf{I}_n + \mathbf{E}_{l+1})$, $(\mathbf{I}_n + \mathbf{E}_{l+2}), \ldots, (\mathbf{I}_n + \mathbf{E}_M)$ *are invertible.*

(ii) There exists a $(m \times n)-$*matrix* $\mathbf{F}_M'$ *such that* $\mathbf{F}_M\mathbf{F}_M' = \mathbf{I}_n$.

(iii) There exists at least one $k \in \{1, 2, \ldots, M\}$ *such that* $(\mathbf{I}_n + \mathbf{E}_k), \ldots, (\mathbf{I}_n + \mathbf{E}_M)$ *are invertible and* $\mathbf{W} = \int_{t_{k-1}}^{t_k} \big[\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\big]^* \mathrm{d}s$ *is positive definite.*

(iv) $\mathbf{W} = \int_{t_M}^{t_f} \big[\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\big]^* \mathrm{d}s$ *is positive definite.*

Proof. The proof is similar to that of Theorem 6.3.1, hence we will not go detail to prove this corollary. However, here we are providing a control function that steers the state of system (6.2.1) given in eq (6.2.8) from $\mathbf{x}_0$ to $\mathbf{x}_f$. For the case (i),

$$\mathbf{u}(t) := \begin{cases} \mathbf{F}_l'\mathbf{\Phi}(t_l, t_f)\left(\displaystyle\prod_{j=M}^{l+1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\left\{\mathbf{x}_f - \displaystyle\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_f, t_0)\mathbf{x}_0\right\}, & \text{at } t = t_l, \\ \mathbf{0}, \quad \text{for } t \in [t_0, t_f] \setminus \{t_l\}, \end{cases} \tag{6.3.8}$$

for the case (ii),

$$\mathbf{u}(t) := \begin{cases} \mathbf{F}_M'\mathbf{\Phi}(t_M, t_f)\left\{\mathbf{x}_f - \displaystyle\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_f, t_0)\mathbf{x}_0\right\}, & \text{at } t = t_M, \\ \mathbf{0}, \quad \text{for } t \in [t_0, t_f] \setminus \{t_M\}, \end{cases} \tag{6.3.9}$$

for the case (iii),

$$\mathbf{u}(t) := \begin{cases} \big[\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big]^*\mathbf{W}^{-1}\left(\prod_{j=M}^{k}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\left\{\mathbf{x}_f - \prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0\right\}, \\ \qquad\qquad \text{for } t \in (t_{k-1},t_k), \\ \mathbf{0}, \quad \text{for } t \in [t_0,t_f]\setminus(t_{k-1},t_k), \end{cases} \tag{6.3.10}$$

and finally for the case (iv), the following control function steers the system (6.2.1) from $\mathbf{x}_0$ to $\mathbf{x}_f$.

$$\mathbf{u}(t) := \begin{cases} \big[\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big]^*\mathbf{W}^{-1}\left\{\mathbf{x}_f - \prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0\right\}, \quad \text{for } t \in (t_{k-1},t_k), \\ \mathbf{0}, \quad \text{for } t \in [t_0,t_f]\setminus(t_{k-1},t_k). \end{cases} \tag{6.3.11}$$

□

Corollary 6.3.2 (Sufficient conditions)**.** *If* $\mathbf{E}_k\mathbf{\Phi}(t,s) = \mathbf{\Phi}(t,s)\mathbf{E}_k,\ \forall\, k = 1,2,\ldots,M,$ *in system* (6.2.1)*, then the sufficient conditions for null controllability of system* (6.2.1) *on* $[t_0,t_f]$ *given in Theorem 6.3.2 reduces to*

(i) There exists at least one $l \in \{1,2,\ldots,M\}$ *and a* $(m\times n)-$*matrix* $\mathbf{F}_l'$*, such that* $\mathbf{F}_l\mathbf{F}_l' = \mathbf{I}_n$.

(ii) The matrix $\mathbf{W} = \int_{t_{k-1}}^{t_k}\big[\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big]^*\mathrm{d}s$ *is positive definite for some* $k \in \{1,\ldots,(M+1)\}$.

Proof. The proof is similar to Theorem 6.3.2. Under case (i), consider the control function as

$$\mathbf{u}(t) = \begin{cases} -\,\mathbf{F}_l'\prod_{j=l}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_l,t_0)\mathbf{x}_0, \quad \text{at } t = t_l, \\ \mathbf{0}, \quad \text{for } t \in [t_0,t_f]\setminus\{t_l\}, \end{cases} \tag{6.3.12}$$

in order to steer the state of system (6.2.1) given in eq (6.2.8) from $\mathbf{x}_0$ to $\mathbf{0}$.

For the case (ii), define a control function

$$\mathbf{u}(t) = \begin{cases} -\,\big[\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big]^*\mathbf{W}^{-1}\prod_{j=k-1}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0, \quad \text{for } t \in (t_{k-1},t_k), \\ \mathbf{0}, \quad \text{for } t \in [t_0,t_f]\setminus(t_{k-1},t_k), \end{cases} \tag{6.3.13}$$

for which eq (6.2.8) satisfies $\mathbf{x}(t_f) = \mathbf{0}$. □

The theorems and corollaries introduced so far in this section provides the sufficient conditions under which the system (6.2.1) is controllable. We now investigate the necessary and sufficient for the controllability for the linear system (6.2.1) under the condition $(\mathbf{I}_n+\mathbf{E}_k)$ are non-singular for all $k = 1, 2, \ldots, M$. We introduce the following positive semidefinite $(n \times n)-$symmetric matrices:

$$\left.\begin{aligned}
\mathbf{W}_k = \mathbf{W}(t_{k-1}, t_k) &:= \int_{t_{k-1}}^{t_k} \left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\right] \\
&\qquad \times \left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\right]^* \mathrm{d}s, \\
\mathbf{W}_{M+1} = \mathbf{W}(t_M, t_f) &:= \int_{t_M}^{t_f} \left[\Big(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\Big)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s)\right] \\
&\qquad \times \left[\left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s)\right]^* \mathrm{d}s, \\
\mathbf{V}_1 = \mathbf{V}(t_0, t_1) &:= \int_{t_0}^{t_1} \big[\mathbf{\Phi}(t_0, s)\mathbf{B}(s)\big]\big[\mathbf{\Phi}(t_0, s)\mathbf{B}(s)\big]^* \mathrm{d}s \\
\mathbf{V}_{k+1} = \mathbf{V}(t_k, t_{k+1}) &:= \int_{t_k}^{t_{k+1}} \left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s)\right] \\
&\qquad \times \left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s)\right]^* \mathrm{d}s, \\
\mathbf{G}_k &:= \left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)^{-1} \mathbf{F}_k\right]\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)^{-1} \mathbf{F}_k\right]^*, \\
\mathbf{H}_k &:= \left[\Big(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\Big)^{-1} \mathbf{\Phi}(t_0, t_k)\mathbf{F}_k\right]\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, t_k)\mathbf{F}_k\right]^*.
\end{aligned}\right\} \tag{6.3.14}$$

Theorem 6.3.3 (Necessary and sufficient condition)**.** *If all* $(\mathbf{I}_n+\mathbf{E}_k)$ *are non-singular, then an impulsive system* (6.2.1) *is controllable on* $[t_0, t_f]$ *if and only if*

$$\operatorname{rank}\big([\mathbf{W}_1,\ \mathbf{W}_2,\ \ldots, \mathbf{W}_{M+1},\ \mathbf{G}_1, \ldots, \mathbf{G}_M]\big) = n. \tag{6.3.15}$$

In addition, if $\mathbf{E}_k$ *commutes with the state transition matrix* $\mathbf{\Phi}(t, s)$*, then the above necessary* *and sufficient condition for controllability reduces to*

$$\operatorname{rank}\big([\mathbf{V}_1,\, \ldots, \mathbf{V}_{M+1},\ \mathbf{H}_1, \ldots, \mathbf{H}_M]\big) = n. \tag{6.3.16}$$

Proof. First we prove that the condition (6.3.15) is a necessary and sufficient condition for controllability of the system (6.2.1), where it is given that all $(\mathbf{I}_n + \mathbf{E}_k)$ are non-singular. The necessity of the condition (6.3.15) can be proved by contradiction. For this, let the system (6.2.1) be controllable on $[t_0, t_f]$, but assume that

$$\operatorname{rank}\big([\mathbf{W}_1,\, \mathbf{W}_2,\, \ldots, \mathbf{W}_{M+1},\ \mathbf{G}_1, \ldots, \mathbf{G}_M]\big) < n.$$

Then a homogeneous system

$$\Big[\mathbf{W}_1 \quad \mathbf{W}_2 \cdots \mathbf{W}_{M+1} \quad \mathbf{G}_1 \cdots \mathbf{G}_M\Big]^* \mathbf{z} = \mathbf{0}$$

have at least one non-zero solution $\mathbf{z} \in \mathbb{R}^n$. Further this non-zero solution also satisfies the equations: $\mathbf{W}_k\, \mathbf{z} = \mathbf{0}$, $\mathbf{W}_{M+1}\, \mathbf{z} = \mathbf{0}$ and $\mathbf{G}_k\, \mathbf{z} = \mathbf{0}$. Therefore we have $\mathbf{z}^*\mathbf{W}_k\, \mathbf{z} = 0$, $\mathbf{z}^*\mathbf{W}_{M+1}\, \mathbf{z} = 0$, $\mathbf{z}^*\mathbf{G}_k\, \mathbf{z} = 0$, for all $k = 1, 2, \ldots, M$. Now,

$$\begin{cases} \mathbf{z}^*\mathbf{W}_k\, \mathbf{z} = 0 \Longrightarrow \displaystyle\int_{t_{k-1}}^{t_k} \left\| \mathbf{z}^* \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s) \right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0, \\ \mathbf{z}^*\mathbf{W}_{M+1}\, \mathbf{z} = 0 \Longrightarrow \displaystyle\int_{t_M}^{t_f} \left\| \mathbf{z}^* \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s) \right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0, \\ \mathbf{z}^*\mathbf{G}_k\, \mathbf{z} = 0 \Longrightarrow \left\| \mathbf{z}^* \left(\displaystyle\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k \right\|^2_{\mathbb{R}^{1\times m}} = 0, \end{cases}$$

for all $k = 1, 2, \ldots, M$.

Since the integrands in the above integrals are non-negative continuous functions over their domains, hence it follows that

$$\begin{cases} \mathbf{z}^* \left(\displaystyle\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s) = \mathbf{0}, \ \ \forall\, s \in (t_{k-1}, t_k), \\ \mathbf{z}^* \left(\displaystyle\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s) = \mathbf{0}, \ \ \forall\, s \in (t_M, t_f], \\ \mathbf{z}^* \left(\displaystyle\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k = \mathbf{0}, \end{cases} \tag{6.3.17}$$

for all $k = 1, 2, \ldots, M$.

Now as the system (6.2.1) is controllable on $[t_0, t_f]$, hence in particular it is null controllable and therefore there exists a control function $\mathbf{u}(\cdot) \in \mathcal{PC}$ that steers the state of the system (6.2.1) given in eq (6.2.2) from $\mathbf{x}(t_0) = \mathbf{z}$ to $\mathbf{x}(t_f) = \mathbf{0}$. Therefore, we get

$$
\begin{aligned}
\mathbf{0} = \mathbf{x}(t_f) = \mathbf{\Phi}(t_f, t_M)\Bigg\{ & \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{z} \\
& + \sum_{i=1}^{M} \left(\prod_{j=M}^{i+1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right) (\mathbf{I}_n + \mathbf{E}_i) \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
& + \sum_{i=2}^{M} \prod_{j=M}^{i} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{F}_M\mathbf{u}(t_M) \Bigg\} + \int_{t_M}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
= {} & \mathbf{\Phi}(t_f, t_M) \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \\
& \times \Bigg\{ \mathbf{z} + \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
& + \int_{t_M}^{t_f} \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\
& + \sum_{k=1}^{M} \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k\mathbf{u}(t_k) \Bigg\}.
\end{aligned}
$$

Premultiply the above equation with $\mathbf{z}^*\big[\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\big]^{-1}\mathbf{\Phi}(t_M, t_f)$ and using the results of (6.3.17), we obtain $\mathbf{z}^*\mathbf{z} = 0 \implies \mathbf{z} = \mathbf{0}$, a contradiction. Therefore if the system (6.2.1) is controllable, then

$$\operatorname{rank}\big([\mathbf{W}_1,\ \mathbf{W}_2,\ \ldots,\mathbf{W}_{M+1},\ \mathbf{G}_1,\ldots,\mathbf{G}_M]\big) = n.$$

To prove the sufficiency of condition (6.3.15), let us assume that $\operatorname{rank}\big([\mathbf{W}_1,\ \mathbf{W}_2,\ \ldots,\mathbf{W}_{M+1},\ \mathbf{G}_1,\ldots,\mathbf{G}_M]\big) = n$. Denote $\mathbf{W} := \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{M+1} + \mathbf{G}_1 + \cdots + \mathbf{G}_M$, then

$$\operatorname{rank}\big([\mathbf{W}_1,\ \mathbf{W}_2,\ \ldots,\mathbf{W}_{M+1},\ \mathbf{G}_1,\ldots,\mathbf{G}_M]\big) = \operatorname{rank}(\mathbf{W}),$$

therefore $\mathbf{W}$ is positive definite (Lemma 3.3.1 of Chapter 3). Now for a given initial state

$\mathbf{x}_0 \in \mathbb{R}^n$ and a final state $\mathbf{x}_f \in \mathbb{R}^n$ for the system (6.2.1), define the control function:

$$\mathbf{u}(t) := \begin{cases} \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, t)\mathbf{B}(t) \right\}^* \mathbf{W}^{-1} \\ \quad \times \left\{ -\mathbf{x}_0 + \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t_f)\mathbf{x}_f \right\}, \quad t \in (t_{k-1}, t_k), \\ \left\{ \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t)\mathbf{B}(t) \right\}^* \mathbf{W}^{-1} \\ \quad \times \left\{ -\mathbf{x}_0 + \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t_f)\mathbf{x}_f \right\}, \quad t \in (t_M, t_f], \\ \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k \right\}^* \mathbf{W}^{-1} \\ \quad \times \left\{ -\mathbf{x}_0 + \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t_f)\mathbf{x}_f \right\}, \quad t = t_k, \end{cases} \tag{6.3.18}$$

where $k = 1, 2, \ldots, M$. Now using eq (6.3.18) in eq (6.2.2), we get $\mathbf{x}(t_f) = \mathbf{x}_f$, showing that system (6.2.1) is controllable on $[t_0, t_f]$.

Now we show that (6.3.16) is necessary and sufficient condition for controllability of the system (6.2.1) on $[t_0, t_f]$, under the condition $(\mathbf{I}_n + \mathbf{E}_k)$ are invertible and $\mathbf{E}_k\mathbf{\Phi}(t, s) = \mathbf{\Phi}(t, s)\mathbf{E}_k$, for all k.

Similar to the first part of this theorem, the necessity of the condition (6.3.16) can be proved by contradiction. For this let the system (6.2.1) be controllable on $[t_0, t_f]$, but assume that

$$\operatorname{rank}\big([\mathbf{V}_1, \ldots, \mathbf{V}_{M+1}, \ \mathbf{H}_1, \ldots, \mathbf{H}_M]\big) < n.$$

But then, there exists a non-zero vector $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{V}_1 \mathbf{z} = \mathbf{0}$, $\mathbf{V}_{k+1} \mathbf{z} = \mathbf{0}$, $\mathbf{H}_k \mathbf{z} = \mathbf{0}$. Hence $\mathbf{z}^*\mathbf{v}_1 \mathbf{z} = 0$, $\mathbf{z}^*\mathbf{V}_{k+1} \mathbf{z} = 0$, $\mathbf{z}^*\mathbf{H}_k \mathbf{z} = 0$, for all $k = 1, 2, \ldots, M$. Now,

$$\begin{cases} \mathbf{z}^*\mathbf{V}_1 \mathbf{z} = 0 \Longrightarrow \displaystyle\int_{t_0}^{t_1} \left\| \mathbf{z}^*\mathbf{\Phi}(t_0, s)\mathbf{B}(s) \right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0, \\ \mathbf{z}^*\mathbf{V}_{k+1} \mathbf{z} = 0 \Longrightarrow \displaystyle\int_{t_k}^{t_{k+1}} \left\| \mathbf{z}^* \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s) \right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0, \\ \mathbf{z}^*\mathbf{H}_k \mathbf{z} = 0 \Longrightarrow \left\| \mathbf{z}^* \left(\displaystyle\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \mathbf{\Phi}(t_0, t_k)\mathbf{F}_k \right\|^2_{\mathbb{R}^{1\times m}} = 0, \end{cases}$$

for all $k = 1, 2, \ldots, M$.

Since the integrands in the above integrals are non-negative continuous functions over their domains, hence it follows that

$$\begin{cases} \mathbf{z}^*\mathbf{\Phi}(t_0, s)\mathbf{B}(s) = \mathbf{0}, \ \forall\, s \in [t_0, t_1), \\ \mathbf{z}^*\left(\prod_{j=k}^{1}(\mathbf{I}+\mathbf{E}_j)\right)^{-1}\mathbf{\Phi}(t_0, s)\mathbf{B}(s) = \mathbf{0}, \ \forall\, s \in (t_k, t_{k+1}), \\ \mathbf{z}^*\left(\prod_{j=k}^{1}(\mathbf{I}+\mathbf{E}_j)\right)^{-1}\mathbf{\Phi}(t_0, t_k)\mathbf{F}_k = \mathbf{0}, \end{cases} \tag{6.3.19}$$

for all $k = 1, 2, \ldots, M$.

Now as the system (6.2.1) is controllable on $[t_0, t_f]$, hence in particular it is null controllable. Therefore there exists a control function $\mathbf{u}(\cdot) \in \mathcal{PC}$ that steers the state of the system (6.2.1) given in eq (6.2.8) from $\mathbf{x}(t_0) = \mathbf{z}$ to $\mathbf{x}(t_f) = \mathbf{0}$. Therefore we get

$$\begin{aligned} \mathbf{0} = &\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f, t_0)\mathbf{z} + \sum_{i=1}^{M}\prod_{j=M}^{i}(\mathbf{I}+\mathbf{E}_j)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{i=2}^{M}\prod_{j=M}^{i}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f, t_{i-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{\Phi}(t_f, t_M)\mathbf{F}_M\mathbf{u}(t_M) \\ &+ \int_{t_M}^{t_f}\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ = &\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f, t_0)\Bigg\{\mathbf{z} + \int_{t_0}^{t_1}\mathbf{\Phi}(t_0, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{k=1}^{M}\int_{t_k}^{t_{k+1}}\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{\Phi}(t_0, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{k=1}^{M}\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{\Phi}(t_0, t_k)\mathbf{F}_k\mathbf{u}(t_k)\Bigg\}. \end{aligned}$$

Premultiply the above expression with $\mathbf{z}^*\mathbf{\Phi}(t_0, t_f)\big(\prod_{j=M}^{1}(\mathbf{I}+\mathbf{E}_j)\big)^{-1}$ and using (6.3.19), we obtain $0 = \mathbf{z}^*\mathbf{z} \implies \mathbf{z} = \mathbf{0}$, a contradiction, and hence if the system (6.2.1) is controllable, then $\mathrm{rank}\big([\mathbf{V}_1, \ldots, \mathbf{V}_{M+1}, \ \mathbf{H}_1, \ldots, \mathbf{H}_M]\big) = n$.

For the converse, let $\mathrm{rank}\big([\mathbf{V}_1, \ldots, \mathbf{V}_{M+1}, \ \mathbf{H}_1, \ldots, \mathbf{H}_M]\big) = n$, so that $\mathbf{W} := \mathbf{V}_1 + \cdots + \mathbf{V}_{M+1} + \mathbf{H}_1 + \cdots + \mathbf{H}_M$ is positive definite. Now in order to steer the state of the system (6.2.1) given in eq (6.2.8) from $\mathbf{x}_0$ to $\mathbf{x}_f$, we apply the following control function:

$$\mathbf{u}(t) := \begin{cases} \big[\mathbf{\Phi}(t_0,t)\mathbf{B}(t)\big]^*\mathbf{W}^{-1}\Big\{-\mathbf{x}_0 + \Big(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\Big)^{-1}\mathbf{x}_f\Big\}, \ t \in [t_0,t_1), \\ \Big\{\Big(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\Big)^{-1}\mathbf{\Phi}(t_0,t)\mathbf{B}(t)\Big\}^*\mathbf{W}^{-1} \\ \quad \times\Big\{-\mathbf{x}_0 + \Big(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\Big)^{-1}\mathbf{x}_f\Big\}, \quad t \in (t_k,t_{k+1}), \\ \Big\{\Big(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\Big)^{-1}\mathbf{\Phi}(t_0,t_k)\mathbf{F}_k\Big\}^*\mathbf{W}^{-1} \\ \quad \times\Big\{-\mathbf{x}_0 + \Big(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\Big)^{-1}\mathbf{x}_f\Big\}, \quad t = t_k. \end{cases} \tag{6.3.20}$$

where $k = 1, 2, \dots, M$. □

6.4 Controllability results for a time-invariant system

In this section, we reduce the controllability condition of the Section 6.3 for the time-invariant system (6.2.1) under some assumptions on the system components. The following theorems accomplishes this. Here necessary and sufficient condition for controllability are proved separately.

Theorem 6.4.1 (Necessary conditions). *Let all* $(\mathbf{I}+\mathbf{E}_k)$ *be non-singular matrices and each* $\mathbf{E}_k$ *commutes with* $\mathbf{A}$. *If an impulsive system* (6.2.1) *is controllable on* $[t_0, t_f]$, *then the following conditions are true.*

(i) $\mathrm{rank}(\mathrm{P}) = n$, *where*

$$\begin{aligned} \mathrm{P} := \Big\{ & \mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}, \\ & (\mathbf{I}_n+\mathbf{E}_1)^{-1}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big), \dots, \\ & \Big(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\Big)^{-1}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big), \\ & (\mathbf{I}_n+\mathbf{E}_1)^{-1}\big(\mathbf{F}_1,\ \mathbf{AF}_1,\ \mathbf{A}^2\mathbf{F}_1, \dots, \mathbf{A}^{n-1}\mathbf{F}_1\big), \dots, \\ & \Big(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\Big)^{-1}\big(\mathbf{F}_M,\ \mathbf{AF}_M,\ \mathbf{A}^2\mathbf{F}_M, \dots, \mathbf{A}^{n-1}\mathbf{F}_M\big)\Big\}. \end{aligned} \tag{6.4.1}$$

(ii) $\text{rank}(\mathrm{Q}) = n, \quad \forall\, \lambda \in \sigma(\mathbf{A})$, *where*

$$\begin{aligned}\mathrm{Q} := \Bigg\{ &(\lambda \mathbf{I}_n - \mathbf{A}),\ \mathbf{B},\ \big[\lambda \mathbf{I}_n - (\mathbf{I}_n + \mathbf{E}_1)^{-1}\mathbf{A}\big],\ \big[(\mathbf{I}_n + \mathbf{E}_1)^{-1}\mathbf{B}\big],\\ &\left[\lambda \mathbf{I}_n - \left(\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{A}\right],\ \left[\left(\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{B}\right], \ldots,\\ &\left[\lambda \mathbf{I}_n - \left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{A}\right],\ \left[\left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{B}\right],\\ &\mathbf{F}_1,\ \big[(\mathbf{I}_n + \mathbf{E}_1)^{-1}\mathbf{F}_1\big],\ \mathbf{F}_2,\ \left[\left(\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{F}_2\right], \ldots,\\ &\mathbf{F}_M,\ \left[\left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{F}_M\right]\Bigg\}.\end{aligned} \tag{6.4.2}$$

Proof. First let us show that condition (6.4.1) is necessary for the controllability of the system (6.2.1), by letting $\text{rank}(\mathrm{P}) < n$. Then there exists a non-zero vector $\mathbf{z} \in \mathbb{R}^n$ such that

$$\begin{cases} \mathbf{z}^* \mathbf{A}^l \mathbf{B} = \mathbf{0}, \\ \mathbf{z}^* \left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{A}^l \mathbf{B} = \mathbf{0}, \\ \mathbf{z}^* \left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{A}^l \mathbf{F}_k = \mathbf{0}. \end{cases} \tag{6.4.3}$$

for all $l = 0, 1, 2, \ldots, (N-1)$ and $k = 1, 2, \ldots, M$. From (6.3.14) and using (6.4.3), we have

$$\begin{aligned} \mathbf{z}^* \mathbf{V}_1 &= \int_{t_0}^{t_1} \mathbf{z}^* \big[e^{\mathbf{A}(t_0 - s)}\mathbf{B}\big]\big[e^{\mathbf{A}(t_0 - s)}\mathbf{B}\big]^* \mathrm{d}s \\ &= \int_{t_0}^{t_1} \left(\sum_{l=0}^{n-1} f_l(t_0 - s)\mathbf{z}^* \mathbf{A}^l \mathbf{B}\right)\big[e^{\mathbf{A}(t_0 - s)}\mathbf{B}\big]^* \mathrm{d}s = \mathbf{0}, \\ \mathbf{z}^* \mathbf{V}_{k+1} &= \int_{t_k}^{t_{k+1}} \mathbf{z}^* \left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0 - s)}\mathbf{B}\right]\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0 - s)}\mathbf{B}\right]^* \mathrm{d}s \\ &= \int_{t_k}^{t_{k+1}} \left[\sum_{l=0}^{n-1} f_l(t_0 - s)\mathbf{z}^* \left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{A}^l \mathbf{B}\right]\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0 - s)}\mathbf{B}\right]^* \mathrm{d}s \\ &= \mathbf{0}, \end{aligned}$$

$$
\begin{aligned}
\mathbf{z}^*\mathbf{H}_k &= \mathbf{z}^*\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0-t_k)}\mathbf{F}_k\right]\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0-t_k)}\mathbf{F}_k\right]^*\\
&= \left[\sum_{l=0}^{n-1} f_l(t_0-s)\mathbf{z}^*\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{A}^l\mathbf{F}_k\right]\left[\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0-t_k)}\mathbf{F}_k\right]^*\\
&= \mathbf{0}.
\end{aligned}
$$

Therefore we proved that $\mathbf{z}^*\big([\mathbf{V}_1,\ldots,\mathbf{V}_{M+1},\,\mathbf{H}_1,\ldots,\mathbf{H}_M]\big) = \mathbf{0}$ for some non-zero vector $\mathbf{z}$, implying that $\operatorname{rank}\big([\mathbf{V}_1,\,\ldots,\mathbf{V}_{M+1},\,\mathbf{H}_1,\ldots,\mathbf{H}_M]\big) < n$. Hence by Theorem 6.3.3, the system (6.2.1) is not controllable on $[t_0, t_f]$.

Now we prove that condition (6.4.2) is necessary for the controllability of system (6.2.1). This we prove by showing that the rank condition in (6.4.1) implies rank condition in (6.4.2). For this, let there exists some $\lambda \in \sigma(\mathbf{A})$ such that $\operatorname{rank}(\mathrm{Q}) < n$. But then there exists $\mathbf{z}(\neq \mathbf{0}) \in \mathbb{R}^n$ such that

$$
\begin{cases}
\mathbf{z}^*(\lambda\mathbf{I}_n-\mathbf{A}) = \mathbf{0},\\
\mathbf{z}^*\mathbf{B} = \mathbf{0},\\
\mathbf{z}^*\left\{\lambda\mathbf{I}_n-\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{A}\right\} = \mathbf{0},\\
\mathbf{z}^*\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{B} = \mathbf{0},\\
\mathbf{z}^*\mathbf{F}_k = \mathbf{0},\\
\mathbf{z}^*\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{F}_k = \mathbf{0},\ \ \forall\, k = 1,2,\ldots,M.
\end{cases}
\tag{6.4.4}
$$

With the repeated use of equations given in (6.4.4), finally we arrive at $\operatorname{rank}(\mathrm{P}) < n$, proving that system (6.2.1) is not controllable on $[t_0, t_f]$. □

Remark 6.4.1. *The rank conditions given in* (6.4.1) *and* (6.4.2) *are necessary for the controllability of the system* (6.2.1) *under the said assumptions, but not sufficient, as the* *following examples of a 2-D system with one impulse confirms:*

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(t), \quad t \in [0,2] \setminus \{1\}, \\ \mathbf{x}(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \Delta(\mathbf{x}(1)) &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \mathbf{u}(1). \end{aligned}\right\} \tag{6.4.5}$$

In this system $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ *and* $\mathbf{F}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. *Clearly* $\mathbf{AE}_1 = \mathbf{E}_1\mathbf{A}$ *and* $(\mathbf{I}_2 + \mathbf{E}_1)$ *is invertible. The eigenvalues of* $\mathbf{A}$ *are* $\lambda = 1, 2$. *Now one can verify that*

$$\text{rank}\Big\{\lambda\mathbf{I}_2 - \mathbf{A},\ \mathbf{B},\ \big[\lambda\mathbf{I}_2 - (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\mathbf{A}\big],\ \big[(\mathbf{I}_2 + \mathbf{E}_1)^{-1}\mathbf{B}\big],\ \mathbf{F}_1,\ \big[(\mathbf{I}_2 + \mathbf{E}_1)^{-1}\mathbf{F}_1\big]\Big\} = 2,$$

for both $\lambda = 1$ *and* 2. *However we see that,*

$$\text{rank}\Big\{\mathbf{B},\ \mathbf{AB},\ (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\big\{\mathbf{B},\ \mathbf{AB}\big\},\ (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\big\{\mathbf{F}_1,\ \mathbf{AF}_1\big\}\Big\} = 1,$$

implying by Theorem 6.4.1 that, the system (6.4.5) *is not controllable on* $[0,2]$. *This example also shows that the rank condition* (6.4.2) *need not imply the rank condition* (6.4.1).

Consider another 2-D system with one impulse as

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{u}(t), \quad t \in [0,2] \setminus \{1\}, \\ \mathbf{x}(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \Delta(\mathbf{x}(1)) &= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(1). \end{aligned}\right\} \tag{6.4.6}$$

In this system $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ *and* $\mathbf{F}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *Clearly* $\mathbf{AE}_1 = \mathbf{E}_1\mathbf{A}$ *and* $(\mathbf{I}_2 + \mathbf{E}_1)$ *is invertible. And we see that*

$$\text{rank}\Big\{\mathbf{B},\ \mathbf{AB},\ (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\big\{\mathbf{B},\ \mathbf{AB}\big\},\ (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\big\{\mathbf{F}_1,\ \mathbf{AF}_1\big\}\Big\} = 2.$$

Now the solution to the system (6.4.6) *at any time* $t \in (1,2]$ *is found to be*

$$\mathbf{x}(t) = e^{\mathbf{A}(t-1)} \begin{bmatrix} \mathbf{u}(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t-1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}(1) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}(1) \\ (t-1)\mathbf{u}(1) \end{bmatrix}.$$

Clearly there is no control $\mathbf{u}(t)$ *that steers the state of* (6.4.6) *from* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *to* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, *implying that* (6.4.6) *is not controllable on* $[0, 2]$.

Theorem 6.4.2 (Sufficient conditions)**.** *Under one of the following conditions, an impulsive system* (6.2.1) *is controllable on* $[t_0, t_f]$.

(i) $\operatorname{rank}\big([\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]\big) = n.$

(ii) $\operatorname{rank}\big(\lambda \mathbf{I}_n - \mathbf{A},\ \mathbf{B}\big) = n,$ *for every* $\lambda \in \sigma(\mathbf{A})$.

Proof. (a) First we prove case (i). Let $\operatorname{rank}\big([\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]\big) = n$, but assume that the system (6.2.1) is not controllable. Then $\int_{t_M}^{t_f} \big[e^{\mathbf{A}(t_f - s)}\mathbf{B}\big]\big[e^{\mathbf{A}(t_f - s)}\mathbf{B}\big]^* \mathrm{d}s$ is singular by Theorem 6.3.1, therefore there exists a non-zero vector, say $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{z}^* \int_{t_M}^{t_f} \big[e^{\mathbf{A}(t_f - s)}\mathbf{B}\big]\big[e^{\mathbf{A}(t_f - s)}\mathbf{B}\big]^* \mathrm{d}s\, \mathbf{z} = 0$$

which can be written as

$$\int_{t_M}^{t_f} \big\|\mathbf{z}^* e^{\mathbf{A}(t_f - s)}\mathbf{B}\big\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0.$$

The integrand in the above integral is a continuous non-negative function on $(t_M, t_f]$, therefore

$$\mathbf{z}^* e^{\mathbf{A}(t_f - s)}\mathbf{B} = \mathbf{0}, \ \ \forall\, s \in (t_M, t_f].$$

Now at $s = t_f$, we have $\mathbf{z}^*\mathbf{B} = \mathbf{0}$. Further differentiating the above equation with respect to s and substituting $s = t_f$, to get $\mathbf{z}^*\mathbf{AB} = \mathbf{0},\ \mathbf{z}^*\mathbf{A}^2\mathbf{B} = \mathbf{0}, \dots, \mathbf{z}^*\mathbf{A}^{n-1}\mathbf{B} = \mathbf{0}$. Hence we can write

$$\mathbf{z}^*\big(\mathbf{B},\, \mathbf{AB},\, \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) = \mathbf{0}.$$

This implies $\operatorname{rank}\big([\mathbf{B},\, \mathbf{AB},\, \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]\big) < n$, which is a contradiction. Hence the system (6.2.1) is controllable.

(b) Now consider the case (ii). Here we show that

$$\operatorname{rank}\big([\lambda \mathbf{I}_n - \mathbf{A},\ \mathbf{B}]\big) = n,\ \forall\, \lambda \in \sigma(\mathbf{A}),$$

is equivalent to

$$\operatorname{rank}\big([\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]\big) = n.$$

To show $\operatorname{rank}\big([\lambda \mathbf{I}_n - \mathbf{A},\ \mathbf{B}]\big) = n$ implies $\operatorname{rank}\big([\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]\big) = n$, assume

$$0 < \operatorname{rank}\big([\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]\big) = r < n,$$

and prove that there exists at least one $\lambda \in \sigma(\mathbf{A})$ such that $\text{rank}\big([\lambda \mathbf{I}_n - \mathbf{A},\ \mathbf{B}]\big) < n$. This is done as follows:

Let $\mathbf{T}$ be a non-singular operator such that the transformation $\mathbf{y}(t) = \mathbf{T}^{-1}\mathbf{x}(t)$ converts the system (6.2.1) into normal form (see p.101 of [106]). To this end, we have

$$\left.\begin{aligned} \dot{\mathbf{y}}(t) &= \mathbf{T}^{-1}\dot{\mathbf{x}}(t) = \mathbf{T}^{-1}\big(\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\big) = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y}(t) + \big(\mathbf{T}^{-1}\mathbf{B}\big)\mathbf{u}(t), \\ \Delta\mathbf{y}(t_k) &= \mathbf{T}^{-1}\Delta\mathbf{x}(t_k) = \mathbf{T}^{-1}\big(\mathbf{E}_k\mathbf{x}(t_k) + \mathbf{F}_k\mathbf{u}(t_k)\big) = \big(\mathbf{T}^{-1}\mathbf{E}_k\big)\mathbf{x}(t_k) + \big(\mathbf{T}^{-1}\mathbf{F}_k\big)\mathbf{u}(t_k). \end{aligned}\right\} \tag{6.4.7}$$

The system (6.4.7) is in normal form, provided if we

(i) assume that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\mathbf{T}^{-1}\mathbf{B}$ are of the form

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix} \text{ and } \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix} \tag{6.4.8}$$

such that $\mathbf{A}_{11}$ is $(r \times r)-$matrix, $\mathbf{B}_{11}$ is $(r \times m)-$matrix with $r < n$ and

(ii) show that $\text{rank}\big([\mathbf{B}_{11}, \mathbf{A}_{11}\mathbf{B}_{11}, \ldots, \mathbf{A}_{11}^{r-1}\mathbf{B}_{11}]\big) = r$.

Now, from eq (6.4.8) we have

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix}.$$

Further

$$(\mathbf{T}^{-1}\mathbf{A}\mathbf{T})^2 = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})(\mathbf{T}^{-1}\mathbf{A}\mathbf{T}) = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12} + \mathbf{A}_{12}\mathbf{A}_{22} \\ \mathbf{O} & \mathbf{A}_{22}^2 \end{bmatrix},$$

which implies

$$\mathbf{T}^{-1}\mathbf{A}^2\mathbf{T} = \begin{bmatrix} \mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12} + \mathbf{A}_{12}\mathbf{A}_{22} \\ \mathbf{O} & \mathbf{A}_{22}^2 \end{bmatrix}.$$

Then

$$\mathbf{T}^{-1}\mathbf{A}^2\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12} + \mathbf{A}_{12}\mathbf{A}_{22} \\ \mathbf{O} & \mathbf{A}_{22}^2 \end{bmatrix} \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12} + \mathbf{A}_{12}\mathbf{A}_{22} \\ \mathbf{O} & \mathbf{A}_{22}^2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}^2\mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix}.$$

Continuing this, we get

$$\mathbf{T}^{-1}\mathbf{A}^{n-1}\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11}^{n-1}\mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix}.$$

Therefore we have

$$\mathbf{T}^{-1}\{\mathbf{B},\,\mathbf{AB},\ldots,\mathbf{A}^{n-1}\mathbf{B}\} = \{\mathbf{T}^{-1}\mathbf{B},\ \mathbf{T}^{-1}\mathbf{AB},\ldots,\mathbf{T}^{-1}\mathbf{A}^{n-1}\mathbf{B}\}$$
$$= \begin{bmatrix} \mathbf{B}_{11} & \mathbf{A}_{11}\mathbf{B}_{11} & \cdots & \mathbf{A}_{11}^{n-1}\mathbf{B}_{11} \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \end{bmatrix}.$$

Hence $\operatorname{rank}\Big(\mathbf{T}^{-1}\{\mathbf{B},\,\mathbf{AB},\ldots,\mathbf{A}^{n-1}\mathbf{B}\}\Big) = \operatorname{rank}\begin{bmatrix} \mathbf{B}_{11} & \mathbf{A}_{11}\mathbf{B}_{11} & \ldots & \mathbf{A}_{11}^{n-1}\mathbf{B}_{11} \\ \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} \end{bmatrix}$, which implies $\operatorname{rank}\big([\mathbf{B}_{11},\ \mathbf{A}_{11}\mathbf{B}_{11},\ldots,\mathbf{A}_{11}^{n-1}\mathbf{B}_{11}]\big) = \operatorname{rank}\big([\mathbf{B},\,\mathbf{AB},\ldots,\mathbf{A}^{n-1}\mathbf{B}]\big) = r$.
Since $\mathbf{A}_{11}$ is $(r\times r)-$matrix and $\mathbf{B}_{11}$ is $(r\times m)-$matrix, therefore from Cayley–Hamilton theorem we have

$$\operatorname{rank}\big([\mathbf{B}_{11},\,\mathbf{A}_{11}\mathbf{B}_{11},\ldots,\mathbf{A}_{11}^{r-1}\mathbf{B}_{11}]\big) = r.$$

This proves that system (6.4.7) is in normal form.

Now, let $\omega_0 \in \mathbb{R}^{n-r}$ be an eigen vector of $\mathbf{A}_{22}^*$ corresponds to an eigenvalue λ of $\mathbf{A}_{22}^*$, i.e. $\mathbf{A}_{22}^*\,\omega_0 = \lambda\,\omega_0$. Further, as $\mathbf{A}_{22}^*$ is a real matrix, so λ^* is also its eigenvalue, and hence eigenvalue of $\mathbf{A}_{22}$, and therefore, also an eigenvalues of $\mathbf{A}$.

By defining a vector $\omega^* := \big[\mathbf{0}\ \ \omega_0^*\big]\mathbf{T}^{-1} \neq \mathbf{0} \in \mathbb{R}^{1\times n}$, we compute

$$\begin{aligned}
\omega^*\mathbf{B} &= \big[\mathbf{0}\ \ \omega_0^*\big]\mathbf{T}^{-1}\mathbf{B} \\
&= \big[\mathbf{0}\ \ \omega_0^*\big]\begin{bmatrix}\mathbf{B}_{11}\\ \mathbf{O}\end{bmatrix} \\
&= \mathbf{0} \\
\text{and}\quad \omega^*\mathbf{A} &= \big[\mathbf{0}\ \ \omega_0^*\big]\mathbf{T}^{-1}\mathbf{A} \\
&= \big[\mathbf{0}\ \ \omega_0^*\big]\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{O} & \mathbf{A}_{22}\end{bmatrix}\mathbf{T}^{-1} \\
&= \big[\mathbf{0}\ \ \ \omega_0^*\mathbf{A}_{22}\big]\mathbf{T}^{-1} \\
&= \big[\mathbf{0}\ \ \ \lambda^*\omega_0^*\big]\mathbf{T}^{-1} \\
&= \lambda^*\big[\mathbf{0}\ \ \ \omega_0^*\big]\mathbf{T}^{-1} \\
&= \lambda^*\omega^*.
\end{aligned}$$

This proves that, there exists an eigenvalue λ of $\mathbf{A}$ such that $\mathbf{B}^*\omega = \mathbf{0}$ and $\mathbf{A}^*\omega = \lambda\,\omega$. Combining these two results, we can write $\begin{bmatrix}\mathbf{A}^* - \lambda\mathbf{I}_n\\ \mathbf{B}^*\end{bmatrix}\omega = \mathbf{0} \in \mathbb{R}^{m+n}$ with $\omega \neq \mathbf{0}$. This implies $\operatorname{rank}\big([\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}]\big) < n$.

Conversely, to prove $\operatorname{rank}\big([\mathbf{B},\ \mathbf{AB},\ldots,\mathbf{A}^{n-1}\mathbf{B}]\big) = n$ implies $\operatorname{rank}\big([\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}]\big) = n$, we assume that

$$0 \leq \operatorname{rank}\big([\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}]\big) < n,\ \text{ for some } \lambda \in \sigma(\mathbf{A}),$$

and show that $\text{rank}\big([\mathbf{B},\ \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]\big) < n$. But this assumption forces us to write

$$0 \leq \text{rank}\begin{pmatrix} \lambda^*\mathbf{I}_n - \mathbf{A}^* \\ \mathbf{B}^* \end{pmatrix} < n,$$

which implies that the homogeneous system: $\begin{pmatrix} \lambda^*\mathbf{I}_n - \mathbf{A}^* \\ \mathbf{B}^* \end{pmatrix}\omega = \mathbf{0} \in \mathbb{R}^{(m+n)\times n}$ has a non-trivial solution $\omega \in \mathbb{R}^n$. That is, with some non-zero vector ω, we have

$$\omega^*\mathbf{A} = \lambda\omega^* \ \text{ and } \ \omega^*\mathbf{B} = \mathbf{0} \in \mathbb{R}^{1\times m}. \tag{6.4.9}$$

With the repeated use of eqs (6.4.9), one would write

$$\begin{aligned} \omega^*\big(\mathbf{B},\ \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\big) &= \big(\omega^*\mathbf{B},\ \omega^*\mathbf{AB}, \ldots, \omega^*\mathbf{A}^{n-1}\mathbf{B}\big) \\ &= \big(\mathbf{0},\ \mathbf{0}, \ldots, \mathbf{0}\big) \\ &= \mathbf{0} \in \mathbb{R}^{1\times mn}, \ \text{ with } \omega \neq \mathbf{0}. \end{aligned}$$

This is equivalent to saying that the matrix $\big(\mathbf{B},\ \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\big)$ has linearly dependent rows and hence $\text{rank}\big([\mathbf{B},\ \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]\big) < n$.

Therefore, finally we proved that $\text{rank}\big([\lambda\mathbf{I}_n - \mathbf{A},\ \ \mathbf{B}]\big) = n$, $\forall\, \lambda \in \sigma(\mathbf{A})$, is equivalent to $\text{rank}\big([\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]\big) = n$. □

Remark 6.4.2. *This remark gives the procedure to compute a non-singular operator* $\mathbf{T}$ *which makes the system* (6.4.7) *to get into normal form.*

Let $\mathbf{T}^{-1} = \begin{bmatrix} \tau_1^T & \tau_2^T & \ldots & \tau_n^T \end{bmatrix}^T$, *where* $\tau_1,\ \tau_2, \ldots, \tau_n \in \mathbb{R}^{1\times n}$ *are linearly independent vectors to be determined. Since* $\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix} \Longrightarrow \begin{bmatrix} \tau_1\mathbf{B} \\ \vdots \\ \tau_r\mathbf{B} \\ \tau_{r+1}\mathbf{B} \\ \vdots \\ \tau_n\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{O} \end{bmatrix} \Longrightarrow \begin{bmatrix} \tau_1\mathbf{B} \\ \vdots \\ \tau_r\mathbf{B} \end{bmatrix} = \mathbf{B}_{11}$ & $\begin{bmatrix} \tau_{r+1}\mathbf{B} \\ \vdots \\ \tau_n\mathbf{B} \end{bmatrix} = \mathbf{O}$, *from which we can determine* $\tau_1,\ \tau_2, \ldots, \tau_n$ *and hence* $\mathbf{T}^{-1}$.

Remark 6.4.3. *The conditions given in Theorem 6.4.2 are sufficient for the controllability of the system* (6.2.1)*, but not necessary, as the following example of a 2-D system with one* *impulse confirms:*

$$\left.\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\mathbf{x}(t) + \begin{bmatrix}0 & 0 & 0\\1 & 2 & 3\end{bmatrix}\mathbf{u}(t), \quad t \in [0,2]\setminus\{1\},\\ \mathbf{x}(0) &= \begin{bmatrix}0\\0\end{bmatrix},\\ \Delta(\mathbf{x}(1)) &= \begin{bmatrix}1 & 1\\0 & 3\end{bmatrix}\mathbf{x}(1) + \begin{bmatrix}0 & 1 & 2\\1 & 0 & 1\end{bmatrix}\mathbf{u}(1).\end{aligned}\right\} \tag{6.4.10}$$

In this system $\mathbf{A} = \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$, $\mathbf{B} = \begin{bmatrix}0 & 0 & 0\\1 & 2 & 3\end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix}1 & 1\\0 & 3\end{bmatrix}$ *and* $\mathbf{F}_1 = \begin{bmatrix}0 & 1 & 2\\1 & 0 & 1\end{bmatrix}$. *Then we see that, there exists a* $(3 \times 2)-$*matrix* $\mathbf{F}_1' = \begin{bmatrix}1 & 0\\3 & -2\\-1 & 1\end{bmatrix}$ *such that* $\mathbf{F}_1\mathbf{F}_1' = \mathbf{I}_2$, *and hence the system* (6.4.10) *is controllable on* $[0,2]$ *by condition (i) of Theorem 6.3.1. However we observe that for this system* $\operatorname{rank}\big([\mathbf{B},\ \mathbf{AB}]\big) = 1 < 2$.

6.5 Conclusions

In this chapter, a dynamical control system modelled by an $n-$dimensional linear impulsive ordinary differential equations are considered. Various necessary conditions and sufficient conditions are established for controllability. The obtained results are further reduced to the corresponding time-invariant case of the system and subsequently obtained a Kalman's type matrix rank condition and a rank condition which employs the eigenvalues of the system matrix under various assumptions on the system components. Further it has been proven that, for the impulsive systems, the null controllability need not implies controllability, unlike for the non-impulsive linear systems.

Chapter 7

Controllability of a networked impulsive systems

7.1 Introduction

The research on controllability of complex networked systems has been extensively taken up over the last half a century, because of their ubiquity encountered in nature and society. Although various criteria for the state controllability of the individual systems are well developed, but the controllability issues become more complicated and challenging when it applies to the large-scale networks due to their structural complexity. In [78], analytical tools have been developed to study the controllability of an arbitrary complex directed network, by identifying the set of driver nodes with time-dependent control that can guide the system's entire dynamics. The questions of whether a networked system is almost uncontrollable was addressed in [31]. The controllability approachment of the networked system using only one driving signal was proposed in [110]. In [116], LTI-networked systems are considered, and investigated their controllability issues by allowing its every subsystem possessing different dynamics. Recently, in [108], authors studied the state controllability of the networked higher-dimensional LTI-systems with higher-dimensional connections for the multi-input/multi-output settings, and the influences of the network topology, node-system dynamics, the external control input, and the inner interactions on the controllability is investigated. In addition, it is also proved that, the interactions among the states of nodes not only can lead controllable nodes to form an uncontrollable network, but also can assemble uncontrollable nodes into a controllable network. Further simplifications are performed in [109] for a special setting of such systems, but with one-dimensional communications, as in many practical situations the less transmitted information is more economical.

All these investigations points that the subject of controllability of networked systems has become a topic of active pursuit, and till date, no research is conducted on such systems with impulses. This motivates to study controllability of networked systems with impulses. In this chapter, there are four sections. Section 7.2 contains a general mathematical model of the impulsive networked LTI-systems whose controllability are to be investigated. In Section 7.3 some sufficient and necessary conditions for controllability are derived. Numerical examples are given in Section 7.4 to demonstrate the theoretical results. Finally, Section 7.5 contains the conclusions of this chapter.

7.2 The networked impulsive system

Consider a networked linear time-invariant impulsive systems with $N-$nodes, where each node system is of $n-$dimension (where $N, n \in \mathbb{N}$, and to avoid trivial situations, assume that $N \geq 2$). Specifically, the dynamical system corresponding to the node i is described by

$$\left.\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}\mathbf{x}_i(t) + \sum_{j=1}^{N} \beta_{ij}\mathbf{H}\mathbf{y}_j(t) + \delta_i \mathbf{B}\mathbf{u}_i(t),\ t \in [t_0, \infty) \setminus \{t_k : k = 1, 2, \ldots\}, \\ \Delta\big(\mathbf{x}_i(t_k)\big) &= \mathbf{D}_k\mathbf{x}_i(t_k) + \mathbf{E}_k\mathbf{u}_i(t_k), \\ \mathbf{y}_i(t) &= \mathbf{C}\mathbf{x}_i(t), \quad i = 1, 2, \ldots, N, \end{aligned}\right\} \tag{7.2.1}$$

in which

(i) $\mathbf{x}_i(t) \in \mathbb{R}^n$ is the state vector,

(ii) $\beta_{ij} \in \mathbb{R}$ represents the communication channel between two nodes i and j with $\beta_{ii} = 0$ and $\beta_{ij} \neq 0$ if there is a communication from node j to node i, but otherwise $\beta_{ij} = 0$,

(iii) $\mathbf{H} \in \mathbb{R}^{n\times p}$ denotes the inner coupling matrix from the output of one node to the state of another node,

(iv) $\mathbf{y}_i(t) \in \mathbb{R}^p$, $p \in \mathbb{N}$, is the output vector of node i,

(v) $\delta_i = 1$ if node i is under control, but otherwise $\delta_i = 0$,

(vi) $\mathbf{u}_i(t) \in \mathbb{R}^m$, $m \in \mathbb{N}$, is an external control input vector to the node i and

(vii) $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D}_k \in \mathbb{R}^{n\times n}$, $\mathbf{E}_k \in \mathbb{R}^{n\times m}$ are the constant matrices.

Let us denote the network topology of the networked system (7.2.1) which is characterized by β_{ij} by

$$\mathbf{L} := \big(\beta_{ij}\big) \in \mathbb{R}^{N\times N}$$

and the external input channels characterized by δ_i by

$$\mathbf{\Lambda} := \mathrm{diag}(\delta_1, \ldots, \delta_N) \in \mathbb{R}^{N\times N}.$$

Also denote $\mathbf{X}(t) = \left[\mathbf{x}_1^T(t)\ \mathbf{x}_2^T(t)\cdots\mathbf{x}_N^T(t)\right]^T \in \mathbb{R}^{nN}$ the whole state of the networked system (7.2.1) and $\mathbf{U}(t) = \left[\mathbf{u}_1^T(t)\ \mathbf{u}_2^T(t)\cdots\mathbf{u}_N^T(t)\right]^T \in \mathbb{R}^{mN}$ the total external control input. Then the networked system (7.2.1) can be rewritten in a compact form as

$$\left.\begin{aligned} \dot{\mathbf{X}}(t) &= \mathbf{\Theta}\,\mathbf{X}(t) + \mathbf{\Psi}\mathbf{U}(t), \quad t \in [t_0, \infty) \setminus \{t_k : k = 1, 2, \ldots\}, \\ \Delta\mathbf{X}(t_k) &= \mathbf{\Omega}_k\mathbf{X}(t_k) + \mathbf{\Gamma}_k\mathbf{U}(t_k), \end{aligned}\right\} \tag{7.2.2}$$

with

$$\begin{aligned} \mathbf{\Theta} &= \big(\mathbf{I}_N \otimes \mathbf{A} + \mathbf{L} \otimes \mathbf{HC}\big) \in \mathbb{R}^{nN\times nN}, \\ \mathbf{\Psi} &= \big(\mathbf{\Lambda} \otimes \mathbf{B}\big) \in \mathbb{R}^{nN\times mN}, \\ \mathbf{\Omega}_k &= \big(\mathbf{I}_N \otimes \mathbf{D}_k\big) \in \mathbb{R}^{nN\times nN} \\ \text{and } \mathbf{\Gamma}_k &= \big(\mathbf{I}_N \otimes \mathbf{E}_k\big) \in \mathbb{R}^{nN\times mN}. \end{aligned}$$

Note here that, the system (7.2.2) is obtained from the networked system (7.2.1), hence the controllability properties of both the systems are same. Therefore it is enough to study the controllability of (7.2.2) as we are familiar with such systems in Chapter 6. First let us record this fact as a

Proposition 7.2.1. *The networked system* (7.2.1) *is controllable if and only if the system* (7.2.2) *is controllable.*

7.3 Controllability results

The sufficient conditions for the controllability of the system (7.2.1) are mentioned in the following theorem. We assume that there are $M-$impulses in the time interval $[t_0, t_f]$.

Theorem 7.3.1 (Sufficient conditions)**.** *If one of the following conditions holds, then the impulsive networked system* (7.2.1) *is controllable on* $[t_0, t_f]$.

(i) There exists at least a $l \in \{1, 2, \ldots, (M-1)\}$ *and a* $(m \times n)-$*matrix* $\mathbf{E}_l'$ *such that* $\mathbf{E}_l\mathbf{E}_l' = \mathbf{I}_n$ *and* $(\mathbf{I}_n + \mathbf{D}_{l+1})$, $(\mathbf{I}_n + \mathbf{D}_{l+2}), \ldots, (\mathbf{I}_n + \mathbf{D}_M)$ *are invertible.*

(ii) There exists a $(m \times n)-$*matrix* $\mathbf{E}_M'$ *such that* $\mathbf{E}_M\mathbf{E}_M' = \mathbf{I}_n$.

(iii) For every λ *the eigenvalue of* $\mathbf{\Theta}$, *the matrix solution* $\mathbf{F} \in \mathbb{C}^{N\times n}$ *of both equations:*

$$\mathbf{F}(\lambda\mathbf{I}_n - \mathbf{A}) - \mathbf{L}^T\mathbf{FHC} = \mathbf{O} \in \mathbb{C}^{N\times n} \ \ \textit{and} \ \ \mathbf{\Lambda}^T\mathbf{FB} = \mathbf{O} \in \mathbb{C}^{N\times m}$$

is $\mathbf{F} = \mathbf{O}$, *the zero matrix of size* $(N \times n)$.

Proof. (a) First we consider case (i). Assume that the hypothesis are true. Then there exists a matrix $\mathbf{\Gamma}'_l = \big(\mathbf{I}_N \otimes \mathbf{E}'_l\big) \in \mathbb{R}^{mN\times nN}$ such that $\mathbf{\Gamma}_l\mathbf{\Gamma}'_l = \mathbf{I}_{nN}$ and $(\mathbf{I}_{nN} + \mathbf{\Omega}_{l+1})$, $(\mathbf{I}_{nN} + \mathbf{\Omega}_{l+2}), \ldots, (\mathbf{I}_{nN} + \mathbf{\Omega}_M)$ are invertible. Therefore system (7.2.2) is controllable by Theorem 6.3.1 of Chapter 6. Hence networked impulsive system (7.2.1) is controllable.

(b) If we assume that there exists a $(m\times n)-$matrix $\mathbf{E}'_M$ such that $\mathbf{E}_M\mathbf{E}'_M = \mathbf{I}_n$, then the matrix $\mathbf{\Gamma}'_M = \big(\mathbf{I}_N \otimes \mathbf{E}'_M\big) \in \mathbb{R}^{mN\times nN}$ satisfies $\mathbf{\Gamma}_M\mathbf{\Gamma}'_M = \mathbf{I}_{nN}$. Therefore by Theorem 6.3.1, system (7.2.2), and hence system (7.2.1) is controllable.

(c) Recall from Chapter 6, Theorem 6.4.2 that if for every λ the eigenvalue of $\mathbf{\Theta}$, $\text{rank}(\lambda\mathbf{I}_{nN} - \mathbf{\Theta},\, \mathbf{\Psi}) = nN$, then system (7.2.2), and hence (7.2.1) is controllable. This is equivalent to saying that

$$\text{rank}\left(\begin{bmatrix} \lambda\mathbf{I}_{nN} - \mathbf{\Theta} & \mathbf{\Psi} & \mathbf{O} \\ \mathbf{I}_N \otimes \mathbf{C} & \mathbf{O} & \mathbf{I}_{pN} \end{bmatrix}\right) = (n+p)N. \tag{7.3.1}$$

Since

$$\begin{bmatrix} \mathbf{I}_{nN} & \mathbf{L}\otimes\mathbf{H} \\ \mathbf{O} & \mathbf{I}_{pN} \end{bmatrix} \begin{bmatrix} \lambda\mathbf{I}_{nN} - \mathbf{\Theta} & \mathbf{\Psi} & \mathbf{O} \\ \mathbf{I}_N \otimes \mathbf{C} & \mathbf{O} & \mathbf{I}_{pN} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_N \otimes (\lambda\mathbf{I}_n - \mathbf{A}) & \mathbf{\Lambda}\otimes\mathbf{B} & \mathbf{L}\otimes\mathbf{H} \\ \mathbf{I}_N \otimes \mathbf{C} & \mathbf{O} & \mathbf{I}_{pN} \end{bmatrix} = \mathbf{Q} \tag{7.3.2}$$

having a size of $\big((n+p)N \times (m+n+p)N\big)$, in view of condition (7.3.1), $\mathbf{Q}$ must have a full rank $= (n\ +\ p)\ N$. Define $\xi := [\xi_1\ \xi_2 \cdots \xi_N] \in \mathbb{C}^{1\times nN}$ with each $\xi_i \in \mathbb{C}^{1\times n}$ and $\eta := [\eta_1\ \eta_2 \cdots \eta_N] \in \mathbb{C}^{1\times pN}$ with each $\eta_i \in \mathbb{C}^{1\times p}$. Then $\mathbf{Q}$ has a full rank if and only if the solution to the following three equations:

$$\begin{aligned} \xi\big(\mathbf{I}_N \otimes (\lambda\mathbf{I}_n - \mathbf{A})\big) + \eta(\mathbf{I}_N \otimes \mathbf{C}) = \mathbf{0} \in \mathbb{C}^{1\times nN}, \\ \xi(\mathbf{\Lambda}\otimes\mathbf{B}) = \mathbf{0} \in \mathbb{C}^{1\times mN}, \\ \xi(\mathbf{L}\otimes\mathbf{H}) + \eta = \mathbf{0} \in \mathbb{C}^{1\times pN}, \end{aligned}$$

is $\xi = \mathbf{0}$ and $\eta = \mathbf{0}$.

Denote $\mathbf{F} := \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \in \mathbb{C}^{N\times n}$, and $\mathbf{P} := \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_N \end{bmatrix} \in \mathbb{C}^{N\times p}$. Then the above three equations can be transformed into the following:

$$\begin{aligned} \mathbf{F}(\lambda\mathbf{I}_n - \mathbf{A}) + \mathbf{P}\mathbf{C} = \mathbf{O} \in \mathbb{C}^{N\times n}, \\ \mathbf{\Lambda}^T\mathbf{F}\mathbf{B} = \mathbf{O} \in \mathbb{C}^{N\times m}, \\ \mathbf{L}^T\mathbf{F}\mathbf{H} + \mathbf{P} = \mathbf{O} \in \mathbb{C}^{N\times p} \end{aligned}$$

or

$$\left.\begin{aligned}\mathbf{F}(\lambda\mathbf{I}_n-\mathbf{A})-\mathbf{L}^T\mathbf{FHC}=\mathbf{O}\in\mathbb{C}^{N\times n},\\ \mathbf{\Lambda}^T\mathbf{FB}=\mathbf{O}\in\mathbb{C}^{N\times m}.\end{aligned}\right\}\tag{7.3.3}$$

Hence $\mathbf{Q}$ has a full rank if and only if the system (7.3.3) possesses only the trivial solution $\mathbf{F}=\mathbf{O}\in\mathbb{C}^{N\times n}$. □

The following theorem gives a necessary controllability criteria for the system (7.2.1).

Theorem 7.3.2 (Necessary condition)**.** *If all* $(\mathbf{I}_n+\mathbf{D}_k)$ *are non-singular, each* $\mathbf{D}_k$ *commutes with* $\mathbf{A}$ *and* $\mathbf{HC}$ *and the networked impulsive system* (7.2.1) *is controllable, then the following condition holds good.*

$$\begin{aligned}\text{rank}\Bigg\{&(\lambda\mathbf{I}_{nN}-\mathbf{\Theta}),\ \ \mathbf{\Psi},\ \ \big[\lambda\mathbf{I}_{nN}-(\mathbf{I}_{nN}+\mathbf{\Omega}_1)^{-1}\mathbf{\Theta}\big],\ \ \big[(\mathbf{I}_{nN}+\mathbf{\Omega}_1)^{-1}\mathbf{\Psi}\big],\\ &\left[\lambda\mathbf{I}_{nN}-\left(\prod_{j=2}^{1}(\mathbf{I}_{nN}+\mathbf{\Omega}_j)\right)^{-1}\mathbf{\Theta}\right],\ \left[\left(\prod_{j=2}^{1}(\mathbf{I}_{nN}+\mathbf{\Omega}_j)\right)^{-1}\mathbf{\Psi}\right],\dots,\\ &\left[\lambda\mathbf{I}_{nN}-\left(\prod_{j=M}^{1}(\mathbf{I}_{nN}+\mathbf{\Omega}_j)\right)^{-1}\mathbf{\Theta}\right],\ \left[\left(\prod_{j=M}^{1}(\mathbf{I}_{nN}+\mathbf{\Omega}_j)\right)^{-1}\mathbf{\Psi}\right],\\ &\mathbf{\Gamma}_1,\ \big[(\mathbf{I}_{nN}+\mathbf{\Omega}_1)^{-1}\mathbf{\Gamma}_1\big],\ \mathbf{\Gamma}_2,\ \left[\left(\prod_{j=2}^{1}(\mathbf{I}_{nN}+\mathbf{\Omega}_j)\right)^{-1}\mathbf{\Gamma}_2\right],\dots,\\ &\mathbf{\Gamma}_M,\ \left[\left(\prod_{j=M}^{1}(\mathbf{I}_{nN}+\mathbf{\Omega}_j)\right)^{-1}\mathbf{\Gamma}_M\right]\Bigg\}=nN,\ \ \text{for all}\,\lambda,\ \text{the eigenvalues of}\,\mathbf{\Theta}.\end{aligned}$$

Proof. One can verify that, $(\mathbf{I}_{nN}+\mathbf{\Omega}_k)$ are non-singular if and only if $(\mathbf{I}_n+\mathbf{D}_k)$ are non-singular; and each $\mathbf{\Omega}_k$ commutes with $\mathbf{\Theta}$ if and only if $\mathbf{D}_k$ commutes with $\mathbf{A}$ and $\mathbf{HC}$. Then the proof of this theorem follows from Theorem 6.4.1. □

7.4 Numerical examples

1. Consider a two-dimensional networked impulsive system with two nodes ($N=2$) and one impulse with $\beta_{12}=1,\ \beta_{21}=0,\ \delta_1=\delta_2=1$. Let $m=3$, $p=2$, the node-system be described by the matrices

$$\mathbf{A}=\begin{bmatrix}2&1\\0&3\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}3&4&1\\0&0&0\end{bmatrix}\ \text{and}\ \mathbf{C}=\begin{bmatrix}2&-3\\0.5&2\end{bmatrix},$$

the inner interactions given by the matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},$$

and the impulse matrices be given by

$$\mathbf{D}_1 = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \text{ and } \mathbf{E}_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

The network topology of this system becomes $\mathbf{L} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and the external input channel is $\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. By calculation we get $\mathbf{\Theta} = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 3 & 1.5 & 6 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{\Psi} = \begin{bmatrix} 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then obviously this networked system is controllable by Theorem 7.3.1, as there exists a $(3 \times 2)-$matrix $\mathbf{E}_1' = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$ such that $\mathbf{E}_1\mathbf{E}_1' = \mathbf{I}_2$. However observe that, for an eigenvalue $\lambda = 3$ of $\mathbf{\Theta}$, $\text{rank}\big(3\mathbf{I}_4 - \mathbf{\Theta},\ \mathbf{\Psi}\big) = 3 < 4$. Alternatively, there exists a non-zero solution matrix $\mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ to the system of matrix equations given in (7.3.3) with an eigenvalue $\lambda = 3$ of the matrix $\mathbf{\Theta}$.

2. Consider another two-dimensional networked impulsive system with two nodes and one impulse with $\beta_{12} = 1, \beta_{21} = 0,\ \delta_1 = \delta_2 = 1$. Let $m = 1$, $p = 2$, the node-system be described by the matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix},$$

the inner interactions by the matrix

$$\mathbf{H} = \begin{bmatrix} 2 & -1 \\ 0 & 0.5 \end{bmatrix},$$

and the impulse matrices are

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \text{ and } \mathbf{E}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

The network topology of this system is $\mathbf{L} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and the external input channel is $\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Further $\mathbf{\Theta} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $\mathbf{\Psi} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$. This networked system is controllable by Theorem 7.3.1, as for every eigenvalue λ of $\mathbf{\Theta}$, $\text{rank}\big(\lambda\mathbf{I}_4 - \mathbf{\Theta},\ \mathbf{\Psi}\big) = 4$. Alternatively, the only solution matrix $\mathbf{F}$ to the system of matrix equations (7.3.3) for every eigenvalue λ of the matrix $\mathbf{\Theta}$ is $\mathbf{F} = \mathbf{O}$. However, there does not exists $(1 \times 2)-$matrix $\mathbf{E}_1'$ such that $\mathbf{E}_1\mathbf{E}_1' = \mathbf{I}_2$.

3. For the verification of the Theorem 7.3.2, a two-dimensional networked impulsive system with two nodes and one impulse with $\beta_{12} = 1$, $\beta_{21} = 0$, $\delta_1 = 0$ and $\delta_2 = 1$ is considered in which $m = 1$, $p = 2$, the node-system are described by the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix},$$

the inner interactions by the matrix

$$\mathbf{H} = \begin{bmatrix} 0.5 & -0.5 \\ 1 & 3 \end{bmatrix},$$

and the impulse are

$$\mathbf{D}_1 = \mathbf{I}_2 \text{ and } \mathbf{E}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The network topology of this system is $\mathbf{L} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and the external input channel is $\mathbf{\Lambda} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We compute $\mathbf{\Theta} = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{\Psi} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$, $\mathbf{\Omega}_1 = \mathbf{I}_4$ and $\mathbf{\Gamma}_1 =$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here $\mathbf{D}_1$ commutes with both $\mathbf{A}$ and $\mathbf{HC}$, also $(\mathbf{I}_2 + \mathbf{D}_1)$ is non-singular.
Notice that for an eigenvalue $\lambda = 0$ of $\boldsymbol{\Theta}$,

$$\text{rank}\big([\lambda \mathbf{I}_4 - \boldsymbol{\Theta},\ \boldsymbol{\Psi},\ \lambda \mathbf{I}_4 - (\mathbf{I}_4 + \boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Theta},\ (\mathbf{I}_4 + \boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Psi},\ \boldsymbol{\Gamma}_1,\ (\mathbf{I}_4 + \boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Gamma}_1]\big) = 3 < 4,$$

then by applying Theorem 7.3.2, we say that this networked system is uncontrollable.

7.5 Conclusions

In this chapter, the networked $n-$dimensional linear impulsive systems modelled with ordinary differential equations are considered with N number of nodes, $N \geq 2$. The necessary condition for controllability is derived in terms of system matrices under some special properties of system parameters. Further an easy-to-verify sufficient controllability result is provided in terms of algebraic matrix equations. The results are verified with examples.

Summary of the thesis and Future work

In this part, we discuss the main contributions of the thesis and future research work based on the present work. The main objective of the thesis is to investigate the controllability analysis of some classes of finite-dimensional deterministic dynamical systems on a continuous time-scale. The fixed-point theorems—Schauder's fixed-point theorem, Banach contraction principle are used in establishing the controllability results. The contributions of the thesis are summarized as follows.

In **Chapter 3**, the controllability issues for an $n-$dimensional semilinear impulsive dynamical control system with multiple constant time-delays in control are addressed. For three different classes of nonlinearities and impulse functions of this system, namely, uniform boundedness, Lipschitz continuity and linear growth continuity, the controllability conditions are obtained by employing Schauder's fixed-point theorem and Banach contraction principle under the assumptions that, the semilinear system admits a unique solution on a given time-interval and the corresponding linear part of the system is controllable. Such analysis for semilinear impulsive systems with delay in control is new. In **Chapter 4**, sufficient conditions for the controllability of an $n-$dimensional fractional-order $\alpha \in (0,1)$ semilinear control system having multiple constant time-delays in control are established. Here also, the controllability conditions are obtained for three different classes of nonlinearities. The results are obtained by using Schauder's fixed-point theorem and Banach contraction principle. Though controllability analysis for fractional semilinear systems is available in the literature, analysis for such systems with delay in control is new. In **Chapter 5**, a dynamical control system modelled by an $(n \times n)-$dimensional linear impulsive matrix Lyapunov ordinary differential equations having multiple constant time-delays in its control function is considered. The controllability conditions are derived for certain classes of admissible control functions. Further, these controllability conditions are applied to special cases of systems such as systems without impulses–with delays; with impulses–without delays; and without impulses–without delays. In each of such cases, the obtained controllability results coincide with the results of the existing works available in the literature. In **Chapter 6**, a dynamical control system modelled by an $n-$dimensional linear impulsive ordinary differential equations are considered. Various necessary conditions and sufficient conditions for controllability of the system are obtained. The results are further reduced to the corresponding time-invariant system, and subsequently obtained a Kalman's type matrix rank condition and a rank condition in terms of the eigenvalues of the system matrix under certain assumptions on the system components. Further it has been proven that, for the impulsive systems, the null controllability need not imply controllability, unlike for the non-impulsive linear systems. Next, some of the results of this chapter are applied to the networked impulsive systems considered in **Chapter 7**. A sufficient condition for controllability is also given in terms of two algebraic matrix equations.

Apart from the results presented in the thesis, there are few interesting and challenging problems which forces one to do further research. They are briefly stated as follows. It will be interesting to consider the variable time-delays and distributed time delays in control function for the controllability investigation of semilinear impulsive systems discussed in Chapter 3, the fractional semilinear systems considered in Chapter 4, and the matrix Lyapunov systems of Chapter 5. Similarly only few papers are available on the constrained controllability problems of semilinear systems [22, 61] and one can analyze the controllability of impulsive nonlinear systems with constrained controls and delayed controls. Further, in this thesis only delays in control are considered. One can also look in the direction of controllability for impulsive systems with different kinds of delays in state and controls. To the best of our knowledge, upto now there is no work is reported on the controllability of fractional-order impulsive semilinear systems. Also, an interesting and challenging research is to investigate the higher-dimensional networked systems with delays in state/control.

List of Publications

1. Vijayakumar S. Muni and Raju K. George, "Controllability of semilinear impulsive control systems with multiple time delays in control," *IMA J. Math. Control & Inform.*, 2018, DOI: 10.1093/imamci/dny011.
2. Vijayakumar S. Muni, V. Govindaraj, and Raju K. George, "Controllability of fractional order semilinear systems with a delay in control," *Indian J. Math.*, vol. 60, no. 2, pp. 311–335, Aug. 2018.
3. Vijayakumar S. Muni and Raju K. George, "Controllability of linear impulsive matrix Lyapunov differential systems with delays in the control function," *Kybernetika*, vol. 54, no. 4, pp. 664–698, Sep. 2018.
4. Vijayakumar S. Muni and Raju K. George, "Controllability of linear impulsive systems—an eigenvalue approach." (communicated).

List of Papers Presented

1. "Controllability of fractional order semilinear systems with multiple time-delays in control," *International Conference on Algebra and Applied Analysis*, Aug. 9–11, 2018, Integral University, Lucknow, India.
2. "Controllability of networked higher-dimensional systems with one-dimensional communications having multiple constant time-delays in control," *International Conference on Advances in Pure and Applied Mathematics*, Sep. 6–8, 2018, Madurai Kamaraj University, Madurai, India.
3. "A PBH-rank condition for the controllability of linear impulsive systems," *International Conference on Recent Advances in Pure and Applied Mathematics*, Oct. 23–25, 2018, Delhi Technological University, New Delhi, India.
4. "Controllability of complex networks involving impulses and time-delay controls," *Second National Conference on Control and Inverse Problems (CIP),* March 1–2, 2019, Central University of Tamil Nadu, Thiruvarur, India.

Conferences Attended

1. "International Conference on Nonlinear Dynamical Systems," March 24–26, 2016, held at Bharathiar University, Coimbatore, India.
2. "International Conference on Mathematical Analysis and its Applications," Nov. 28–Dec. 2, 2016 held at IIT-Roorkee, India.

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Appendix A

The Matlab codes for the compuatational tests of Chapter 5

1. **Matlab code for the Figure 5.1 in Example 5.5.**

clear all
clc
syms t

$$t1 = 0 : 0.001 : 0.1;$$
$$y1 = 31.2 * \exp(-2 * t1) - 54.0563 * \exp(-4 * t1);$$
$$z1 = -2.326 * \exp(-t1) + 6.1598 * \exp(-3 * t1);$$

plot $(t1,\ y1)$
plot $(t1,\ z1)$

$$t2 = 0.1 : 0.001 : 0.3;$$
$$y2 = 31.2 * \exp(-2 * t2) - 27.0283 * \exp(-4 * t2);$$
$$z2 = -2.326 * \exp(-t2) + 3.08 * \exp(-3 * t2);$$

plot $(t2,\ y2)$
plot $(t2,\ z2)$

$$t3 = 0.3 : 0.001 : 0.5;$$
$$y3 = 15.6 * \exp(-2 * t3) - 27.028 * \exp(-4 * t3);$$
$$z3 = -1.163 * \exp(-t3) + 3.08 * \exp(-3 * t3);$$

plot $(t3,\ y3)$

plot $(t3,\ z3)$

$$t4 = 0.5 : 0.001 : 0.6;$$
$$y4 = 15.6 * \exp(-2 * t4) - 27.028 * \exp(-4 * t4);$$
$$z4 = -1.163 * \exp(-t4) + 3.08 * \exp(-3 * t4);$$

plot $(t4,\ y4)$
plot $(t4,\ z4)$

$$t5 = 0.6 : 0.001 : 0.8;$$
$$y5 = 15.6 * \exp(-2 * t5);$$
$$z5 = -1.163 * \exp(-t5);$$

plot $(t5,\ y5)$
plot $(t5,\ z5)$

$$t6 = 0.8 : 0.001 : 1;$$
$$y6 = 0 * \text{ones}(\text{size}(t6));$$
$$z6 = 0 * \text{ones}(\text{size}(t6));$$

plot $(t6,\ y6)$
plot $(t6,\ z6)$

$$t = [t1\ \ t2\ \ t3\ \ t4\ \ t5\ \ t6];$$
$$U11 = [y1\ \ y2\ \ y3\ \ y4\ \ y5\ \ y6];$$
$$U12 = [z1\ \ z2\ \ z3\ \ z4\ \ z5\ \ z6];$$

plot (t, $U11$, '.blue', t, $U12$, '.red')
legend ('$U_{11}(t)$', '$U_{12}(t)$', 'Location', 'southeast')
xlabel ('time(t) \rightarrow')
ylabel ('$U_{11}(t)$, $U_{12}(t)$ \rightarrow')
grid on
print –depsc CF1

2. Matlab code for the Figure 5.2 in Example 5.5.

clear all
clc

syms t

$$t1 = 0 : 0.001 : 0.2;$$
$$v1 = 0.5 * (\exp(2 * t1) - 1);$$
$$w1 = 0.8 * (\exp(t1) - 1.25 * t1 - 1);$$
$$y1 = 0.25 * (\exp(4 * t1) - 1);$$
$$z1 = -0.0222 * (\exp(3 * t1) + 15.0135 * t1 - 1);$$

plot ($t1$, $v1$)
plot ($t1$, $w1$)
plot ($t1$, $y1$)
plot ($t1$, $z1$)

$$t2 = 0.2 : 0.001 : 0.3;$$
$$v2 = 20.05 * \exp(-4 * t2) - 11.629 * \exp(-2 * t2) - 0.6456 * \exp(2 * t2);$$
$$w2 = -2.806 * \exp(-3 * t2) + 1.4198 * \exp(-t2) + 0.289 * \exp(t2);$$
$$y2 = 0.25 * (\exp(4 * t2) - 1);$$
$$z2 = -0.0222 * (\exp(3 * t2) + 15.136 * t2 - 1);$$

plot ($t2$, $v2$)
plot ($t2$, $w2$)
plot ($t2$, $y2$)
plot ($t2$, $z2$)

$$t3 = 0.3 : 0.001 : 0.4;$$
$$v3 = 10.025 * \exp(-4 * t3) - 11.634 * \exp(-2 * t3) + 1.0125 * \exp(2 * t3);$$
$$w3 = -1.403 * \exp(-3 * t3) + 1.4198 * \exp(-t3) - 1.3258 * \exp(t3);$$
$$y3 = 0.25 * (\exp(4 * t3) - 1);$$
$$z3 = -0.0222 * (\exp(3 * t3) + 15.136 * t3 - 1);$$

plot ($t3$, $v3$)
plot ($t3$, $w3$)
plot ($t3$, $y3$)
plot ($t3$, $z3$)

$$t4 = 0.4 : 0.001 : 0.5;$$
$$v4 = 10.025 * \exp(-4 * t4) - 11.689 * \exp(-2 * t4) + 1.0406 * \exp(2 * t4);$$

$$w4 = -1.403 * \exp(-3 * t4) + 1.4198 * \exp(-t4) - 0.1326 * \exp(t4);$$

$$y4 = 33.469 * \exp(-4 * t4) - 11.4464 * \exp(-2 * t4) - 0.1272 * \exp(4 * t4);$$

$$z4 = -3.409 * \exp(-3 * t4) + 0.8693 * \exp(-t4) + 0.0784 * \exp(3 * t4);$$

plot $(t4, v4)$
plot $(t4, w4)$
plot $(t4, y4)$
plot $(t4, z4)$

$$t5 = 0.5 : 0.001 : 0.7;$$

$$v5 = 10.025 * \exp(-4 * t5) - 5.8175 * \exp(-2 * t5) + 0.1584 * \exp(2 * t5);$$

$$w5 = -1.403 * \exp(-3 * t5) + 0.7102 * \exp(-t5) + 0.3286 * \exp(t5);$$

$$y5 = 16.734 * \exp(-4 * t5) - 11.5716 * \exp(-2 * t5) + 0.1138 * \exp(4 * t5);$$

$$z5 = -1.7042 * \exp(-3 * t5) + 0.8674 * \exp(-t5) + 0.0194 * \exp(3 * t5);$$

plot $(t5, v5)$
plot $(t5, w5)$
plot $(t5, y5)$
plot $(t5, z5)$

$$t6 = 0.7 : 0.001 : 0.8;$$

$$v6 = 10.025 * \exp(-4 * t6) - 5.8175 * \exp(-2 * t6) + 0.1584 * \exp(2 * t6);$$

$$w6 = -1.403 * \exp(-3 * t6) + 0.7102 * \exp(-t6) + 0.3286 * \exp(t6);$$

$$y6 = 16.734 * \exp(-4 * t6) - 5.785 * \exp(-2 * t6) + 0.0268 * \exp(4 * t6);$$

$$z6 = -1.7042 * \exp(-3 * t6) + 0.4334 * \exp(-t6) + 0.046 * \exp(3 * t6);$$

plot $(t6, v6)$
plot $(t6, w6)$
plot $(t6, y6)$
plot $(t6, z6)$

$$t7 = 0.8 : 0.001 : 1.0$$

$$v7 = -5.8178 * \exp(-2 * t7) + 0.2415 * \exp(2 * t7);$$

$$w7 = 0.7102 * \exp(-t7) + 0.2715 * \exp(t7);$$

$$y7 = 16.734 * \exp(-4 * t7) - 5.785 * \exp(-2 * t7) + 0.0268 * \exp(4 * t7);$$

$$z7 = -1.7042 * \exp(-3 * t7) + 0.4334 * \exp(-t7) + 0.046 * \exp(3 * t7);$$

plot ($t7$, $v7$)
plot ($t7$, $w7$)
plot ($t7$, $y7$)
plot ($t7$, $z7$)

$$t = [t1\ \ t2\ \ t3\ \ t4\ \ t5\ \ t6\ \ t7];$$
$$X11 = [v1\ \ v2\ \ v3\ \ v4\ \ v5\ \ v6\ \ v7];$$
$$X12 = [w1\ \ w2\ \ w3\ \ w4\ \ w5\ \ w6\ \ w7];$$
$$X21 = [y1\ \ y2\ \ y3\ \ y4\ \ y5\ \ y6\ \ y7];$$
$$X22 = [z1\ \ z2\ \ z3\ \ z4\ \ z5\ \ z6\ \ z7];$$

plot (t, $X11$, '.blue', t, $X12$, '.red', t, $X21$, '.green', t, $X22$, '.yellow')
legend ('$X_{11}(t)$', '$X_{12}(t)$', '$X_{21}(t)$', '$X_{22}(t)$', 'Location', 'southeast')
xlabel ('time(t) \rightarrow')
ylabel ('$X_{11}(t)$, $X_{12}(t)$, $X_{21}(t)$, $X_{22}(t)$ \rightarrow')
grid on
print –depsc CT1

3. Matlab code for the Figure 5.3 in Example 5.5.

clear all
clc
syms t

$$t1 = 0 : 0.001 : 0.5;$$
$$y1 = -60.11 * \exp(-4 * t1) + 40.4515 * \exp(-2 * t1);$$
$$z1 = 2.4901 * \exp(-3 * t1) - 0.2351 * \exp(-t1);$$

plot ($t1$, $y1$)
plot ($t1$, $z1$)

$$t2 = 0.5 : 0.001 : 0.6;$$
$$y2 = -60.11 * \exp(-4 * t2) + 40.4515 * \exp(-2 * t2);$$
$$z2 = 2.4901 * \exp(-3 * t2) - 0.2351 * \exp(-t2);$$

plot ($t2$, $y2$)
plot ($t2$, $z2$)

$$t3 = 0.6 : 0.001 : 0.8;$$

$$y3 = 40.4515 * \exp(-2 * t3);$$
$$z3 = -0.2351 * \exp(-t3);$$

plot ($t3$, $y3$)
plot ($t3$, $z3$)

$$t4 = 0.8 : 0.001 : 1;$$
$$y4 = 0 * \text{ones}(\text{size}(t4));$$
$$z4 = 0 * \text{ones}(\text{size}(t4));$$

plot ($t4$, $y4$)
plot ($t4$, $z4$)

$$t = [t1 \;\; t2 \;\; t3 \;\; t4];$$
$$U11 = [y1 \;\; y2 \;\; y3 \;\; y4];$$
$$U12 = [z1 \;\; z2 \;\; z3 \;\; z4];$$

plot (t, $U11$, '.blue', t, $U12$, '.red')
legend ('$U_{11}(t)$', '$U_{12}(t)$', 'Location', 'southeast')
xlabel ('time(t) \rightarrow')
ylabel ('$U_{11}(t)$, $U_{12}(t)$ \rightarrow')
grid on
print –depsc CF2

4. Matlab code for the Figure 5.4 in Example 5.5.

clear all
clc
syms t

$$t1 = 0 : 0.001 : 0.2;$$
$$v1 = 0.5 * (\exp(2 * t1) - 1);$$
$$w1 = 0.8 * (\exp(t1) - 1.25 * t1 - 1);$$
$$y1 = 0.25 * (\exp(4 * t1) - 1);$$
$$z1 = -0.0222 * (\exp(3 * t1) + 15.0135 * t1 - 1);$$

plot ($t1$, $v1$)
plot ($t1$, $w1$)

plot $(t1,\ y1)$
plot $(t1,\ z1)$

$$t2 = 0.2 : 0.001 : 0.4;$$
$$v2 = 22.2946 * \exp(-4 * t2) - 15.086 * \exp(-2 * t2) + 0.2285 * \exp(2 * t2);$$
$$w2 = -1.1342 * \exp(-3 * t2) + 0.1436 * \exp(-t2) + 0.3947 * \exp(t2);$$
$$y2 = 0.25 * (\exp(4 * t2) - 1);$$
$$z2 = -0.0222 * (\exp(3 * t2) + 15.0135 * t2 - 1);$$

plot $(t2,\ v2)$
plot $(t2,\ w2)$
plot $(t2,\ y2)$
plot $(t2,\ z2)$

$$t3 = 0.4 : 0.001 : 0.8;$$
$$v3 = 22.2954 * \exp(-4 * t3) - 15.086 * \exp(-2 * t3) + 0.22825 * \exp(2 * t3);$$
$$w3 = -1.1342 * \exp(-3 * t3) + 0.1436 * \exp(-t3) + 0.3947 * \exp(t3);$$
$$y3 = 37.216 * \exp(-4 * t3) - 15.005 * \exp(-2 * t3) + 0.0437 * \exp(4 * t3);$$
$$z3 = -1.378 * \exp(-3 * t3) + 0.0877 * \exp(-t3) + 0.0516 * \exp(3 * t3);$$

plot $(t3,\ v3)$
plot $(t3,\ w3)$
plot $(t3,\ y3)$
plot $(t3,\ z3)$

$$t4 = 0.8 : 0.001 : 1.0;$$
$$v4 = -15.086 * \exp(-2 * t4) + 0.4117 * \exp(2 * t4);$$
$$w4 = 0.1436 * \exp(-t4) + 0.3484 * \exp(t4);$$
$$y4 = 37.216 * \exp(-4 * t4) - 15.005 * \exp(-2 * t4) + 0.0437 * \exp(4 * t4);$$
$$z4 = -1.378 * \exp(-3 * t4) + 0.0877 * \exp(-t4) + 0.0516 * \exp(3 * t4);$$

plot $(t4,\ v4)$
plot $(t4,\ w4)$
plot $(t4,\ y4)$
plot $(t4,\ z4)$

$$t = [t1\ \ t2\ \ t3\ \ t4];$$

$$X11 = [v1 \;\; v2 \;\; v3 \;\; v4];$$
$$X12 = [w1 \;\; w2 \;\; w3 \;\; w4];$$
$$X21 = [y1 \;\; y2 \;\; y3 \;\; y4];$$
$$X22 = [z1 \;\; z2 \;\; z3 \;\; z4];$$

plot (t, $X11$, '.blue', t, $X12$, '.red', t, $X21$, '.green', t, $X22$, '.yellow')
legend ('$X_{11}(t)$', '$X_{12}(t)$', '$X_{21}(t)$', '$X_{22}(t)$', 'Location', 'southeast')
xlabel ('time(t) \rightarrow')
ylabel ('$X_{11}(t)$, $X_{12}(t)$, $X_{21}(t)$, $X_{22}(t)$ \rightarrow')
grid on
print –depsc CT2

5. Matlab code for the Figure 5.5 in Example 5.5.

clear all
clc
syms t

$$t1 = 0 : 0.001 : 0.1;$$
$$y1 = 0 * \text{ones}(\text{size}(t1));$$
$$z1 = 0 * \text{ones}(\text{size}(t1));$$

plot ($t1$, $y1$)
plot ($t1$, $z1$)

$$t2 = 0.1 : 0.001 : 0.3;$$
$$y2 = 0.4122 * \exp(-4 * t2);$$
$$z2 = 0.40846 * \exp(-3 * t2);$$

plot ($t2$, $y2$)
plot ($t2$, $z2$)

$$t3 = 0.3 : 0.001 : 0.5;$$
$$y3 = 2.9556 * \exp(-2 * t3) + 0.4122 * \exp(-4 * t3);$$
$$z3 = 2.4668 * \exp(-t3) + 0.40846 * \exp(-3 * t3);$$

plot ($t3$, $y3$)
plot ($t3$, $z3$)

$$t4 = 0.5 : 0.001 : 0.6;$$
$$y4 = 2.9556 * \exp(-2 * t4) + 0.4122 * \exp(-4 * t4);$$
$$z4 = 2.4668 * \exp(-t4) + 0.40846 * \exp(-3 * t4);$$

plot ($t4$, $y4$)
plot ($t4$, $z4$)

$$t5 = 0.6 : 0.001 : 0.8;$$
$$y5 = 2.9556 * \exp(-2 * t5);$$
$$z5 = 2.4668 * \exp(-t5);$$

plot ($t5$, $y5$)
plot ($t5$, $z5$)

$$t6 = 0.8 : 0.001 : 1;$$
$$y6 = 0 * \text{ones}(\text{size}(t6));$$
$$z6 = 0 * \text{ones}(\text{size}(t6))$$

plot ($t6$, $y6$)
plot ($t6$, $z6$)

$$t = [t1\ \ t2\ \ t3\ \ t4\ \ t5\ \ t6];$$
$$U11 = [y1\ \ y2\ \ y3\ \ y4\ \ y5\ \ y6];$$
$$U12 = [z1\ \ z2\ \ z3\ \ z4\ \ z5\ \ z6];$$

plot (t, $U11$, '.blue', t, $U12$, '.red')
legend ('$U_{11}(t)$', '$U_{12}(t)$', 'Location', 'southeast')
xlabel ('time(t) \rightarrow')
ylabel ('$U_{11}(t)$, $U_{12}(t)$ \rightarrow')
grid on
print –depsc CF3

6. Matlab code for the Figure 5.6 in Example 5.5.

clear all
clc

syms t

$$t1 = 0 : 0.001 : 0.2;$$

$$v1 = 0.5 * (\exp(2 * t1) - 1);$$

$$w1 = 0.8 * (\exp(t1) - 1.25 * t1 - 1);$$

$$y1 = 0.25 * (\exp(4 * t1) - 1);$$

$$z1 = -0.0222 * (\exp(3 * t1) + 15.0135 * t1 - 1);$$

plot $(t1, v1)$
plot $(t1, w1)$
plot $(t1, y1)$
plot $(t1, z1)$

$$t2 = 0.2 : 0.001 : 0.3;$$

$$v2 = 0.1648 * \exp(2 * t2);$$

$$w2 = -0.0187 * \exp(t2);$$

$$y2 = 0.25 * (\exp(4 * t2) - 1);$$

$$z2 = -0.0222 * (\exp(3 * t2) + 15.136 * t2 - 1);$$

plot $(t2, v2)$
plot $(t2, w2)$
plot $(t2, y2)$
plot $(t2, z2)$

$$t3 = 0.3 : 0.001 : 0.4;$$

$$v3 = 0.19 * \exp(2 * t3) - 0.153 * \exp(4 * t3);$$

$$w3 = -0.186 * \exp(-3 * t3) + 0.0373 * \exp(t3);$$

$$y3 = 0.25 * (\exp(4 * t3) - 1);$$

$$z3 = -0.0222 * (\exp(3 * t3) + 15.136 * t3 - 1);$$

plot $(t3, v3)$
plot $(t3, w3)$
plot $(t3, y3)$
plot $(t3, z3)$

$$t4 = 0.4 : 0.001 : 0.5;$$

$$v4 = 0.19 * \exp(2 * t4) - 0.153 * \exp(4 * t4);$$

$$w4 = -0.186 * \exp(-3 * t4) + 0.0373 * \exp(t4);$$
$$y4 = 0.1995 * \exp(4 * t4);$$
$$z4 = -0.0557 * \exp(3 * t4);$$

plot ($t4$, $v4$)
plot ($t4$, $w4$)
plot ($t4$, $y4$)
plot ($t4$, $z4$)

$$t5 = 0.5 : 0.001 : 0.7;$$
$$v5 = -0.153 * \exp(-4 * t5) - 1.103 * \exp(-2 * t5) + 0.1568 * \exp(2 * t5);$$
$$w5 = -0.186 * \exp(-3 * t5) - 1.5064 * \exp(-t5) + 0.5794 * \exp(t5);$$
$$y5 = -0.255 * \exp(-4 * t5) + 0.0047 * \exp(4 * t5);$$
$$z5 = -0.226 * \exp(-3 * t5) + 0.0113 * \exp(3 * t5);$$

plot ($t5$, $v5$)
plot ($t5$, $w5$)
plot ($t5$, $y5$)
plot ($t5$, $z5$)

$$t6 = 0.7 : 0.001 : 0.8;$$
$$v6 = -0.153 * \exp(-4 * t6) - 1.103 * \exp(-2 * t6) + 0.1568 * \exp(2 * t6);$$
$$w6 = -0.186 * \exp(-3 * t6) - 1.5064 * \exp(-t6) + 0.5794 * \exp(t6);$$
$$y6 = -0.255 * \exp(-4 * t6) - 1.096 * \exp(-2 * t6) + 0.0211 * \exp(4 * t6);$$
$$z6 = -0.226 * \exp(-3 * t6) - 0.9204 * \exp(-t6) + 0.067 * \exp(3 * t6);$$

plot ($t6$, $v6$)
plot ($t6$, $w6$)
plot ($t6$, $y6$)
plot ($t6$, $z6$)

$$t7 = 0.8 : 0.001 : 1.0;$$
$$v7 = -1.1023 * \exp(-2 * t7) + 0.1555 * \exp(2 * t7);$$
$$w7 = -1.5064 * \exp(-t7) + 0.5718 * \exp(t7);$$
$$y7 = -0.255 * \exp(-4 * t7) - 1.096 * \exp(-2 * t7) + 0.0211 * \exp(4 * t7);$$
$$z7 = -0.226 * \exp(-3 * t7) - 0.9204 * \exp(-t7) + 0.067 * \exp(3 * t7);$$

plot ($t7$, $v7$)
plot ($t7$, $w7$)
plot ($t7$, $y7$)
plot ($t7$, $z7$)

$$t = [t1\ \ t2\ \ t3\ \ t4\ \ t5\ \ t6\ \ t7];$$
$$X11 = [v1\ \ v2\ \ v3\ \ v4\ \ v5\ \ v6\ \ v7];$$
$$X12 = [w1\ \ w2\ \ w3\ \ w4\ \ w5\ \ w6\ \ w7];$$
$$X21 = [y1\ \ y2\ \ y3\ \ y4\ \ y5\ \ y6\ \ y7];$$
$$X22 = [z1\ \ z2\ \ z3\ \ z4\ \ z5\ \ z6\ \ z7];$$

plot (t, $X11$, '.blue', t, $X12$, '.red', t, $X21$, '.green', t, $X22$, '.yellow')
legend ('$X_{11}(t)$', '$X_{12}(t)$', '$X_{21}(t)$', '$X_{22}(t)$', 'Location', 'southeast')
xlabel ('time(t) \rightarrow')
ylabel ('$X_{11}(t)$, $X_{12}(t)$, $X_{21}(t)$, $X_{22}(t)$ \rightarrow')
grid on
print –depsc CT3