Virtual Element Method for Time Dependent Problems on Polygonal Meshes

A thesis submitted
in partial fulfillment for the degree of

Doctor of Philosophy

by

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APRIL 2018

Certificate

This is to certify that the thesis entitled **Virtual Element Method for Time Dependent Problems on Polygonal Meshes** submitted by **Dibyendu Adak**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfilment for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work carried out by him under my supervision. The contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Declaration

I declare that this report titled **Virtual Element Method for Time Dependent Problems on Polygonal Meshes** submitted in partial fulfilment of the Degree of **Doctor of Philosophy** is a record of original work carried out by me under the supervision of **Dr. E. Natarajan**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgement have been made wherever the findings of others have been cited.

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Acknowledgments

I am deeply indebted to my supervisor, Dr. E. Natarajan for his enthusiasm, motivation, immense patience and continuous support throughout the duration of PhD at Indian Institute of Space Science and Technology. He has bestowed an adequate amount of time for several long discussion. I honestly acknowledge the way he taught me implementation of Virtual Element Methods. I would like to thank him for his continuous support and inspiration at stressful time of PhD.

I owe to my research committee members Prof. Satyajit Ray, Prof. Raju K. George, Dr. Deepak T.G. and Prof. Subrahamanian Moosath (HoD) for their valuable and helpful comments about my research works. I would like to thanks to our staff members, Anish J., Abdul Karim, Pournami and Nisha for their friendly behaviour which inspired me each single day I spent in IIST. Moreover, I specially thank to Abdul Karim for his helpful suggestion to learn LaTeX, Xfig and many other tools in initial days of my research work. I thank to my dear friends and colleagues Sarath Babu, Abhishek Chakraborty and Swagath Ranjan Das for their help and for creating cheerful environment in IIST.

I am extremely thankful to my parents Shri. Ashis Adak and Smt. Tapati Devi and my youngest uncle Pulak Adak for their endless support, love, and encouragement to continue my research work. I can not imagine to complete my work without them. There is no word to express my gratitude for their dedication to lead me in an authentic way of my life.

I acknowledge the IIST research fund, Government of India for financial support to pursue research at IIST and to attend some international conferences in India. Also, I am thankful to Department of Mathematics at IIST for providing me all kind of equipment to continue my research.

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Abstract

This work is concerned with the numerical approximation of time dependent partial differential equations(PDE) in the context of virtual element method(VEM). Basic aspects of VEM are revealed in Chapter-1 dissecting elliptic PDE. Fundamental theoretical framework as well as computational aspect of VEM are developed based on two reliable projection operators-L^2 projection operator $\Pi^0_{k,K}$ and energy projection operator $\Pi^\nabla_{k,K}$(defined locally). Basis functions are constructed virtually, can be thought of a solution of some PDEs which determine the dimension of virtual element spaces. The method is designed in such a way that it does not require explicit information about basis functions. Informations provided by degrees of freedom (DoF) are enough to evaluate basis functions that exempt us from hectic polynomial integration. Moreover, a polynomial spaces sitting inside VEM space ensure optimal order of convergence. The second but not secondary advantage of VEM is that discrete formulation satisfies Galerkin approximation like FEM. Hence, many fundamental properties of FEM can be incorporated in VEM. In Chapter-1, we briefly study basic properties of VEM and computation of projection operators $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ (defined in same chapter). Moreover, using analogous idea form [1], we modify the virtual element space without changing DoF that ensure computation of L^2 projection operator $\Pi^0_{k,K}$.

In Chapter-2 and Chapter-3, we propose a numerical method to approximate semilinear parabolic and hyperbolic equations respectively. The method is designed by exploiting L^2 projection operator $\Pi^0_{k,K}$ in order to evaluate non-linear load term. We consider modified virtual element space for ensuring optimal order of convergence in H^1 and L^2 norm for semi-discrete and fully-discrete schemes. Furthermore, fully-discrete scheme reduces to non-linear system of equations which can be solved by employing Newton method. Since, Newton method is computationally expensive, we present linearised scheme that ensures optimal order of convergence. For time discretization, we employ backward Euler scheme and Newmark scheme for parabolic and hyperbolic equations respectively. We have conducted numerical experiments on polygonal meshes to illustrate the performance of the proposed scheme and validate the theoretical findings.

In Chapter-4, we have studied the convection dominated diffusion reaction equation using SUPG stabilizers for both FEM and VEM. We initiate our discussion by exploring new finite element for convection dominated diffusion reaction equations. We con-

sider two different bilinear forms and examine convergence behaviour in mesh-dependent norm. Numerical experiments were conducted to emphasize theoretical results. Later, we extend this discussion for time dependent convection dominated diffusion reaction equation in the context of VEM. In the present scheme, VEM is used for space discretization whereas Crank-Nicolson scheme is employed for time discretization. Since the model problem is convection dominated, we add additional SUPG type stabilizer in order to obtain stable numerical solutions. Both semi and fully discretize schemes are analyzed and convergence analysis is carried out in mesh dependent norm and in L^2 norm. A set of numerical examples is presented in order to judge the computational efficiency of the proposed scheme and also to validate our theoretical findings.

In Chapter-5, we address time dependent Stokes equation using VEM. Velocity is approximated using $H1$ conforming discrete inf-sup stable virtual element space and pressure is approximated by discontinuous piecewise polynomial space. We approximate non-stationary part and right hand side load term exploiting vector valued L^2 projection operator. Following analogous idea as [2], we modify the VEM space such that L^2 projection operator $\Pi^0_{k,K}$ is computable. Moreover, we introduce discrete Stokes projection to pursue convergence analysis for semi-discrete case.

In light of the above works, we have drawn some conclusions in Chapter-6 and pointed out some future works in same chapter.

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Abbreviations

VEM	Virtual Element Methods
FEM	Finite Element Methods
PDE	Partial Differential Equations
CDR	Convection Diffusion Reaction
DoF	Degrees of Freedom

Chapter 1

Introduction

1.1 Motivation

In this work, we review the primary features of virtual Element Method (VEM) and its application to provide efficient numerical schemes for time dependent partial differential equations (PDEs). Virtual Element Methods is a new technology originated from mimetic finite difference method introduced as a generalization of the finite element method on polyhedral or polygonal meshes for solving partial differential equations. Keeping in mind the applicability of polygonal meshes, recently attempts have been made to develop certain technologies which make use of polygonal meshes, for instance, see [3, 4, 5]. The origin of VEM can be traced back to classical mimetic finite difference methods[3, 6, 7] and from their subsequent mathematical frameworks and settings. Unlike finite element method, the VEM is applicable for unstructured polygonal meshes making it more suitable for complex geometry usually appeared in real life problem. Furthermore, VEM has several advantages including independent of particular shape of element like FEM, irrespective of complex construction of basis function, simplicity in computing with higher accuracy with possibility of easy extension to higher dimension. In computation, bilinear forms require only integration of polynomial over polygonal elements. Unlike the finite element methods on polygonal meshes(see related work in [8]), VEM does not seek explicit informations of shape functions associated with the finite dimensional spaces. The crucial property of VEM is that this method satisfies **patch-test** which is desirable by engineers and scientists.

The primary focus of VEM is to develop an efficient numerical technique which is well suited on polygons in two space dimension and polyhedral in three space dimension. Meshes with curve boundary can be better approximated by polygonal and polyhedral elements than standard quadrilateral, triangular elements. Moreover, tetrahedra, hexahedra in 3D and hexagonal element in 2D have more rotational symmetries with respect to centroid. Therefore, these elements can be utilized more in topology optimization avoiding distorted elements. In recent technology, scientists, engineers employ polyhedral elements for geophysical flow simulations for capturing geometric objects with various sizes. Geophysical flow simulation has direct application including analysis of fresh water reservoirs, geothermal energy extraction. Furthermore, propagation of high-risk waste buried in earth can be controlled using geophysical flows simulations. In simulator, prismatic polyhedral elements are employed to reduce coding complexity [6]. The VEM can be applied to approximate wide range of model problems with distorted elements which makes VEM more acceptable method to resolve subsurface problems. Basically, exploitation of distorted polygonal mesh makes mesh adaptation algorithms more efficient and have major application in practical problems including dynamic contract problems, PDEs with sliding computational domains. Furthermore, VEM can contribute significantly in various fields including magnetostatic, fluid mechanics and structural mechanics. The deformation of a shape regular mesh produces elements with strongly curved boundary which can be addressed by VEM more efficiently exploiting polygonal and polyhedral elements. Analogous issue arises in simulation of compressible and visco-elastic flows using Lagrangian schemes where computational domain is moving with fluids. The above mentioned features of VEM invite young researchers to contribute efficient numerical schemes in order to approximate model problems arisen in different fields of science and engineering.

1.2 Theoretical and computational Aspects

1.2.1 Theoretical Development

This section is dedicated to design and analyze the fundamental construction of virtual element space. In [9], Da veiga et al. have introduced basic framework of H^1 conforming virtual element method dissecting Poisson's equation. The dimension of the virtual element space is determined by some PDE which is satisfied by the basis functions associated with the space. Unlike FEM, basis functions can be enumerated with the help of informations provided by degrees of freedom. Before depicting basic virtual space, we describe basic discretization of bounded convex polygonal domain $\Omega \subset \mathbb{R}^2$. Let $\{\mathcal{T}_h\}$ be a family of decomposition of Ω (call polygonal mesh hereafter) formed by polygonal elements (referred as polygon) K with $h = \max\limits_{K\in\mathcal{T}_h}(h_K)$, where h_K is the diameter of K. With respect to this partition, let ε_h denotes the set of edges of $\mathcal{T}_h$, and by ε_h^0 and ε_h^∂, we will refer to the set of interior and boundary edges, respectively. In order to satisfy local interpolation approximation properties and stability of discrete bilinear forms, we make the following reasonable assumptions on the polygonal mesh $\mathcal{T}_h$:

- $\mathbf{A_1}$: K is star-shaped with respect to a ball of radius greater than ρh_K,

- $\mathbf{A_2}$: any two vertexes in K are at least $c h_K$ apart, where h_K is the diameter of K

and ρ and c are uniform positive constants[10, 11].

For each polygon $K \in \mathcal{T}_h$, we define local virtual element space as

$$V^k(K) := \Big\{ v \in H^1(K) \cap C^0(\partial K) : v|_e \in \mathbb{P}_k(e)\ \forall e \in \partial K,\ \Delta v \in \mathbb{P}_{k-2}(K) \Big\}, \quad (1.1)$$

and the global virtual element space is defined as

$$V_h^k := \Big\{ v \in L^2(\Omega) : v|_K \in V^k(K) \Big\}.$$

Let χ_K be the set of degrees of freedom associated with the virtual element space $V^k(K)$, which are defined by

- $(\mathbf{D_1})$ Values of v at $V(K)$ vertexes of K.
- $(\mathbf{D_2})$ For $k > 1$, the values of v at $k-1$ uniformly spaced points on each edge e.
- $(\mathbf{D_3})$ For $k > 1$, the moments $\frac{1}{|K|} \int_K \mathcal{P}(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} \quad \forall\, \mathcal{P}(\mathbf{x}) \in \mathcal{M}_{k-2}(K)$.

Here, $\mathcal{M}_{k-2}(K)$ stands for the set of scaled monomials defined in the following manner:

$$\mathcal{M}_{k-2}(K) = \Big\{ \Big(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \Big)^{\mathbf{s}}, |\mathbf{s}| \leq k-2 \Big\}, \quad (1.2)$$

and for a multi-index s, we follow usual convention $|\mathbf{s}| := s_1 + s_2$ and $\mathbf{x}^{\mathbf{s}} := x_1^{s_1}\, x_2^{s_2}$. Notice that $\mathcal{M}_{k-2}(K)$ is a basis for $\mathbb{P}_{k-2}(K)$. Moreover, we emphasise that the the virtual element space $V^k(K)$ is unisolvent w.r.t. degrees of freedom χ_K. In light of the above discussion, we introduce two projectors $\Pi^{\nabla}_{k,K}$ and $\Pi^{0}_{k,K}$. We first define elliptic projection operator $\Pi^{\nabla}_{k,K} : H^1(K) \to \mathbb{P}_k(K)$ by

$$\begin{cases} \displaystyle\int_K \nabla(\Pi^{\nabla}_{k,K} u - u) \cdot \nabla q = 0 \quad \forall\, q \in \mathbb{P}_k(K) \\ P^0(\Pi^{\nabla}_{k,K} u - u) = 0, \end{cases} \quad (1.3)$$

where $P^0 u$ is orthogonal L^2 projection operator on constant polynomial space, defined as

$$\begin{cases} P^0 u := \displaystyle\frac{1}{n(K)} \sum_{i=1}^{n(K)} u(V_i) \quad \text{for } k = 1 \\ P^0 u := \displaystyle\frac{1}{|K|} \int_K u \quad \forall\, k \geq 2, \end{cases} \quad (1.4)$$

where $n(K)$ denotes total number of vertices of K and the local L^2 projection $\Pi^0_{k,K}$: $H^1(K) \to \mathbb{P}_k(K)$ be represented as

$$\left(\Pi^0_{k,K} q - q, p_k\right)_{0,K} = 0 \quad \forall\, p_k \in \mathbb{P}_k(K). \tag{1.5}$$

Globally, the projection operator is constituted as

$$(\Pi^0_k q)|_K = \Pi^0_{k,K}(q), \text{for all} q \ \in L^2(\Omega). \tag{1.6}$$

Employing these two projectors, we approximate the model problem. Unlike FEM, we first define discrete formulation locally on each polygon $K \in \mathcal{T}_h$. Global formulation is obtained by adding local contributions. We consider the following model problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \ \ u = 0 \text{ on } \ \partial\Omega. \end{cases} \tag{1.7}$$

The variational formulation of the model problem(1.7) with zero Dirichlet boundary condition is given by;
Find $u \in V_0 := H^1_0(\Omega)$ such that

$$a(u, v) = (f, v)\, \forall\, v \in V_0, \tag{1.8}$$

where $(\cdot, \cdot)$ denotes $L^2(\Omega)$ inner-product and the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega.$$

Employing local elliptic projection operator $\Pi^\nabla_{k,K}$, we define discrete formulation.

For each polygon $K \in \mathcal{T}_h$, discrete bilinear form $a^K_h(\cdot, \cdot) : V^k(K) \times V^k(K) \to \mathbb{R}$ is defined as follows

$$a^K_h(u_h, v_h) := a^K\left(\Pi^\nabla_{k,K}(u_h), \Pi^\nabla_{k,K}(v_h)\right) + S^K_a\left((I - \Pi^\nabla_{k,K})u_h, (I - \Pi^\nabla_{k,K})v_h\right), \tag{1.9}$$

where $u_h, v_h \in V^k(K)$ and $S^K_a(\cdot, \cdot)$ is a symmetric positive definite bilinear form which ensures stability of discrete bilinear form $a^K_h(\cdot, \cdot)$. Moreover, $S^K_a(\cdot, \cdot)$ is spectrally equivalent to identity matrix and scales same as $a^K(\cdot, \cdot)$. Finally, we introduce the global bilinear form $a_h(\cdot, \cdot) : V^k_h \times V^k_h \to \mathbb{R}$ by adding local contribution as

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h)\; \forall\, u_h, v_h \in V_h^k. \tag{1.10}$$

VEM does not provide explicit information about basis functions, hence direct computation of load term $\int_\Omega f\ v_h$, where $v_h \in V_h^k$ is not possible. In view of this issue, exploiting L^2 projection operator $\Pi^0_{k-2,K}$, we approximate load term ensuring optimal order of convergence. We define the right-hand side load term on each element $K \in \mathcal{T}_h$ as follows:

$$f_h(\mathbf{x})|_K := \Pi^0_{k-2,K}(f(\mathbf{x})), \tag{1.11}$$

where $\mathcal{T}_h$ denotes polygonal mesh. As a consequence, employing orthogonality property of L^2 projection operator $\Pi^0_{k-2,K}$ load term reduces to

$$\begin{aligned}(f_h, v_h) &= \sum_{K \in \mathcal{T}_h} \int_K f_h\ v_h = \sum_{K \in \mathcal{T}_h} \int_K \Pi^0_{k-2,K}(f)\ v_h \\ &= \sum_{K \in \mathcal{T}_h} \int_K f\ \Pi^0_{k-2,K} v_h.\end{aligned} \tag{1.12}$$

Since, force function f is known and $\Pi^0_{k-2,K} v_h$ is computable over $V^k(K)$, we can evaluate (1.12) with the help of degrees of freedom. In [9], authors shown that this choice has optimal order of convergence. In contrast with FEM, VEM needs approximation properties of local polynomial projection operator in order to ensure optimal order of convergence for discrete bilinear form. The construction of these projection operators demand analogous idea like FEM [10, 12]. Moreover, the construction can be generalized over polygonal domains that are finite union of star-shaped domains.

Local approximation: For every function $z \in H^s(K)$ with $2 \leq s \leq k+1$ there exists a polynomial $z_\pi \in \mathbb{P}_k(K) \subset V^k(K)$ such that

$$\|z - z_\pi\|_{0,K} + h_K\, |z - z_\pi|_{1,K} \leq\ C\, h_K^s\, |z|_{s,K}, \tag{1.13}$$

where C is a positive constant that only depends on the polynomial degree k and the mesh regularity constant ρ. Furthermore, we propose the approximation property of following local interpolation operator.

Interpolation operator: On VEM space $V^k(K)$, following mechanism from [9] we can devise interpolation operator $I_h^K : H^s(K) \to V^k(K)$ satisfying optimal approximation. Unlike FEM, VEM does not demand explicit construction of interpolation operator. Exploiting degrees of freedom associated with the VEM space, we can ensure existence and approximation property of interpolation operator. In continuation of the above discussion, we briefly formulate fundamental idea of construction. Let us consider the dimension of virtual element space $V^k(K)$ is N^{dof}. From the construction of virtual element space, we can justify there exists Lagrange type of basis functions ϕ_i satisfying $dof_i(\phi_j) = \delta_{ij}$ for $1 \leq i, j \leq N^{dof}$, where dof_i denotes degrees of freedom. Then the framework of VEM space confirms that for any function $v \in H^1(K)$, there exists a function $I_h^K v \in V^k(K)$ such that

$$dof_i(v) = dof_i(I_h^K v) \qquad \forall i = 1, \ldots, N^{dof},\ dof_i \in \chi_K.$$

Now we can apply classical approximation property.

For every $h > 0$, every $K \in \mathcal{T}_h$, every s with $2 \leq s \leq k+1$, and every $v \in H^s(K)$, the interpolant $I_h^K v \in V^k(K)$ satisfies

$$\|v - I_h^K v\|_{0,K} + h_K |v - I_h^K v|_{1,K} \leq C\, h_K^s\, |v|_{s,K}, \tag{1.14}$$

where C is independent of local mesh size h and depends on mesh regularity constant ρ. Now, we define the global interpolation operator I_h as

$$I_h(v_h)|_K := I_h^K(v_h|_K).$$

However, for the accomplishment of theoretical convergence analysis, we demand optimal order L^2 projection operator over VEM space.(By optimal order L^2 projection operator, we mean k^{th} order L^2 projection operator over VEM space consisting of polynomial space at most of degree k). Consequently, borrowing idea from [1], we first construct auxiliary VEM space and later we recast global VEM space (of course consist of local VEM space following same fashion as FEM). We first define auxiliary space

$$\mathcal{Q}^k(K) := \Big\{ v \in H^1(K) \cap C^0(\partial K) : v|_e \in \mathbb{P}_k(e)\ \forall e \in \partial K,\ \Delta v \in \mathbb{P}_k(K) \Big\}. \tag{1.15}$$

Employing elliptic projection operator $\Pi_{k,K}^{\nabla}$, we introduce local virtual element space as

$$\mathcal{Z}^k(K) := \Big\{ v \in \mathcal{Q}^k(K) : \int_K (\Pi_{k,K}^{\nabla} v) q = \int_K v\, q\ \forall q \in \mathbb{P}_k/\mathbb{P}_{k-2}(K),\ K \in \mathcal{T}_h \Big\}, \tag{1.16}$$

where $\mathbb{P}_k/\mathbb{P}_{k-2}(K)$ denotes the linear space spanned by the scaled monomials of degree k and $k-1$ on K. The global virtual element space is defined as

$$\mathcal{Z}_h^k := \Big\{ v \in H_0^1(\Omega) : v|_K \in \mathcal{Z}^k(K) \Big\}. \tag{1.17}$$

It seems that the dimension of local virtual element space $\mathcal{Z}^k(K)$ is more and therefore requires more number of degrees of freedom. But, this prediction is not true. χ_K forms degrees of freedom for the modified virtual element space $\mathcal{Z}^k(K)$. On virtual element space $\mathcal{Z}^k(K)$, the orthogonal L^2 projection operator $\Pi^0_{k,K}$ is computable. Employing $\Pi^0_{k,K}$ operator, we can formulate local bilinear form $m_h^K(\cdot,\cdot) : \mathcal{Z}^k(K) \times \mathcal{Z}^k(K) \to \mathbb{R}$ for each polygon K and for all $u_h, v_h \in \mathcal{Z}^k(K)$ as

$$m_h^K(u_h, v_h) := \Big(\Pi^0_{k,K} u_h, \Pi^0_{k,K} v_h\Big)_K + S_m^K\Big((I - \Pi^0_{k,K})u_h, (I - \Pi^0_{k,K})v_h\Big). \tag{1.18}$$

Global form is defined as

$$m_h(v, w) = \sum_{K \in \mathcal{T}_h} m_h^K(v, w) \quad \forall\, v, w \in \mathcal{Z}_h^k.$$

Moreover, the discrete bilinear form $a_h^K(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ is meaningful and computable over virtual element space $\mathcal{Z}^k(K)$. In the formulation (1.18), the non-polynomial part $S_m^K(\cdot,\cdot)$ ensures stability of discrete bilinear form which scales same as polynomial part. We note that by the construction of the bilinear forms $a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$, it is clear that these forms possess two following fundamental properties polynomial consistency and stability, in the following sense, for more details we refer to [12, 9] .

Polynomial-consistency: For all $h > 0$ and for all $K \in \mathcal{T}_h$, the bilinear form $a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$ defined in (1.9),(1.18) satisfy the following consistency property (with respect to polynomials $\mathbb{P}_k(K)$)

$$\begin{aligned} a_h^K(p, v) &= a^K(p, v) \quad \forall\, p \in \mathbb{P}^k(K),\ \forall\, v \in \mathcal{Z}^k(K); \\ m_h^K(p, v) &= (p, v)_K \quad \forall\, p \in \mathbb{P}^k(K),\ \forall\, v \in \mathcal{Z}^k(K). \end{aligned}$$

Stability: There exist four positive constants $\alpha_*, \alpha^*, \beta_*$ and β^*, independent of h but depending on the shape regularity of the partition such that for all $v \in \mathcal{Z}^k(K)$

$$\begin{aligned} \alpha_*\, a^K(v, v) &\leq a_h^K(v, v) \leq \alpha^*\, a^K(v, v); \\ \beta_*\, (v, v)_K &\leq m_h^K(v, v) \leq \beta^*\, (v, v)_K, \end{aligned} \tag{1.19}$$

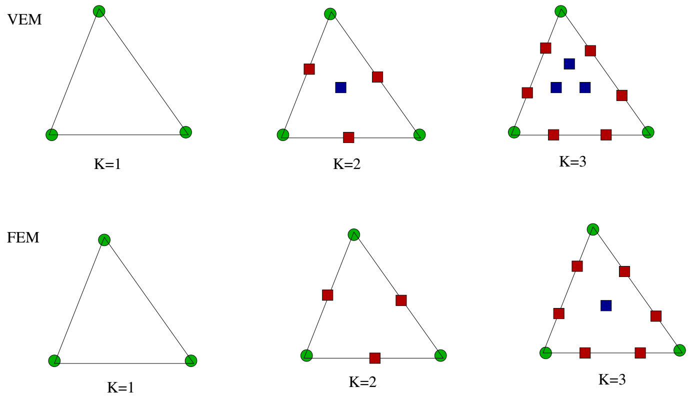


Figure 1.1: Degrees of freedom of a triangular cell for k=1,2,3; $(\mathbf{D_1})$ degrees of freedom and $(\mathbf{D_2})$ degrees of freedom are marked by green circle and red square respectively, cell moments are marked by a blue square.

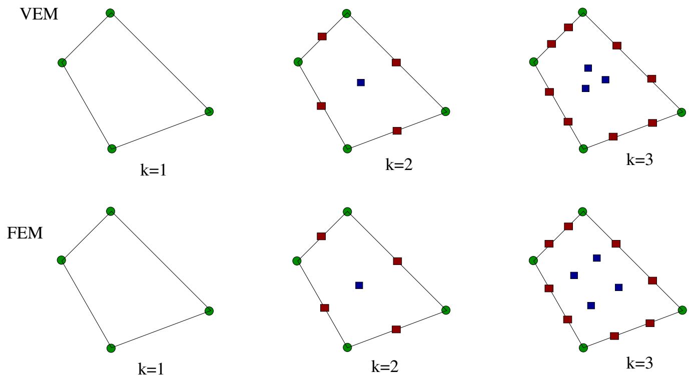


Figure 1.2: Degrees of freedom of a quadrilateral cell for k=1,2,3; $(\mathbf{D_1})$ and $(\mathbf{D_2})$ degrees of freedom are marked by green circle and red square, respectively; cell moments are marked by a blue square.

hold. Furthermore, in order to deduce direct comparison between VEM and FEM in terms of degrees of freedom, we depict the following Figures 1.1-1.3.

In contrast to FEM, convergence analysis of elliptic equation does not follow usual way. We need some additional tools in order to pursue error estimations. It is well-known that together with Cea's lemma, interpolation operator are utilized in order to derive error estimates with optimal order of convergence. This framework can not be incorporated in VEM directly. The primary issue related with VEM is that discrete solution u_h is not simply a polynomial function. Therefore, we desire some polynomial projection operator (1.13) which ensures optimal order of convergence. Furthermore, we begin the discussion with introducing two projectors $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ which assist to frame convergence analysis. These operators satisfies the following approximation properties

Lemma 1. *There exists a positive constant $C = C(\rho, k)$ such that, for all $K \in \mathcal{T}_h$ and all*

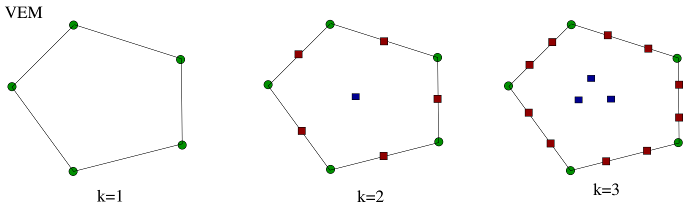


Figure 1.3: Degrees of freedom of a pentagonal cell for k=1,2,3; $(\mathbf{D_1})$ and $(\mathbf{D_2})$ degrees of freedom are marked by green circle and red square, respectively; cell moments are marked by a blue square.

functions $v \in H^{k+1}(K)$ *defined on* K*, it holds*

$$\begin{aligned} \|v - \Pi^0_{k,K} v\|_{m.K} \leq &\ C\, h_K^{s-m}\, |v|_{s,K}, \quad m, s \in \mathbb{N},\ m \leq s \leq k+1, \\ \|v - \Pi^\nabla_{k,K} v\|_{m,K} \leq &\ C\, h_K^{s-m}\, |v|_{s,K}, \quad m, s \in \mathbb{N},\ m \leq s \leq k+1, s \geq 1. \end{aligned} \tag{1.20}$$

Employing the two projectors $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$, optimal order of convergence are derived in L^2 and H^1 norm.

1.2.2 Computational Issue

In Finite element methods, we decompose the domain Ω into triangle or quadrilateral elements in order to obtain straightforward basis functions which reduce computational complexity. However, in VEM, we do not require direct computation of basis functions. Employing two projection operators $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ (projectors are defined in local sense, where k denotes order of virtual element space and K denotes polygonal element), we compute basis functions. Computation of mass and stiffness matrices depend on computation of three fundamental matrices **B**,**D**,**G** which are constructed in [13].

The projection operator $\Pi^\nabla_{k,K}$ acts from local virtual element space $\mathcal{Z}^k(K)$ to polynomial subspace $\mathbb{P}_k(K)$. We symbolize the matrix representation of the projection operator $\Pi^\nabla_{k,K}$ in the basis of $\mathcal{M}_k(K)$, by $\mathbf{A}^*$ and in the canonical basis of $\mathcal{Z}^k(K)$ by $\mathbf{A}^\nabla$ which can be evaluated employing analogous idea as [13].

Computation of the local stiffness matrix We first split the basis function ϕ into polynomial parts $\Pi^\nabla\phi$ and non-polynomial part $(I - \Pi^\nabla)\phi$ as

$$\phi = \Pi^\nabla\phi + (I - \Pi^\nabla)\phi. \tag{1.21}$$

A simple manipulation yields,

$$
\begin{aligned}
(\mathbf{S}_K)_{ij} &= (\nabla\phi_i, \nabla\phi_j) \\
&= (\nabla\Pi^\nabla\phi_i, \nabla\Pi^\nabla\phi_j)_{0,K} + (\nabla(I-\Pi^\nabla)\phi_i, \nabla(I-\Pi^\nabla)\phi_j)_{0,K} \\
&\quad + \underbrace{(\nabla\Pi^\nabla\phi_i, \nabla(I-\Pi^\nabla)\phi_j)}_{I} + \underbrace{(\nabla(I-\Pi^\nabla)\phi_i, \nabla\Pi^\nabla\phi_j)}_{II}.
\end{aligned} \tag{1.22}
$$

An application of orthogonality property of projection operator Π^∇ reduces the terms I and II to zero. Hence, we acquire

$$
(\mathbf{S}_K)_{ij} = (\nabla\Pi^\nabla\phi_i, \nabla\Pi^\nabla\phi_j)_{0,K} + (\nabla(I-\Pi^\nabla)\phi_i, \nabla(I-\Pi^\nabla)\phi_j)_{0,K}, \tag{1.23}
$$

where the first term ensures consistency and second term preserves stability. The first term can be evaluated with the help of $\mathbf{A}^*$ and $\mathbf{G}$ matrices as

$$
(\nabla\Pi^\nabla\phi_i, \nabla\Pi^\nabla\phi_j)_{0,K} = [\mathbf{A}^*\bar{\mathbf{G}}\mathbf{A}^*]_{ij}, \tag{1.24}
$$

where

$$
\bar{\mathbf{G}} := \begin{bmatrix} (0)_{1,n_k} \\ (\mathbf{G})_{n_k-1,n_k} \end{bmatrix}.
$$

As we have promised that direct calculation of non-polynomial part or stabilization part is not possible, we will assume the following rough approximation.

$$
\begin{aligned}
\Big(\nabla(I-\Pi^\nabla)\phi_i, \nabla(I-\Pi^\nabla)\phi_j\Big)_{0,K} &\approx \sum_{\omega=1}^{N^{dof}} \mathrm{dof}_\omega\Big((I-\Pi^\nabla)\phi_i\Big)\, \mathrm{dof}_\omega\Big((I-\Pi^\nabla)\phi_j\Big) \\
&= [(\mathbf{I}-\mathbf{A}^\nabla)^{\mathrm{T}}(\mathbf{I}-\mathbf{A}^\nabla)]_{ij}.
\end{aligned} \tag{1.25}
$$

In the above estimation, we have assumed the following irregular approximation $\int_K \nabla\phi_i \cdot \nabla\phi_j = 1$.

Plugging(1.24) and (1.25) into (1.23), we have

$$
\mathbf{S}_K = \mathbf{A}^*\bar{\mathbf{G}}\mathbf{A}^* + (\mathbf{I}-\mathbf{A}^\nabla)^{\mathrm{T}}(\mathbf{I}-\mathbf{A}^\nabla). \tag{1.26}
$$

Remark 1. *In this respect, we desire to mention that local stiffness matrix obtained in VEM methods is not same as usual FEM stiffness matrix. Moreover, in order to calculate* $\mathbf{B}$ *matrix, we have considered Gauss-Lobatto quadrature points to capture end points as quadrature points. This allows to employ vertex momentums directly to enumerate edge integration involved in* $\mathbf{B}$ *matrix estimation.*

In order to evaluate mass matrix $\mathbf{M}$, we demand to evaluate matrix representation of operator $\Pi^0_{k,K}$ in terms of monomial basis of $\mathbb{P}_k(K)$ and canonical basis of $\mathcal{Z}^k(K)$. Let us assume that $\mathbf{W}^*$ be the matrix representation of $\Pi^0_{k,K}$ with respect to monomial basis and $\mathbf{W}^0$ be the matrix representation of $\Pi^0_{k,K}$ with respect to canonical basis of $\mathcal{Z}^k(K)$. Following idea from [13], we can evaluate $\mathbf{W}^*$, $\mathbf{W}^0$ matrices.

Computation of Local Mass matrix

Now, we are in a stage to represent the framework of computation of local mass matrix. As usual like FEM, we define the local mass matrix as

$$(\mathbf{M})_{ij} := \int_K \phi_i \phi_j. \tag{1.27}$$

In order to set $\int_K \phi_i\phi_j$ into VEM framework, we first split the basis function into two parts polynomial part $\Pi^0\phi_i$ and non-polynomial part $(I-\Pi^0)\phi_i$. Hence,

$$\begin{aligned}\int_K \phi_i\phi_j == (\Pi^0_{k,K}\phi_i, \Pi^0_{k,K}\phi_j)_{0,K} + ((I-\Pi^0_{k,K})\phi_i, (I-\Pi^0_{k,K})\phi_j)_{0,K} \\ + \underbrace{(\Pi^0_{k,K}\phi_i, (I-\Pi^0_{k,K})\phi_j)_{0,K}}_{I} + \underbrace{((I-\Pi^0_{k,K})\phi_i, \Pi^0_{k,K}\phi_j)_{0,K}}_{II}.\end{aligned} \tag{1.28}$$

Orthogonality property of L^2 projection operator $\Pi^0_{k,K}$ ensures that both $(I)\&(II)$ are zero. Consequently, we conclude with the identity

$$(\mathbf{M})_{ij} := \underbrace{(\Pi^0_{k,K}\phi_i, \Pi^0_{k,K}\phi_j)_{0,K}}_{\text{polyomial part}} + \underbrace{((I-\Pi^0_{k,K})\phi_i, (I-\Pi^0_{k,K})\phi_j)_{0,K}}_{\text{non-polynomial part}} \tag{1.29}$$

Exploiting two matrices $\mathbf{W}$ and $\mathbf{H}$(define in [13]) , we can evaluate polynomial part as

$$(\Pi^0_{k,K}\phi_i, \Pi^0_{k,K}\phi_j)_{0,K} = (\mathbf{W}^{\mathrm{T}}\,\mathbf{H}^{-1}\,\mathbf{W})_{ij}. \tag{1.30}$$

Here, we exempt from exhaustive demonstration since detailed explanation is given in [13].

In order to compute non-polynomial part (II), we deduce the rough approximation

$\int_K \phi_i\, \phi_j \approx |K|$. Hence, the stability part or non-polynomial part can be evaluated as

$$\int_K (I - \Pi^0_{k,K})\phi_i\, (I - \Pi^0_{k,K})\phi_j \approx |K| \sum_{\omega=1}^{N^{dof}} \mathrm{dof}_\omega\Big((I - \Pi^0_{k,K})\phi_i\Big)\, \mathrm{dof}_\omega\Big((I - \Pi^0_{k,K})\phi_j\Big)$$
$$\approx |K|\, [(\mathbf{I} - \mathbf{W^0})^{\mathrm{T}}(\mathbf{I} - \mathbf{W^0})]_{ij}. \tag{1.31}$$

Substituting (1.31) and (1.30) into (1.29), we obtain

$$\mathbf{M} = \mathbf{W}^{\mathrm{T}}\, \mathbf{H}^{-1}\, \mathbf{W} + |K|\, (\mathbf{I} - \mathbf{W^0})^{\mathrm{T}}(\mathbf{I} - \mathbf{W^0}), \tag{1.32}$$

where $|K|$ represents area of polygon.

1.3 Related work and contribution

Time dependent PDEs are used to model many physical phenomenons including Newtonian and non-Newtonian flows, two phase flows and geophysical problems. Therefore, it is worthy to develop efficient numerical schemes for time dependent PDEs which are also computationally cheap. We initiate our discussion with fundamental time-dependent problems like semi-linear parabolic and semi-linear hyperbolic problems and then extend to time-dependent convection dominated diffusion reaction problem and time dependent Stokes equation.

VEM is uncovered first in [9]. In this paper, author has explored the basic framework of conforming virtual element methods. VEM schemes gain popularity soon after the invention because of its robust mathematical foundation, simplicity in implementation and suitable for adaptive method. VEM has been developed rapidly through a series of papers and has been applied for elasticity problems[14, 15, 16], plate bending problem [17], convection diffusion equation for convection dominated case [18], general elliptic equation [19, 20] basic mixed virtual element methods [21, 22], Stokes equation [23, 24, 25], Brinkman's equation [2], posteriori error estimation [26, 27], linear parabolic and hyperbolic equation [11, 28] . In addition to conforming methods, non-conforming method has evolved through a series of papers [29, 12] for Poisson's equation, general elliptic's equation and most recently for Stokes equation respectively.

The main focus of our exploration goes to development of efficient VEM approximations for nonlinear time dependent PDEs. Indeed, there are plenty of numerical approximations available in FEM to provide efficient and accurate solutions for nonlinear PDEs. But, hardly we can incorporate these techniques for nonlinear PDEs in the context of VEM. The primary issue indicates that VEM does not provide adequate informations to evaluate discrete solutions directly. In view of this problem, we propose an elegant VEM approximation for semilinear parabolic and semilinear hyperbolic problems employing L^2 projection operator $\Pi^0_{k,K}$ that ensure optimal order of convergence in L^2 and H^1 norm. In the next work, we have encountered time dependent convection dominated diffusion reaction equation. It is well-known that standard Galerkin methods produce non-physical oscillations for convection dominated diffusion reaction problems. Indeed, there are plenty of stabilizing techniques available in FEM literature in order to stabilize the numerical solution for convection dominated diffusion reaction problems. But, among them SUPG method can be absorbed directly in VEM context. However, stationary convection dominated diffusion problem is studied by Benedetto et al. in [18]. We explore this technique for time dependent convection dominated diffusion reaction problem. The primary focus is to examine basic aspects of stabilization parameter δ_K in the context of VEM. Moreover, providing efficient numerical approximations for time dependent convection dominated diffusion reaction problem over polygonal meshes are always welcomed with immense importances. Furthermore, VEM formulation is defined locally element wise having similarity with nonconforming FEM formulation. Nonconforming FEM formulation has immense importance as it preserves material properties element-wise. Therefore, together with VEM mechanism, we have proposed a new nonconforming FEM formulation for convection dominated diffusion reaction equations exploiting SUPG stabilizer. Moreover, the primary nonconforming VEM formulations are studied for elliptic problems and Stokes problem in [12, 20, 29]. Therefore, exploration of nonconforming VEM technique for convection dominated diffusion reaction equation would be meaningful extension of our work. In conclusion, we attempt to explore VEM approximation for time dependent Stokes problem. We review a VEM space which is discrete inf-sup stable and H^1 conforming. However, the VEM space is not divergence free which can be treated as main drawback. On top of that we introduce discrete Stokes projection which devise basic framework of convergence analysis for semi-discrete case.

1.4 Notations and preliminaries

This section introduces some standard notations which we will exploit throughout the dissertation. Let Ω be a bounded domain in $\mathbb{R}^d$ and $\partial\Omega$ denotes its boundary where $\mathbb{R}^d$ is the d dimensional metric space with Euclidean norm. Furthermore, we let $L^p(\Omega)$ denotes Banach space of equivalence classes of measurable functions $\phi(x)$ equipped with the norm

$$\|\phi\|_{L^p(\Omega)} := \Big(\int_\Omega |\phi(x)|^p \, dx \Big)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ denotes the normed linear space consisting of all essentially bounded functions on Ω with the norm

$$\|\phi\|_\infty = \text{ess} \sup_{x\in\Omega} |\phi(x)| := \inf_{\mathcal{Z}\subset\Omega, |\mathcal{Z}|=0} \sup_{\Omega\backslash\mathcal{Z}} g(x).$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be a d-tuple multi index with non-negative integers α_i and order of α is defined by $|\alpha| = \sum_{i=1}^d \alpha_i$. The $\alpha\ th$ order partial derivative defined as

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Sobolev space of order (s, p) over Ω is denoted by $W^{s,p}(\Omega)$ for non-negative integer s and $1 \le p \le \infty$ and defined as

$$W^{s,p}(\Omega) := \{\phi \in L^p(\Omega) : D^\alpha\phi \in L^p(\Omega), |\alpha| \le s\},$$

equipped with the norm

$$\|\phi\|_{s,p} = \Big(\sum_{|\alpha|\le S} \|D^\alpha\phi\|_{L^p}^p \Big)^{1/p} \quad \forall\, 1 \le p < \infty,$$

for $p = \infty$,

$$\|\phi\|_{s,\infty} = \sup_{|\alpha|\le s} \|D^\alpha\phi\|_{L^\infty(\Omega)}.$$

Semi-norm of order (s, p) is denoted by $|\cdot|_{s,p}$ and defined as

$$|\phi|_{s,p} = \Big(\sum_{|\alpha|=s} \|D^\alpha\phi\|_{L^p}^p \Big)^{1/p} \quad \forall\, 1 \le p < \infty,$$

and for $p = \infty$,

$$|\phi|_{s,\infty} = \sup_{|\alpha|=s} \|D^\alpha\phi\|_{L^\infty(\Omega)}.$$

The algebraic dual space of $H^s(\Omega)$ is denoted by $H^{-s}(\Omega)$ and the corresponding norm is defined by

$$\|\phi\|_{-s} = \sup_{\eta\in H^s(\Omega)/\{0\}} \frac{|(\phi,\eta)|}{\|\eta\|_s}.$$

Let $a, b \in \mathbb{R}$. The linear space $L^q(a, b; W^{s,p}(\Omega)),\ 1 \le q, p \le \infty, s \ge 0$, consists of functions $\psi : [a, b] \to W^{s,p}(\Omega)$ such that $\|\psi(t)\|_{s,p} \in L^q(a, b)$. The space $L^q(a, b; W^{s,p}(\Omega))$ is furnished with the norm

$$\|\phi\|_{L^q(a,b;W^{s,p}(\Omega))} := \Big(\int_a^b \|\phi(t)\|_{s,p}^q dt\Big)^{1/q} \quad 1 \le q < \infty,$$

and for $q = \infty$

$$\|\phi\|_{L^\infty(a,b;W^{s,p}(\Omega))} = \text{ess} \sup_{t\in(a,b)} \|\phi(t)\|_{s,p}.$$

In order to pursue convergence analysis for semi-discrete and fully discrete case revealed in next Chapters, we employ some standard inequalities which are mentioned below.

Lemma 2. ***(Young's Inequality)*** *Let $a, b \ge 0$ be non-negative real numbers and $\epsilon > 0$. Then the following inequality holds:*

$$ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}.$$

Lemma 3. ***(Hölder's inequality)*** *Let $\phi \in L^p(\Omega)$ and $\psi \in L^q(\Omega)$ be two functions where $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds*

$$\Big|\int_\Omega \phi\, \psi\, dx\Big| \le \Big(\int_\Omega |\phi|^p dx\Big)^{1/p} \Big(\int_\Omega |\psi|^q dx\Big)^{1/q}.$$

Lemma 4. ***(Cauchy-Schwarz inequality)*** *Let $(a_i)_{i=1}^N$ and $(b_i)_{i=1}^N$ be two sequences of positive real numbers and $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds*

$$\Big(\sum_{i=1}^N a_i b_i\Big) \le \Big(\sum_{i=1}^N a_i^p\Big)^{1/p} \Big(\sum_{i=1}^N b_i^q\Big)^{1/q}.$$

Lemma 5. ***(Continuous Gronwall inequality)*** *Let f, g, h be piecewise continuous non-negative functions defined on (a, b). Assume that g is non-decreasing function and there is a positive constant C independent of t such that*

$$f(t) + h(t) \le g(t) + C \int_a^t f(s)\, ds \quad \forall\, t \in (a, b).$$

Then

$$f(t) + h(t) \le\ e^{C(t-a)}\, g(t) \forall\, t \in (a, b). \tag{1.33}$$

Lemma 6. ***(Discrete Gronwall inequality)*** *Let $\Delta t, B, C > 0$ and let $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$, $(c_i)_{i=1}^n$, $(d_i)_{i=1}^n$ be sequences of non-negative numbers satisfying*

$$a_n + \Delta t \sum_{i=1}^n b_i \le B + C\Delta t \sum_{i=1}^n a_i + \Delta t \sum_{i=1}^n c_i \quad \forall n \ge 1.$$

Then if $C\Delta t < 1$,

$$a_n + \Delta t \sum_{i=1}^{n} b_i \leq e^{C(n+1)\Delta t} \Big(B + \Delta t \sum_{i=1}^{n} c_i \Big) \quad \forall\, n \geq 1.$$

Lemma 7. ***(Poincaré's inequality)*** *Let Ω be a bounded open set in $\mathbb{R}^d$. Then there exists a positive constant $C = C(\Omega, p)$ such that*

$$\|u\|_{0,p} \leq C\ |u|_{1,p} \quad \forall\, u \in W_0^{1,p}(\Omega).$$

In particular, $u \to |u|_{1,p}$ defines a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{1,p}$. On $H_0^1(\Omega)$, the bilinear form

$$(u, v) \to \int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}$$

defines an inner-product giving rise to the norm $|\cdot|_1$ equivalent to the norm $\|\cdot\|_1$.

Lemma 8. ***(Green's Formula)*** *Let Ω be a bounded open set of $\mathbb{R}^n$ set of class C^1 lying on the same side of its boundary $\partial\Omega$. Let $u, v \in H^1(\Omega)$. Then for $1 \leq i \leq n$,*

$$\int_\Omega u \frac{\partial v}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i}\, v + \int_{\partial\Omega} u\ v\eta_i,$$

where η_i is the i-th component of the outward normal to the boundary $\partial\Omega$.

1.5 Outline of thesis

The thesis is arranged as follows. In Chapter-2, we encounter semi-linear parabolic problem. We propose an elegant technique in order to tackle nonlinear load term exploiting orthogonal L^2 projection operator. Moreover, we highlight some computational techniques to diminish computational complexity. Using analogous idea, we modify the VEM framework for semi-linear hyperbolic problem in Chapter-3. Furthermore, the Chapter is enriched with several numerical examples including Sine-Gordon equation in order to justify the theoretical demonstration. In Chapter-4, we start our discussion by exploring new finite element for stationary convection dominated diffusion reaction equation and then extend to time dependent convection dominated diffusion reaction equations employing SUPG stabilizer in the context of virtual element method. In both cases, we prove error estimations in mesh dependent norm. A detailed discussion about the construction of stabilization parameter δ_K on polygonal meshes is revealed in same chapter. Moreover, several numerical experiments are conducted in order to justify theoretical results. Finally, in Chapter-5, we propose H^1 conforming, discrete inf-sup stable virtual element space for time dependent Stokes equations. Furthermore, we construct discrete Stokes projection to pursue convergence analysis for velocity and pressure. In Chapter-6, we explore some possible future works based on our research and also critical assessments are given in the same Chapter.

Chapter 2

Virtual Element Method For Semilinear Parabolic Problems on Polygonal Meshes

2.1 Introduction

In this chapter, we modify virtual element formulation for semi-linear parabolic problem. Virtual element method for linear parabolic problem is introduced in [11]. We extended the method for for semi-linear case where right hand side force function f is function of exact solution u as well as time t. In contrast with FEM, the basis functions are implicitly known in virtual element method (VEM). Hence, we can not directly evaluate nonlinear load term employing degrees of freedom. This difficulty appears in any kind of nonlinear model problem. As far as non-linearity is concerned, there are few contributions made in this direction. For better understanding, we highlight some VEM papers[30, 31] where fundamental approaches are presented in order to tackle nonlinear term in the context of VEM. In [30], authors examine C^1 virtual element method for non-linear Cahn-Hilliard equation. Their approach is quite different from our outlook. They have approximated non-linear part by computable term (i.e. the term can be evaluated with the help of mass matrix) ensuring optimal order of convergence. Obviously, the theoretical framework contributes some fundamental idea but still this approach may not be applicable to approximate general non-linear terms. In [31], Da veiga et al. encounter non-linearity exploiting projection operators. This idea can be widely used for other non-linear problems. The fundamental idea goes to considering non-linearity of only polynomial part of discrete solution u_h in discrete formulation instead of considering non-linearity of u_h. In our discussion, we employ analogous concept instigating L^2- projection operator $\Pi^0_{k,K}$ to compute non-linear term confirming optimal order of convergence.

We recollect basic contribution in VEM. Conforming VEM, was first introduced in [9] for the Poisson problem, and later the authors in [12] extended these ideas to a non-conforming VEM for the linear elliptic equation. For symmetric formulation, the implementation and analysis are based on finite element analysis; however, the treatment of VEM in non-symmetric formulation (generally occurs for convection-diffusion problems) setting is not an easy task and one need appropriate tools for the establishment of error estimates and uniqueness of the discrete solution. In this direction, attempts have been made in order to generalize the VEM analysis when one deals with non-symmetric formulation, for details, kindly see [19, 22, 20]. Moreover, recently, there are a few contributions which deals with VEM for linear hyperbolic problems, see [28]. On the other hand, there is rich literature available in connection with finite element methods for semilinear parabolic problems, see [32, 33, 34] and references therein. However, as per the best of our knowledge, there are hardly any article on VEM for semilinear parabolic problems, and therefore an attempt has been made in this paper to introduce a new VEM for semilinear parabolic problem. The stationary diffusion part of the model problem is discretized with the help of elliptic projection $\Pi^{\nabla}_{k,K}$ and an external L^2 projection operator $\Pi^0_{k,K}$ is used for the terms that involve time derivatives. Convergence analysis are presented for both semi-discrete and fully discrete schemes. We also conduct numerical experiments for the validation of theoretical results when the exact solution of the investigating problem is known. In order to illustrate the performance of the proposed method, for our numerical experiments we also considered a problem which describe motion of boundaries between phases in alloys, model for wave propagation in nerve axons where the underlying medium is continuous and reasonably smooth and for which the exact solution is not known. In summary following contributions have been made in this work.

- Using L^2 projection operator $\Pi^0_{k,K}$, we have derived optimal error estimates in L^2 and H^1 norms. $\Pi^0_{k,K}$ can be approximated using degrees of freedom of local virtual element space.

- Employed Newton method for the solvability of the resulted non-linear system of equation on polygonal mesh.

- In order to avoid the computational difficulties associated with Newton method, based on non-linear formulation, a fully discrete linearized scheme is introduced and order of convergence is derived which matches with nonlinear scheme.

We have arranged the remainder of this chapter in the following manner. Some basic notations and statements of the governing equation with its corresponding weak formulation are presented Section 2.2. This section also deals with the unique solvability of the continuous problem under some reasonable assumptions on the data which would also be required for deriving error estimates for semi discrete and fully discrete schemes. The virtual element discretization of the model problem is discussed in Section 2.3 and optimal error estimates are derived in Section 2.4.1. Section 2.5 contains several numerical experiments in order to examine the accuracy and performance of the proposed scheme. Finally, based on theoretical and computational observations, some conclusions are made in Section 2.6.

2.2 Governing equations and weak formulations

In view of the several real world applications mentioned earlier, we consider the following time-dependent semi-linear parabolic problem: for a given source function f which depends on u, find $u(x,t) : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} u_t - \Delta u = f(u,t) & \text{in} \quad \Omega, \quad \text{for } t \in (0,T), \\ u = 0 & \text{on } \Gamma = \partial\Omega, \quad \text{for } t \in (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where u_t denotes time derivative of u and $u_0 \in L^2(\Omega)$ denotes initial data. For our subsequent analysis, we assume provisionally $f(u,t)$ is Lipschitz continuous with respect to u i.e.

$$|f(w) - f(v)| \leq K_1(1 + |w| + |v|)^\gamma |w - v|, \forall w, v \in \mathbb{R}, \tag{2.2}$$

with $K_1 > 0$ and $\gamma \geq 0$. Multiplying first equation of (2.1) by $v \in H_0^1(\Omega)$ and integrating by parts, the weak formulation of the problem reads as :find $u(\cdot,t) \in L^2(0,T,H_0^1(\Omega))$ such that

$$\begin{cases} m(u_t, v) + a(u(t), v) =< f(u,t), v > \quad \forall\, v \in H_0^1(\Omega) \text{ for a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where the bilinear forms defined by

$$\begin{cases} m(u,v) = \displaystyle\int_\Omega u\, v\, dx \quad \forall u, v \in L^2(\Omega), \\ a(u,v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v \quad \forall\, u, v \in H^1(\Omega), \end{cases}$$

and $< \cdot, \cdot >$ denotes the duality product in $L^2(\Omega)$. In the light of assumption that f satisfies Lipschitz continuous, one can show that (2.3) admits a unique solution, for details we refer to[34].

2.3 Virtual element formulation

Assumption on mesh regularity

Let us first introduce local and global finite dimensional space (named as virtual element space) which will be used for the approximation of model problem (2.1). For this purpose, we discretize the spatial domain as follows. Let $\{\mathcal{T}_h\}$ be a family of decomposition of Ω

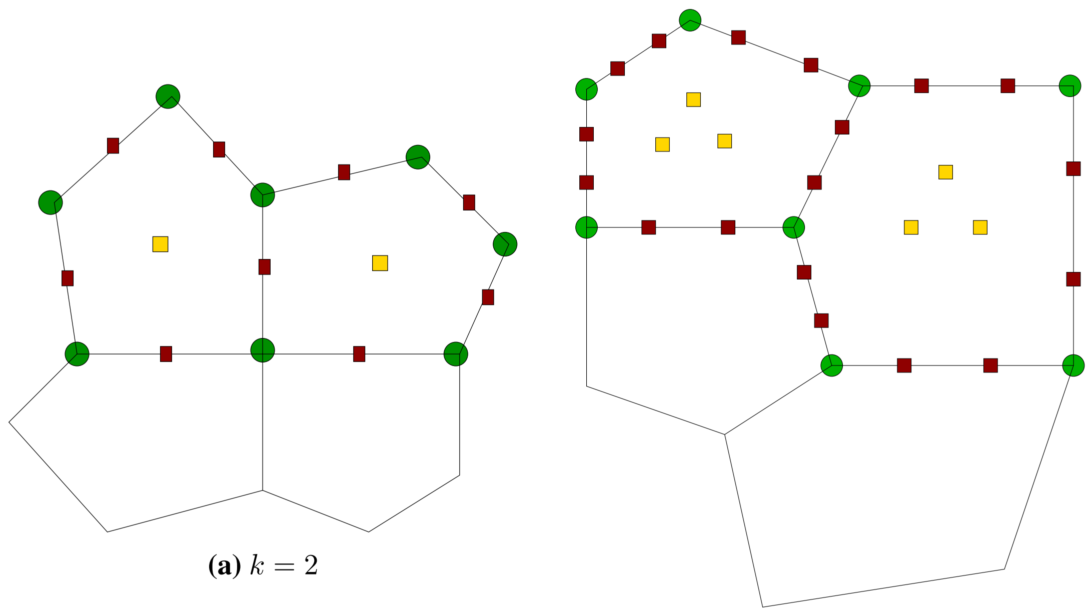
(call polygonal mesh hereafter) formed by polygonal elements (referred as polygon) K with $h = \max_{K \in \mathcal{T}_h}(h_K)$, where h_K is the diameter of K. With respect to this partition, let ε_h denotes the set of edges of $\mathcal{T}_h$, and by ε_h^0 and ε_h^∂, we will refer to the set of interior and boundary edges, respectively. In order to satisfy local interpolation approximation properties and stability of discrete bilinear forms, we make the following reasonable assumptions on the polygonal mesh $\mathcal{T}_h$:

Assumption 1.

$\mathbf{A_1}$: *K is star-shaped with respect to a ball of radius greater than ρh_K,*

$\mathbf{A_2}$: *any two vertexes in K are at least $c h_K$ apart, where h_K is the diameter of K and ρ and c are uniform positive constants[10, 11].*

The above mesh regularity assumption allows VEM to consider more general type of polygonal element which makes VEM more applicable for practical problems. Unlike FEM, VEM does not demand explicit construction of shape functions which reduce computational cost. Moreover, mesh can consist of various type of elements permitting hanging nodes which can not even imagine in FEM. Therefore, one single numerical code can tackle different kind of element. For further demonstration, we represents Figure 2.1.



(a) $k = 2$

(b) $k = 3$

Figure 2.1: Degrees of freedom of polygonal cell for $k = 2, 3$; $(\mathbf{D_1})$ and $(\mathbf{D_2})$ degrees of freedom are indicated by green circles and red squares respectively; cell moments are indicated by yellow squares.

Construction of discrete bilinear form:

In continuation of above discussion, we first recollect the definition of modified virtual element space $\mathcal{Z}^k(K)$ defined in 1.16.

$$\mathcal{Z}^k(K) := \Big\{ v \in \mathcal{Q}^k(K) : \int_K (\Pi^\nabla_{k,K} v) q = \int_K v\, q \; \forall q \in \mathbb{P}_k/\mathbb{P}_{k-2}(K),\ K \in \mathcal{T}_h \Big\}, \quad (2.4)$$

There are several possible choice of DOF for VEM space $\mathcal{Z}^k(K)$. But, we restrict our choice to one defined in previous chapter(Subsection-1.2.1)and the global virtual element space is defined as

$$\mathcal{Z}_h^k := \Big\{ v \in H_0^1(\Omega) : v|_K \in \mathcal{Z}^k(K) \Big\}. \quad (2.5)$$

As mentioned earlier, these spaces (referred as virtual element spaces) contain polynomial as well as non-polynomial functions. We stress that non-polynomial functions are approximated with the help of two projection operators $\Pi^\nabla_{k,K}$ and $\Pi^0_{k,K}$ as discussed in previous chapter. Now, we define the following semi-discrete virtual element formulation corresponding to the continuous problem (2.1): find $u_h \in L^2(0, T, \mathcal{Z}_h^k)$ with $u_{h,t} \in L^2(0, T, \mathcal{Z}_h^k)$ such that

$$\begin{cases} m_h(u_{h,t}(t), v_h) + a_h(u_h(t), v_h) =< f_h(u_h(t), t), v_h > \ \forall\, v_h \in \mathcal{Z}_h^k, \text{ for a.e. } t \in (0, T), \\ u_h(0) = u_{h,0}. \end{cases} \quad (2.6)$$

Here, $u_{h,0}$ and $f_h(u_h, t)$ are certain approximations of the $u_h(0)$ and $f(u_h, t)$ which will be defined at a later stage, and the global discrete bilinear forms $a_h(\cdot,\cdot) : \mathcal{Z}_h^k \times \mathcal{Z}_h^k \to \mathbb{R}$ and $m_h(\cdot,\cdot) : \mathcal{Z}_h^k \times \mathcal{Z}_h^k \to \mathbb{R}$ are defined as

$$\begin{aligned} a_h(v, w) &= \sum_{K \in \mathcal{T}_h} a_h^K(v, w) \quad \forall\, v, w \in \mathcal{Z}_h^k, \\ m_h(v, w) &= \sum_{K \in \mathcal{T}_h} m_h^K(v, w) \quad \forall\, v, w \in \mathcal{Z}_h^k, \end{aligned}$$

where, $a_h^K(\cdot,\cdot) : \mathcal{Z}^k(K) \times \mathcal{Z}^k(K) \to \mathbb{R}$ and $m_h^K(\cdot,\cdot) : \mathcal{Z}^k(K) \times \mathcal{Z}^k(K) \to \mathbb{R}$ are local computable discrete bilinear forms, in view of [11], we defined them as follows:

$$a_h^K(u_h, v_h) := a^K(\Pi^\nabla_{k,K}(u_h), \Pi^\nabla_{k,K}(v_h)) + s_a^K((I - \Pi^\nabla_{k,K})u_h, (I - \Pi^\nabla_{k,K})v_h) \ \forall\, u_h, v_h \in \mathcal{Z}^k(K) \quad (2.7)$$

and

$$m_h^K(u_h, v_h) := m^K(\Pi^0_{k,K}u_h, \Pi^0_{k,K}v_h) + s_m^K((I-\Pi^0_{k,K})u_h, (I-\Pi^0_{k,K})v_h) \;\forall\, u_h, v_h \in \mathcal{Z}^k(K), \tag{2.8}$$

where $m^K(u,v) := \int_K u\, v \;\forall\, u, v \in L^2(K)$. The stabilization terms $s_a^K(\cdot,\cdot)$ and $s_m^K(\cdot,\cdot)$ are symmetric bilinear forms whose matrix representation in the canonical basis functions $\{\phi_i\}$ of $\mathcal{Z}^k(K)$ can be taken as identity matrix and the identity matrix multiplied by h_K^2, respectively. The non-linear forcing term $f_h(u_h, t)$ and the bilinear form $m_h^K(\cdot,\cdot)$ can be constructed using L^2 orthogonal projection operator

$$\Pi^0_{k,K} : L^2(K) \to \mathbb{P}_k(K). \tag{2.9}$$

We note that by the construction of the bilinear forms $a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$, it is clear that these forms possess two following fundamental properties polynomial consistency and stability, in the following sense, for more details we refer to [12, 9] .

Polynomial-consistency: For all $h > 0$ and for all $K \in \mathcal{T}_h$, the bilinear form $a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$ defined in (2.7),(2.8) satisfy the following consistency property (with respect to polynomials $\mathbb{P}_k(K)$)

$$\begin{aligned} a_h^K(p, v) &= a^K(p, v) \quad \forall\, p \in \mathbb{P}^k(K),\; \forall\, v \in \mathcal{Z}^k(K); \\ m_h^K(p, v) &= m^K(p, v) \quad \forall\, p \in \mathbb{P}^k(K),\; \forall\, v \in \mathcal{Z}^k(K). \end{aligned} \tag{2.10}$$

Stability: There exist four positive constants $\alpha_*, \alpha^*, \beta_*$ and β^*, independent of h but depending on the shape regularity of the partition such that for all $v \in \mathcal{Z}^k(K)$

$$\begin{aligned} \alpha_*\, a^K(v, v) &\leq a_h^K(v, v) \leq \alpha^*\, a^K(v, v); \\ \beta_*\, m^K(v, v) &\leq m_h^K(v, v) \leq \beta^*\, m^K(v, v), \end{aligned} \tag{2.11}$$

hold. Condition (2.11) ensures that $s_a^K(v,v)$(resp. $s_m^K(v,v)$) scales like $a^K(v,v)$(resp. $m^K(v,v)$). Next, with the help of L^2- projection operator, we approximate the non-linear force function.

Computation of nonlinear force function: For each element K, we define $f_h(u_h, t)$ as follows:

$$f_h(u_h, t)|_K := \Pi^0_{k,K}(f(\Pi^0_{k,K}u_h, t)) \text{ on each } K \in \mathcal{T}_h, \text{ for a.e. } t \in (0, T).$$

Now, orthogonality property of the operator $\Pi^0_{k,K}$ yields

$$
\begin{aligned}
< f_h(u_h), v_h >= \sum_{K\in\mathcal{T}_h}\int_K f_h(u_h)\, v_h &= \sum_{K\in\mathcal{T}_h}\int_K \Pi^0_{k,K}(f(\Pi^0_{k,K}u_h, t))\, v_h \\
&= \sum_{K\in\mathcal{T}_h}\int_K f(\Pi^0_{k,K}u_h, t)\, \Pi^0_{k,K}v_h \\
&= \sum_{K\in\mathcal{T}_h}\int_K f(\sum_{i=1}^{N^K} dof_i(u_h)\, \Pi^0_{k,K}\phi_i, t)\, \Pi^0_{k,K}v_h,
\end{aligned}
\tag{2.12}
$$

where N^K is total number of degrees of freedom (locally) on $\mathcal{Z}^k(K)$. We notice that the last term is computable using degrees of freedom. Hence, we set for $f(u_h, t) \in L^2(0, T; L^2(\Omega))$,

$$
f_h(u_h, t) = \Pi^0_k(f(\Pi^0_k u_h, t)) \text{ for a.e. } t \in (0, T), \tag{2.13}
$$

where the right hand side of the above equation is understood as:

$$
\Pi^0_k(f(\Pi^0_k u_h, t))|_K = \Pi^0_{k,K}(f(\Pi^0_{k,K}u_h, t))\ \forall K \in \mathcal{T}_h.
$$

2.3.1 Fully discrete scheme

For the approximation of time derivative, we employ backward-Euler scheme. Let $U^n \approx u_h(\cdot, t_n)$, n=0,1,. . ., N and $\tau = T/N$. Therefore, fully discrete form corresponding to (2.6) read as: Find $U^n \in \mathcal{Z}^k_h$ such that

$$
m_h\left(\frac{U^n - U^{n-1}}{\tau}, v_h\right) + a_h(U^n, v_h) =< f_h(U^n, t_n), v_h > \quad \forall v_h \in \mathcal{Z}^k_h. \tag{2.14}
$$

For convenience, we assume that $U^0 = u_{h,0}$.

As we can see that the resulted scheme is a non-linear system at each time step. Therefore, one needs to solve a non-linear system at each time step and this would be a tedious job. The possible remedy is to linearize the right hand side, i.e., replace $f_h(U^n, t_n)$ by $f_h(U^{n-1}, t_{n-1})$ in (2.14). Therefore, we write our linearized scheme as follows: find $U^n \in \mathcal{Z}^k_h$ such that

$$
\begin{cases}
m_h(\dfrac{U^n - U^{n-1}}{\tau}, v_h) + a_h(U^n, v_h) =< f_h(U^{n-1}, t_{n-1}), v_h > \quad \forall v_h \in \mathcal{Z}^k_h, \\
U^0 = u_{h,0}.
\end{cases}
\tag{2.15}
$$

The matrix form of the above equation is same as for fully discrete nonlinear case except the right hand side (has to be written at $n-1$ level). After linearizing the scheme, we feel that there would be reduction in the order of convergence; however this prediction is no longer true and we present detailed analysis for both linear and non-linear scheme in the next section.

2.3.2 Unique solvability of semi-discrete schemes.

In the light of stability, symmetry of the bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$ and assumption that the source term f is Lipschitz continuous, we can easily show that (2.6) has a unique solution (see also [34, 11]) and which is given by

$$u_h(t) := \sum_{n=1}^{N^h} \Big(m_h(u_{h,0}, \omega_h^n)\, e^{-\lambda_h^n t} + \int_0^t < f_h(u_h, s), \omega_h^n > \, e^{-\lambda_h^n (t-s)}\, ds \Big) \omega_h^n,$$

where $\{\omega_h^n\}_{n=1,\ldots,N^h}$ denote orthonormal basis of $\mathcal{Z}_h^k$ w.r.t $m_h(\cdot,\cdot)$ and $\{\lambda_h^n\}_{n=1,\ldots,N^h}$ are strictly positive number satisfying $a_h(\omega_h^n, v_h) = \lambda_h^n\, m_h(\omega_h^n, v_h)\ \forall\ v_h \in \mathcal{Z}_h^k$; N^h denotes total number of degrees of freedom of $\mathcal{Z}_h^k$.

2.4 Convergence analysis

In this section, we derive error estimates in L^2 and H^1- norms for both semi-discrete and fully discrete scheme (when time derivative is approximated by a finite difference scheme). The main tools in the analysis are borrowed from [11]. Moreover, due to the presence of the non-linear term $f(u)$, the error estimates would depend on certain bounds of unknown solution u_h and $f(u_h)$; and these issues have been reported in [35], where FEM is used for the approximation of semilinear problems. Since it would be difficult to obtain bounds for u_h and also $f(u_h)$ (even these may not be bounded), attention has been paid so that the estimates depends on the regularity of exact solution u and $f(u)$ and the analysis carried out accordingly.

2.4.1 Error analysis for the semi-discrete case

We introduce the energy projection operator $P^h : H_0^1(\Omega) \to \mathcal{Z}_h^k$ defined as

$$a_h(P^h u, v_h) = a(u, v_h) \quad \forall v_h \in \mathcal{Z}_h^k,$$

with the following approximation properties:

Lemma 9. *Let $u \in H_0^1(\Omega)$. Then there exists a unique function $P^h u \in \mathcal{Z}_h^k$ verifying*

$$|P^h u - u|_1 \ \leq \ C\, h^k\, |u|_{k+1}.$$

Moreover, if the domain Ω is convex, the following bound holds

$$\|P^h u - u\|_0 \ \leq \ C\, h^{k+1}\, |u|_{k+1}, \tag{2.16}$$

where C is constant independent of h.

Proof. See [11] for the proof. ■

Now, we prove our main theorem of this section which deals with L^2 error estimates.

Theorem 10. *Let u be the solution of problem (2.3) and let u_h be the solution of problem (2.6). Then there exists a positive constant independent of h such that for all $t \in [0,T]$ following holds*

$$\begin{aligned} \|u_h(\cdot,t) - u(\cdot,t)\|_0 \leq C\, \|u_{h,0} - u_0\|_0 + C\, h^{k+1} \Big(&|u_0|_{k+1} + \|u_t\|_{L^2(0,t,H^{k+1}(\Omega))} \\ &+ \|u\|_{L^2(0,t,H^{k+1}(\Omega))} + \|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))} \Big). \end{aligned} \tag{2.17}$$

Proof. Decompose the error as follows:

$$u_h(\cdot,t) - u(\cdot,t) = u_h(\cdot,t) - P^h u(\cdot,t) + P^h u(\cdot,t) - u(\cdot,t) =: \theta(\cdot,t) + \rho(\cdot,t). \tag{2.18}$$

An application of Lemma (9) enable us to write the following estimates

$$\|\rho(\cdot,t)\|_0 \leq C\, h^{k+1}\, (|u_0|_{k+1} + |u_t|_{L^1(0,t,H^{k+1}(\Omega))}). \tag{2.19}$$

In order to bound the term $\theta(\cdot,t)$, we proceed as follows. Using (2.6) and the fact that time derivative commutes with the energy projection, we write the following error equation $\forall v_h \in \mathcal{Z}_h^k$ in terms of $\theta(\cdot,t)$ and $\rho(\cdot,t)$

$$\begin{aligned} m_h(\theta_t(\cdot,t), v_h) + a_h(\theta(\cdot,t), v_h) \ &= \ < f_h(u_h,t), v_h > - m_h\left(\frac{d}{dt} P^h u(\cdot,t), v_h\right) \\ &\qquad - a_h(P^h u(\cdot,t), v_h) \\ &= \ < f_h(u_h,t) - f(u,t), v_h > + m(u_t(\cdot,t), v_h) \\ &\qquad - m_h(P^h u_t(\cdot,t), v_h) \\ &=: \ T_1 + T_2. \end{aligned} \tag{2.20}$$

In order to bound T_1, we write

$$
\begin{aligned}
< f_h(u_h,t) - f(u,t), v_h >_K =& < \Pi^0_{k,K} f(\Pi^0_{k,K} u_h, t), v_h >_K - < \Pi^0_{k,K} f(\Pi^0_{k,K} u, t), v_h >_K \\
&+ < \Pi^0_{k,K} f(\Pi^0_{k,K} u, t), v_h >_K - < \Pi^0_{k,K} f(u), v_h >_K \\
&+ < \Pi^0_{k,K} f(u), v_h >_K - < f(u,t), v_h >_K .
\end{aligned}
\tag{2.21}
$$

where $< \cdot, \cdot >_K$ denotes the duality product in $L^2(K)$. An application of Cauchy-Schwarz inequality together with Lipschitz continuity of f and boundedness property of the L^2 projection operator $\Pi^0_{k,K}$, yields

$$
\begin{aligned}
< \Pi^0_{k,K} f(\Pi^0_{k,K} u_h, t) &- \Pi^0_{k,K} f(\Pi^0_{k,K} u, t), v_h >_K \\
&\leq \|\Pi^0_{k,K} f(\Pi^0_{k,K} u_h, t) - \Pi^0_{k,K} f(\Pi^0_{k,K} u, t)\|_{0,K} \, \|v_h\|_{0,K} \\
&\leq \; C \, \|u - u_h\|_{0,K} \, \|v_h\|_{0,K}.
\end{aligned}
\tag{2.22}
$$

Using Cauchy-Schwarz inequality, boundedness and approximation property of L^2 projection operator $\Pi^0_{k,K}$, we have

$$
\begin{aligned}
< \Pi^0_{k,K} f(\Pi^0_{k,K} u, t), v_h >_K &- < \Pi^0_{k,K} f(u,t), v_h >_K \\
&\leq \|\Pi^0_{k,K} f(\Pi^0_{k,K} u, t) - \Pi^0_{k,K} f(u,t)\|_{0,K} \times \|v_h\|_{0,K} \\
&\leq h_K^{k+1} \, |u|_{k+1,K} \, \|v_h\|_{0,K}.
\end{aligned}
\tag{2.23}
$$

With the help of Cauchy-Schwarz inequality and approximation property of L^2 projection operator $\Pi^0_{k,K}$, we infer that

$$
< \Pi^0_{k,K} f(u,t), v_h >_K - < f(u,t), v_h >_K \quad \leq \quad C \, h_K^{k+1} \, |f(u,t)|_{k+1,K} \, \|v_h\|_{0,K} \tag{2.24}
$$

Using (2.22),(2.23) and (2.24) in (2.21) and summing over all element K, we immediately have the following bounds for T_1

$$
|T_1| \;\; \leq \;\; C \left(\|u - u_h\|_0 \, \|v_h\|_0 + h^{k+1} |u|_{k+1} \, \|v_h\|_0 + h^{k+1} |f(u,t)|_{k+1} \, \|v_h\|_0 \right) \tag{2.25}
$$

Proceeding analogously to the proof of Theorem 3.1 given in [11], and using Cauchy-

Schwarz inequality and the definition of $m_h(\cdot,\cdot)$, we arrive at

$$
\begin{aligned}
T_2 &= \sum_{K\in\mathcal{T}_h} m^K(u_t(\cdot,t),v_h) - m_h^K(P^h u_t(\cdot,t),v_h) \\
&= \underbrace{\sum_{K\in\mathcal{T}_h} m^K(u_t(\cdot,t)-\Pi^0_{k,K}u_t(\cdot,t),v_h)}_{\le Ch^{k+1}|u_t(\cdot,t)|_{k+1}\|v_h\|_0} + \underbrace{\sum_{K\in\mathcal{T}_h} m_h^K(\Pi^0_{k,K}u_t(\cdot,t)-P^h u_t(\cdot,t),v_h)}_{\le Ch^{k+1}|u_t(\cdot,t)|_{k+1}\|v_h\|_0}.
\end{aligned}
\tag{2.26}
$$

Substituting estimates of T_1 and T_2 derived in (2.25) and (2.26), respectively in (2.20) and choosing $v_h=\theta(\cdot,t)$, we obtain

$$
\begin{aligned}
m_h(\theta_t(\cdot,t),\theta(\cdot,t)) + a_h(\theta(\cdot,t),\theta(\cdot,t)) \le C\,\|\theta(\cdot,t)\|_0\Big(&\|u-u_h\|_0 + h^{k+1}\,|u_t(\cdot,t)|_{k+1} \\
&+ h^{k+1}|u|_{k+1} + h^{k+1}|f(u,t)|_{k+1}\Big).
\end{aligned}
$$

Now using $\|u-u_h\|_0 \le \|\rho(\cdot,t)\|_0 + \|\theta(\cdot,t)\|_0$ and stability property (2.11) of the bilinear form $m_h^K(\cdot,\cdot)$ and $a_h^K(\cdot,\cdot)$ in the above equation, we infer that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta(\cdot,t)\|_0^2 + \|\nabla\theta(\cdot,t)\|_0^2 \le\ C\,\|\theta(\cdot,t)\|_0\Big(&\|\rho(\cdot,t)\|_0 + \|\theta(\cdot,t)\|_0 \\
&+ h^{k+1}|u_t(\cdot,t)|_{k+1} + h^{k+1}|u|_{k+1} + h^{k+1}|f(u,t)|_{k+1}\Big).
\end{aligned}
$$

Hence, we can write

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta(\cdot,t)\|_0^2 \le C\|\theta(\cdot,t)\|_0\Big(&\|\rho(\cdot,t)\|_0 + \|\theta(\cdot,t)\|_0 \\
&+ h^{k+1}|u_t(\cdot,t)|_{k+1} + h^{k+1}|u|_{k+1} + h^{k+1}|f(u,t)|_{k+1}\Big).
\end{aligned}
$$

Using (2.19), Young's inequality and an application of Gronwall's lemma yields

$$
\begin{aligned}
\|\theta(\cdot,t)\|_0 \le \|\theta(\cdot,0)\|_0 + Ch^{k+1}\Big(&|u_0|_{k+1} + \|u_t\|_{L^2(0,t,H^{k+1}(\Omega))} + \|u\|_{L^2(0,t,H^{k+1}(\Omega))} \\
&+ \|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))}\Big).
\end{aligned}
$$

Since

$$
\|\theta(\cdot,0)\|_0 \le C(\|u_{h,0}-u_0\|_0 + h^{k+1}|u_0|_{k+1}),
$$

we have

$$\begin{aligned}\|\theta(\cdot,t)\|_0 \le C\Big(&\|u_{h,0}-u_0\|_0\\ &+h^{k+1}\big(|u_0|_{k+1}+\|u_t\|_{L^2(0,t,H^{k+1}(\Omega))}\\ &+\|u\|_{L^2(0,t,H^{k+1}(\Omega))}+\|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))}\big)\Big).\end{aligned} \tag{2.27}$$

Combining the estimates for $\|\rho(\cdot,t)\|_0$ and $\|\theta(\cdot,t)\|_0$, we have the following desired result

$$\begin{aligned}\|u_h(t)-u(t)\|_0 &\le \|\theta(\cdot,t)\|_0+\|\rho(\cdot,t)\|_0\\ &\le C\|u_{h,0}-u_0\|_0+Ch^{k+1}\Big(|u_0|_{k+1}+\|u_t\|_{L^2(0,t,H^{k+1}(\Omega))}\\ &\quad+\|u\|_{L^2(0,t,H^{k+1}(\Omega))}+\|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))}\Big).\end{aligned} \tag{2.28}$$

■

Next, we derive estimates in H^1- norm.

Theorem 11. *Let u be the solution of problem(2.3) and let u_h be the solution of problem (2.6). Then for all $t \ge 0$, there exists a constant independent of h and t but may depends on u such that the following holds*

$$\begin{aligned}|u(\cdot,t)-u_h(\cdot,t)|_1 &\le C|u_{h,0}-u_0|_1+Ch^k\Big(|u_0|_{k+1}+\|u_t\|_{L^1(0,t,H^{k+1}(\Omega))}\Big)\\ &+Ch^{k+1}\Big(\|u_t\|_{L^2(0,t,H^{k+1}(\Omega))}+\|u\|_{L^2(0,t,H^{k+1}(\Omega))}+\|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))}\Big).\end{aligned}$$

Proof. Proof involves the similar steps used in the derivation of Theorem 10. Again, we split the error in terms of $\rho(\cdot,t)$ and $\theta(\cdot,t)$ and write

$$|u(\cdot,t)-u_h(\cdot,t)|_1 \;\;\le\;\; |\theta(\cdot,t)|_1+|\rho(\cdot,t)|_1. \tag{2.29}$$

An application of lemma (9) provides the following bound for $\rho(\cdot,t)$.

$$|\rho(\cdot,t)|_1 \;\;\le\;\; Ch^k\Big(|u_0|_{k+1}+\|u_t\|_{L^1(0,t,H^{k+1}(\Omega))}\Big). \tag{2.30}$$

Now we proceed to estimate $|\theta(\cdot,t)|_1$. Choosing $v_h=\theta_t(\cdot,t)$ in the error equation (2.20) and using the bounds of T_1 and T_2, we have

$$\begin{aligned}m_h(\theta_t(\cdot,t),\theta_t(\cdot,t))&+a_h(\theta(\cdot,t),\theta_t(\cdot,t))\\ &\le Ch^{k+1}|u_t|_{k+1}\|\theta_t(\cdot,t)\|_0+C\|u-u_h\|_0\|\theta_t(\cdot,t)\|_0\\ &+Ch^{k+1}|u|_{k+1}\|\theta_t(\cdot,t)\|_0+Ch^{k+1}|f(u,t)|_{k+1}\|\theta_t(\cdot,t)\|_0.\end{aligned}$$

Stability property (2.11) of the bilinear forms $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$, enable us to write

$$\beta_*\|\theta_t(\cdot,t)\|_0^2 + \frac{1}{2}\alpha_*\frac{d}{dt}|\theta(\cdot,t)|_1^2 \leq C\,\|\theta_t(\cdot,t)\|_0\,\Big(h^{k+1}|u_t|_{k+1} + \|\theta(\cdot,t)\|_0 + \|\rho(\cdot,t)\|_0 + h^{k+1}|u|_{k+1} + h^{k+1}|f(u,t)|_{k+1}\Big).$$

An application of Young's inequality, yields

$$\frac{d}{dt}|\theta(\cdot,t)|_1^2 \leq Ch^{2(k+1)}\Big(|u_t|_{k+1}^2 + |u|_{k+1}^2 + |f(u,t)|_{k+1}^2\Big) + C\|\theta(\cdot,t)\|_0^2 + C\|\rho(\cdot,t)\|_0^2.$$

Using the bound of $\|\theta(\cdot,t)\|_0$ (derived in equation 2.27) and $\|\rho(\cdot,t)\|_0$ (derived in equation 2.19), and integrating from 0 to t, yields the following

$$|\theta(\cdot,t)|_1^2 \leq |\theta(\cdot,0)|_1^2 + C\|u_{h,0}-u_0\|_0^2 + C\,h^{2k+2}\Big(|u_0|_{k+1}^2 + \|u_t\|_{L^2(0,t,H^{k+1}(\Omega))}^2 + \|u\|_{L^2(0,t;H^{k+1}(\Omega))}^2 + \|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))}^2\Big).$$

Again, we know $|\theta(\cdot,0)|_1 \leq |u_{h,0}-u_0|_1 + Ch^k|u_0|_{k+1}$. Now, an application of the estimate $|\theta(\cdot,0)|_1$ and Cauchy-Schwarz inequality, yield

$$|\theta(\cdot,t)|_1 \leq C|u_{h,0}-u_0|_1 + Ch^k|u_0|_{k+1} + Ch^{k+1}\Big(\|u_t\|_{L^2(0,t,H^{k+1}(\Omega))} + \|u\|_{L^2(0,t,H^{k+1}(\Omega))} + \|f(u,t)\|_{L^2(0,t,H^{k+1}(\Omega))}\Big).$$

Now, using estimates of $|\rho(\cdot,t)|_1$, the rest of the proof can be completed in a standard way (usually done in case of standard finite element methods). ■

2.4.2 Error analysis for the discrete case:

Theorem 12. *Let u be the the solution of the problem (2.3) and let $\{U^n\}_{n\in\mathbb{N}}$ be the sequence of approximate value of $u_h(\cdot,t_n)$. Then the following error estimation holds for $n = 1,\ldots,N$*

$$\|U^n - u(\cdot,t_n)\|_0 \leq C\|u_{h,0}-u_0\|_0 + C\Big(\tau\|u_{tt}\|_{L^1(0,t_n,L^2(\Omega))} + h^{k+1}\big(|u_0|_{k+1} + \|u_t\|_{L^1(0,t_n,H^{k+1}(\Omega))} + \max_{1\leq j\leq n}|u(\cdot,t_j)|_{k+1} + \max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\big)\Big).$$

Proof. Splitting the error as:

$$U^n - u(\cdot, t_n) = U^n - P^h u(\cdot, t_n) + P^h u(\cdot, t_n) - u(\cdot, t_n) =: \theta^n + \rho^n.$$

The following estimate for ρ follows from lemma(9)

$$\|\rho^n\|_0 \leq Ch^{k+1}\Big(|u_0|_{k+1} + \|u_t\|_{L^1(0,t_n,H^{k+1}(\Omega))}\Big).$$

In order to reduce clumsy notations, we define

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\tau}, \tag{2.31}$$

and

$$\bar{\partial}P^h u(\cdot, t_n) := \frac{P^h u(\cdot,t_n) - P^h u(\cdot,t_{n-1})}{\tau},$$

where τ is time step. Now, we write the error equation (2.20) at $t = t_n$

$$\begin{aligned} m_h(\bar{\partial}\theta^n, v_h) + a_h(\theta^n, v_h) =&< f_h(\cdot, t_n) - f(\cdot, t_n), v_h > \\ &+ m(u_t(\cdot, t_n), v_h) - m_h(\bar{\partial}P^h u(\cdot, t_n), v_h) =: T_1^n + T_2^n. \end{aligned} \tag{2.32}$$

Now, T_1^n can be bounded in the similar fashion as T_1

$$T_1^n \leq C\Big(\|U^n - u(\cdot, t_n)\|_0 + h^{k+1}|f(u(t_n), t_n)|_{k+1} + h^{k+1}|u(\cdot, t_n)|_{k+1}\Big)\|v_h\|_0.$$

Now, the following bound of T_2^n can be easily obtained by using consistency, stability and definition of the form $m_h(\cdot,\cdot)$, and simple manipulation of terms

$$\begin{aligned} T_2^n &= m(u_t(\cdot, t_n), v_h) - m_h(\bar{\partial}P^h u(\cdot, t_n), v_h) \\ &= \sum_{K\in\mathcal{T}_h} \Big(m^K(u_t(\cdot, t_n), v_h) - m_h^K(\bar{\partial}P^h u(\cdot, t_n), v_h)\Big) \\ &\leq \frac{C}{\tau}\Big(\|\tau\, u_t(\cdot, t_n) - (u(\cdot, t_n) - u(\cdot, t_{n-1}))\|_0 + h^{k+1}|u(\cdot, t_n) - u(\cdot, t_{n-1})|_{k+1}\Big)\|v_h\|_0 \\ &=: \frac{C}{\tau}(\xi^n + \eta^n)\|v_h\|_0, \end{aligned}$$

Putting $v_h = \theta^n$, using stability and boundedness property of the bilinear form $m_h(\cdot,\cdot)$

and substituting the bounds for T_1^n and T_2^n in the above equation(2.32), we arrive at

$$\|\theta^n\|_0 \leq C\|\theta^{n-1}\|_0 + C\,\tau\|U^n - u(\cdot,t_n)\|_0 + C\,\tau\,h^{k+1}|u(\cdot,t_n)|_{k+1} + C\,\tau\,h^{k+1}|f(u(t_n),t_n)|_{k+1} + C(\xi^n + \eta^n).$$

For small step size τ, it readily follows that

$$\|\theta^n\|_0 \leq C(1 + C\,\tau)\|\theta^{n-1}\|_0 + C\,\tau\|\rho^n\|_0 + C\,\tau\,h^{k+1}|u(\cdot,t_n)|_{k+1} + C\,\tau\,h^{k+1}|f(u(t_n),t_n)|_{k+1} + C(\xi^n + \eta^n).$$

Iterating over n, finally we can conclude that

$$\|\theta^n\|_0 \leq C(1 + C\,\tau)^n\|\theta^0\|_0 + C\,\tau\Big(\sum_{j=1}^{n}(1 + C\tau)^{n-j}\big(\|\rho^j\|_0 + h^{k+1}|u(\cdot,t_j)|_{k+1} + h^{k+1}|f(u(t_j),t_j)|_{k+1}\big)\Big) + C\sum_{j=1}^{n}(1 + C\tau)^{n-j}(\xi^j + \eta^j).$$

Therefore,

$$\|\theta^n\|_0 \leq C\|\theta^0\|_0 + C\,\tau\sum_{j=1}^{n}\Big(\|\rho^j\|_0 + h^{k+1}|u(\cdot,t_j)|_{k+1} + h^{k+1}|f(u(t_j),t_j)|_{k+1}\Big) + C\sum_{j=1}^{n}(\xi^j + \eta^j). \tag{2.33}$$

The following estimates for ξ^j and η^j have been established in [11]

$$\sum_{j=1}^{n}\xi^j \leq \tau\,\|u_{tt}\|_{L^1(0,t_n,L^2(\Omega))}, \qquad \sum_{j=1}^{n}\eta^j \leq h^{k+1}\,\|u_t\|_{L^1(0,t_n,H^{k+1}(\Omega))}. \tag{2.34}$$

Also, in view of lemma(9), we have

$$C\,\tau\sum_{j=1}^{n}\|\rho^j\|_0 \leq C\,h^{k+1}\Big(|u_0|_{k+1} + |u_t|_{L^1(0,t_n,H^{k+1}(\Omega))}\Big). \tag{2.35}$$

Non-linear source term can be easily bounded in the following manner

$$\begin{aligned} C\,\tau\sum_{j=1}^{n}h^{k+1}|f(u(t_j),t_j)|_{k+1} &\leq C\,h^{k+1}\max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\,n\,\tau \\ &\leq C\,h^{k+1}\max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}. \end{aligned} \tag{2.36}$$

Similarly, we can bound

$$C\,\tau\sum_{j=1}^{n} h^{k+1}|u(t_j)|_{k+1} \leq C\,h^{k+1}\max_{1\leq j\leq n}|u(t_j)|_{k+1}. \tag{2.37}$$

Collecting all estimates of (2.34), (2.35), (2.36) &(2.37) in (2.33), and using $U^n - u(\cdot,t_n) = \theta^n + \rho^n$, we obtain

$$\begin{aligned}\|U^n - u(\cdot,t_n)\|_0 \leq C\|u_{h,0} - u_0\|_0 &+ C\,\tau\|u_{tt}\|_{L^1(0,t_n,L^2(\Omega))} \\ &+ C\,h^{k+1}\Big(|u_0|_{k+1} + \|u_t\|_{L^1(0,t_n,H^{k+1}(\Omega))} \\ &+ \max_{1\leq j\leq n}|u(\cdot,t_j)|_{k+1} + \max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\Big).\end{aligned}$$

This completes the proof. ■

2.4.3 Error estimates for linearized scheme

Let us recall that for reducing the non-linear scheme into a linear scheme, we simply replace $f(U^n, t_n)$ by $f(U^{n-1}, t_{n-1})$, and hence our linearized scheme read as follows: Find $U^n \in \mathcal{Z}_h^k$ such that

$$\begin{cases} m_h\left(\dfrac{U^n - U^{n-1}}{\tau}, v_h\right) + a_h(U^n, v_h) =< f_h(U^{n-1}, t_{n-1}), v_h > \quad \forall v_h \in \mathcal{Z}_h^k, \\ U^0 = u_{h,0}. \end{cases} \tag{2.38}$$

As we have promised that there is no reduction in the over all rate of convergence, in this section we shall show that the result of Theorem (12) remains valid for this linearized form of the Backward-Euler VEM.

Theorem 13. *Let u be the solution of the problem(2.3) and let $\{U^n\}_{n\in\mathbb{N}}$ be the sequence of approximate values of $u_h(\cdot,t_n)$. Then the following error estimation holds for $n = 1, 2, \ldots, N$*

$$\begin{aligned}\|U^n - u(\cdot,t_n)\|_0 \leq C\,\|u_{h,0} - u_0\|_0 + C\Big(&\tau\,\|u_{tt}\|_{L^1(0,t_n,L^2(\Omega))} + \tau\,\|u_t\|_{L^\infty(0,t_n,L^2(\Omega))} \\ &+ h^{k+1}\,\big(|u_0|_{k+1} + \|u_t\|_{L^1(0,t_n,H^{k+1}(\Omega))} + \mathop{max}_{1\leq j\leq n}|u(\cdot,t_j)|_{k+1} \\ &+ \mathop{max}_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\big)\Big).\end{aligned}$$

Proof. The proof of this theorem is similar to the previous theorem; however, for the sake of completeness, we provide the outline of the proof. In analogy with the previous theorem, we write the error equation with the help of (2.15) and $U^n - u(\cdot, t_n) = \theta^n + \rho^n$.

$$\begin{aligned} m_h(\bar{\partial}\theta^n, v_h) + a_h(\theta^n, v_h) &=< f_h(U^{n-1}), v_h > - m_h(\bar{\partial} P^h u(\cdot, t_n), v_h) - a(u(\cdot, t_n), v_h) \\ &=< f_h(U^{n-1}, t_{n-1}) - f(u(\cdot, t_n), t_n), v_h > \\ &\quad + \Big(m(u_t(\cdot, t_n), v_h) - m_h(\bar{\partial} P^h u(\cdot, t_n), v_h) \Big) \\ &=: I^n + II^n. \end{aligned}$$

The bound for I^n is followed by using Lipschitz continuity of f with respect to u and standard approximation property of L^2 projection operator

$$\begin{aligned} I^n = < f_h(U^{n-1}, t_{n-1}) - f(u(\cdot, t_n), t_n), v_h > \; &\le C\Big(\|U^{n-1} - u(\cdot, t_n)\|_0 + h^{k+1}|u(\cdot, t_n)|_{k+1} \\ &\quad + h^{k+1}|f(u(t_n), t_n)|_{k+1} \Big) \|v_h\|_0. \end{aligned}$$

A simple manipulation yields

$$\begin{aligned} &\|U^{n-1} - u(\cdot, t_n)\|_0 \\ &= \|U^{n-1} - P^h u(\cdot, t_{n-1}) + P^h u(\cdot, t_{n-1}) - u(\cdot, t_{n-1}) + u(\cdot, t_{n-1}) - u(\cdot, t_n)\|_0 \\ &\le \|\theta^{n-1}\|_0 + \|\rho^{n-1}\|_0 + \|u(\cdot, t_{n-1}) - u(\cdot, t_n)\|_0. \end{aligned}$$

Using the similar arguments used in the bound of T_2^n (in the previous theorem), we have

$$II^n = \Big(m(u_t(\cdot, t_n), v_h) - m_h(\bar{\partial} P^h u(\cdot, t_n), v_h) \Big) \le \frac{C}{\tau}(\xi^n + \eta^n)\|v_h\|_0.$$

Therefore,

$$\begin{aligned} m_h(\frac{\theta^n - \theta^{n-1}}{\tau}, v_h) + a_h(\theta^n, v_h) \le &\Big(\|\theta^{n-1}\|_0 + \|\rho^{n-1}\|_0 + \tau\|\bar{\partial} u(\cdot, t_n)\|_0 \\ &+ Ch^{k+1}|u(\cdot, t_n)|_{k+1} + Ch^{k+1}|f(u(t_n), t_n)|_{k+1} \Big) \|v_h\|_0 \\ &+ \frac{C}{\tau}(\xi^n + \eta^n)\|v_h\|_0, \end{aligned}$$

where $\bar{\partial} u(\cdot, t_n) := \frac{u(\cdot, t_n) - u(\cdot, t_{n-1})}{\tau}$. Putting $v_h = \theta^n$, using stability property of $m_h(\cdot, \cdot)$ and continuity property of bilinear form $m_h(\cdot, \cdot)$, and proceeding as in the proof of previous theorem, we easily achieve following bounds for small values of τ

$$\begin{aligned} \|\theta^n\|_0 \le C\|\theta^0\|_0 &+ C\,\tau\, \|u_{tt}\|_{L^1(0, t_n, L^2(\Omega))} + C\,\tau\, \|u_t\|_{L^\infty(0, t_n, L^2(\Omega))} \\ &+ C\, h^{k+1}\Big(|u_0|_{k+1} + \|u_t\|_{L^1(0, t_n, H^{k+1}(\Omega))} + \max_{1 \le j \le n} |u(\cdot, t_j)|_{k+1} \\ &+ \max_{1 \le j \le n} |f(u(t_j), t_j)|_{k+1} \Big). \end{aligned}$$

Since

$$\|\theta^0\|_0 \;\; \leq \;\; \|u_{h,0} - u_0\|_0 + Ch^{k+1}|u_0|_{k+1},$$

we can write

$$\begin{aligned}\|\theta^n\|_0 \leq C\|u_{h,0} - u_0\|_0 &+ C\,\tau\,\|u_{tt}\|_{L^1(0,t_n,L^2(\Omega))} + C\,\tau\,\|u_t\|_{L^\infty(0,t_n,L^2(\Omega))} \\ &+ C\,h^{k+1}\Big(|u_0|_{k+1} + \|u_t\|_{L^1(0,t_n,H^{k+1}(\Omega))} + \max_{1\leq j\leq n}|u(\cdot,t_j)|_{k+1} \\ &+ \max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\Big).\end{aligned}$$

Combining the estimates of $\|\rho^n\|_0$ and $\|\theta^n\|_0$, we obtain the desired result and hence this completes the rest of the proof. ■

2.5 Numerical Experiments

In this section, we consider two numerical examples: first one for which exact solution is known and help us in confirming the theoretical rate of convergence presented in Section 2.4, and the second one is bistable equation for which exact solution is not known and generally used in describing many real phenomena. For spatial discretization we have taken virtual element method of order $k = 1, 2$ for the first example and $k = 1$ for the second example with implicit Euler method for time discretization. After applying the proposed fully discretized scheme presented in Section 2.3, we obtain system of linear (if linearized approach is used) and nonlinear algebraic equation corresponding to investigating semi-linear parabolic equation. We emphasis that for solving resulted nonlinear system, we have employed Newton iterative procedure; however for linear system (the nonlinear term is evaluated at previous time level) a conjugate gradient method is used. Before proceeding to the numerical test, we would like to discuss the implementation details of the proposed method.

2.5.1 Implementation Aspects

Let N^h denotes total number of degrees of freedom of $\mathcal{Z}_h^k$ and $\alpha_i = dof_i(u_h)$, then u_h can be expressed as:

$$u_h = \sum_{j=1}^{N^h} \alpha_j(t)\;\Phi_j,$$

where Φ_i are global basis functions. Therefore, with the help of (2.12) (also see (2.13)) our fully discrete scheme (2.14) leads to the following nonlinear algebraic equations:

$$\begin{aligned} m_h\left(\sum_{j=1}^{N^h} \alpha_j^n \Phi_j, \Phi_i\right) &- m_h\left(\sum_{j=1}^{N^h} \alpha_j^{n-1} \Phi_j, \Phi_i\right) + \tau\, a_h\left(\sum_{j=1}^{N^h} \alpha_j^n \Phi_j, \Phi_i\right) \\ &= \tau \int_\Omega f\left(\sum_{j=1}^{N^h} \alpha_j^n\, \Pi_k^0 \Phi_j\right) \Pi_k^0 \Phi_i\, dx, \quad i = 1, 2, \cdots N^h. \end{aligned}$$

The above system can be expressed in the following matrix form:

$$(B + \tau A)\boldsymbol{\alpha}^n = B\boldsymbol{\alpha}^{n-1} + \tau \tilde{f}^n, \tag{2.39}$$

where B and A are matrices corresponding to the bilinear form $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ respectively; $\tilde{f}^n$ is the column vector $(\tilde{f}^n)_i =< f(\sum_{j=1}^{N^h} \alpha_j^n\, \Pi_k^0\, \Phi_j), \Pi_k^0\, \Phi_i >$. Since the bilinear form $a_h(\cdot,\cdot)$ is positive semi-definite and $m_h(\cdot,\cdot)$ is positive definite and hence the matrix $(B + \tau A)$ is nonsingular, this would be an evidence for the unique solvability of the fully discrete linear and nonlinear schemes. Now, the nonlinear system (2.39) can be solved directly by Newton iterative scheme.

From the above nonlinear system, we infer that the residual vector is of the form:

$$Res(\boldsymbol{\alpha}^n) := (B + \tau A)\, \boldsymbol{\alpha}^n - B\, \boldsymbol{\alpha}^{n-1} - \tau \tilde{f}^n.$$

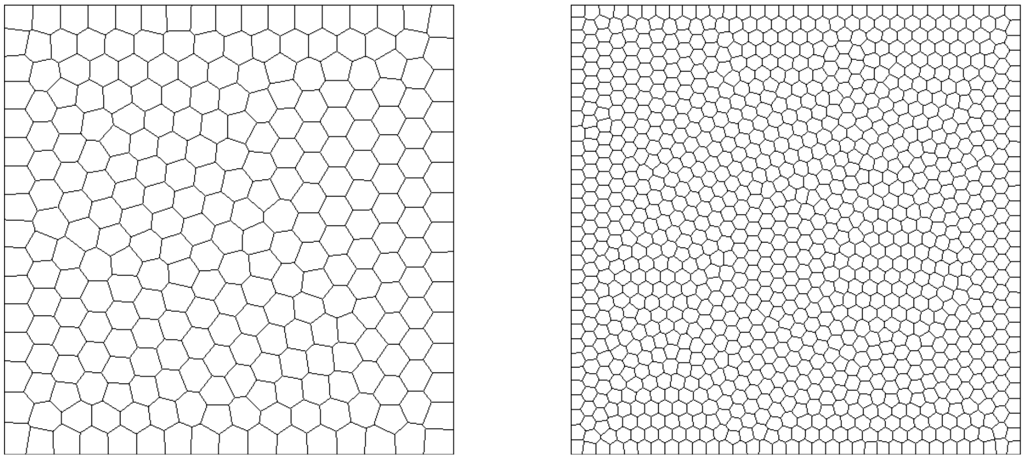
Now, the Jacobian (contains partial derivatives of the residual vectors) used in the Newton's iterations can be expressed as:

$$J_{ij} := \frac{\partial Res(\boldsymbol{\alpha}^n)_i}{\partial(\boldsymbol{\alpha}^n)_j};\ 1 \le i, j \le N^h.$$

We stress that it would be easy to differentiate the first term $(B + \tau A)\boldsymbol{\alpha}^n$, however, differentiation of the non-linear term $\tilde{f}^n$ may not be straight forward and it will be of the following form

$$\frac{\partial(\tilde{f}^n)_i}{\partial(\boldsymbol{\alpha}^n)_j} = \sum_{K \in \mathcal{T}_h} \int_K f'\left(\sum_{j=1}^{N^K} \alpha_j^n\, \Pi_{k,K}^0\, \phi_j\right) \Pi_{k,K}^0 \phi_j\, \Pi_{k,K}^0 \phi_i\, dx.$$

From the above expression, we notice that the nonlinearity of $f(u)$ makes integrand very complicated and also we need to update the Jacobian at each iteration of Newton method which would be not an easy task and in this setting the Newton method becomes very expensive.



(a) $h = 1/10$ **(b)** $h = 1/20$

Figure 2.2: Voronoi meshes

Let u be the exact solution of the semilinear problem and u_h be the numerical solution obtained by VEM method. We have evaluated the error using local projectors $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ in the following way

- L^2-norm error : $e_{h,0} = \sqrt{\sum_{K\in\mathcal{T}_h} \|u - \Pi^0_{k,K} u_h\|^2_{L^2(K)}}$
- H^1-norm error : $e_{h,1} = \sqrt{\sum_{K\in\mathcal{T}_h} |u - \Pi^\nabla_{k,K} u_h|^2_{H^1(K)}}$

2.5.2 Test1

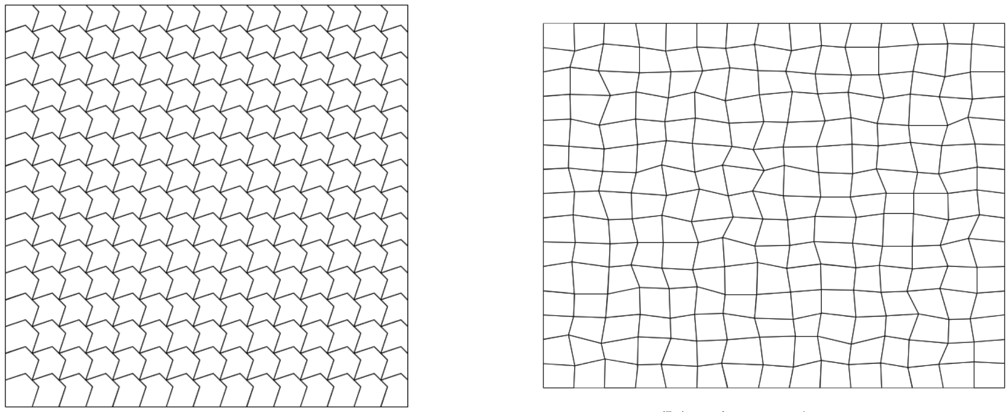
Consider the following problem

$$\frac{\partial u}{\partial t} - \Delta u = (u - u^2) + g(x, y, t) \text{ on } \Omega \times J,$$

where $\Omega = [0, 1]\times[0, 1]$ and $J = (0, 1]$. We choose g so that $u(x, y, t) = \sin(2\pi t)\ \sin(\pi x)$ $\sin(\pi y)$ will become the exact solution to the problem. We have discretized the domain Ω into Polygonal mesh which is generated by Polymesher (cf. [36]), see Figure 2.2. The matrices associated with bilinear forms are constructed by following [13].

Errors $e_{h,0}$ and $e_{h,1}$ are evaluated for different mesh size h and a suitable time step τ. To obtain the optimal rate of convergence we have taken the time step $\tau = \mathcal{O}(h^{k+1})$ for $k = 1, 2$. We denote $r_{h,0}$ and $r_{h,1}$ to be the rate of convergence in L^2-norm and H^1-seminorm respectively.

From Table 2.1 and 2.2, we observe the expected rate of convergence in h which



(a) Non-convex

(b) Distorted-square

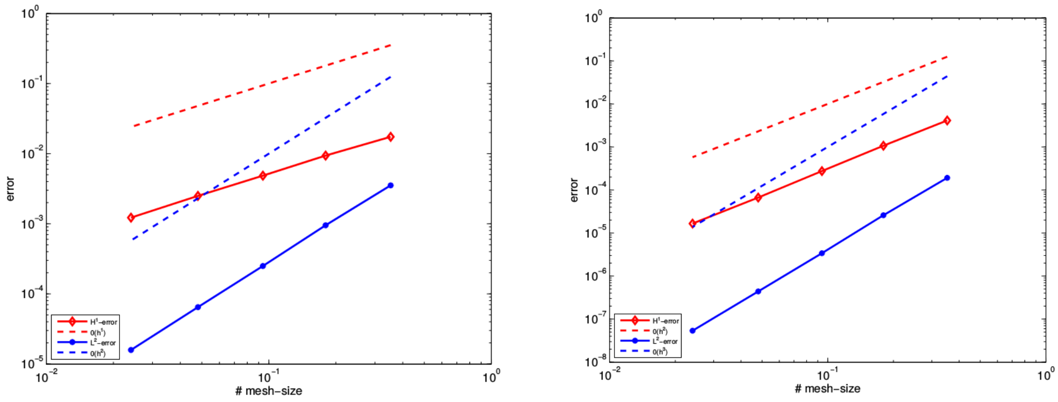
Figure 2.3: Polygonal meshes

Table 2.1
Error table for non-linear scheme for structured-voronoi meshes.

	h	$e_{h,0}$	$r_{h,0}$	$e_{h,1}$	$r_{h,1}$	**COND**
$k=1$	1/5	2.5421e-3	-	1.1388e-2	-	15.9914
	1/10	5.9296e-4	2.10	5.5768e-3	1.03	11.9067
	1/20	1.4319e-4	2.05	2.8667e-3	0.96	11.8744
	1/40	3.6046e-5	1.99	1.4234e-3	1.01	10.5317
	1/80	8.9492e-6	2.01	7.0190e-4	1.02	10.4410
$k=2$	1/5	8.9482e-4	-	3.9939e-3	-	210.5571
	1/10	1.16602e-4	2.94	9.7792e-4	2.03	196.8512
	1/20	1.4778e-5	2.98	2.4279e-4	2.01	174.0129
	1/40	1.8218e-6	3.02	6.1972e-5	1.97	169.2506
	1/80	2.2615e-7	3.01	1.5174e-5	2.03	151.2130

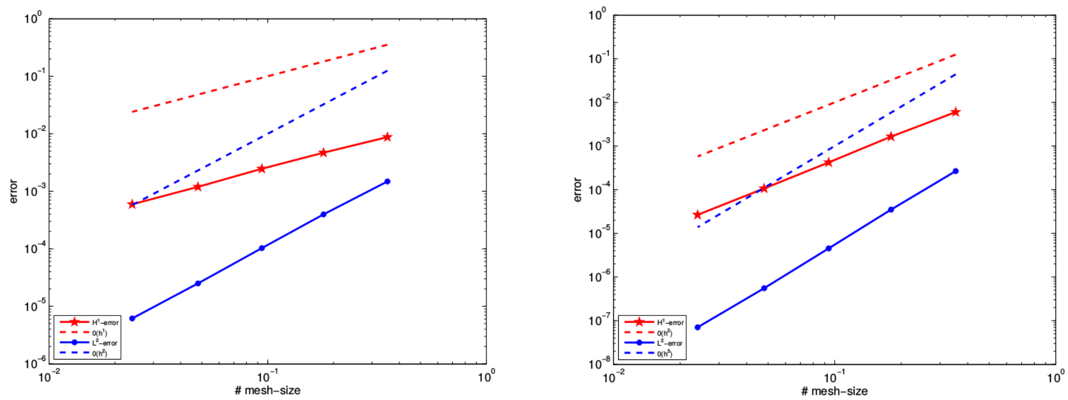
matches with our theoretical findings established in Section 2.4. Furthermore, we examine TEST 1 over distorted square meshes and non-convex meshes. we have constructed distorted square and non-convex meshes over unit square exploiting idea given in [20], see Figure 2.3. Mesh data are exhibited in TABLE 2.3. Convergence rates are displayed in Figure 2.4 and Figure 2.5 respectively.

Also our numerical experiments indicate that both non-linear scheme and linearized scheme have same rate of convergence as predicted by convergence analysis. We would like to mention that one needs to choose a appropriate initial guess in order to ensure the convergence of Newton iterative methods used for solvability of the non-linear system, and since in this example exact solution is known, we have taken our initial guess as the exact solution at $t = 0$. However, the implementation of linearized scheme is much simple, and we also not much bother about the initial guess for the convergence of the conjugate gradient methods employed for the solvability of linear system. This would



(a) $k = 1$ **(b)** $k = 2$

Figure 2.4: Convergence in space variable for distorted square meshes.



(a) $k = 1$ **(b)** $k = 2$

Figure 2.5: Convergence in space variable for non-convex meshes.

Table 2.2
Error table for linearized scheme for structured-voronoi meshes.

	h	$e_{h,0}$	$r_{h,0}$	$e_{h,1}$	$r_{h,1}$	**COND**
$k=1$	1/5	2.5986e-3	-	1.1395e-2	-	15.9914
	1/10	5.9367e-4	2.13	5.5802e-3	1.03	11.9067
	1/20	1.4041e-4	2.08	2.8685e-3	0.96	11.8744
	1/40	3.4860e-5	2.01	1.4243e-3	1.01	10.5317
	1/80	8.6548e-6	2.01	7.0723e-4	1.01	10.4410
$k=2$	1/5	8.9579e-4	-	3.9951e-3	-	210.5571
	1/10	1.1918e-4	2.91	1.0339e-3	1.95	196.8512
	1/20	1.5422e-5	2.95	2.5491e-4	2.02	174.0129
	1/40	1.9144e-6	3.01	6.5066e-5	1.97	169.2506
	1/80	2.3764e-7	3.01	1.6154e-5	2.01	151.2130

Table 2.3
Mesh data

Distorted-square			Non-convex		
Elements	h	Vertices	Elements	h	Vertices
16	.4420	25	9	.5292	28
64	.2316	81	36	.2946	109
256	.1176	289	144	.1473	433
1024	.0599	1089	576	.0736	1729
4096	.0301	4225	2304	.0368	6913

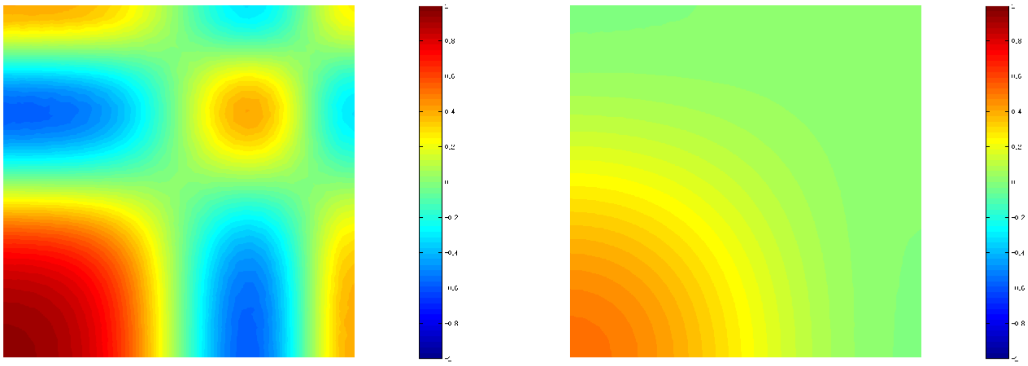
be an advantage if the exact solution is not known, and this we will consider in our next numerical example.

2.5.3 Test2

In this test, we consider the Allen-Cahn (or) Bistable equation-generally a reaction diffusion PDEs for which we do not have the exact solution. This model problem describes physical phenomenon of motions of boundaries between phases in alloys.

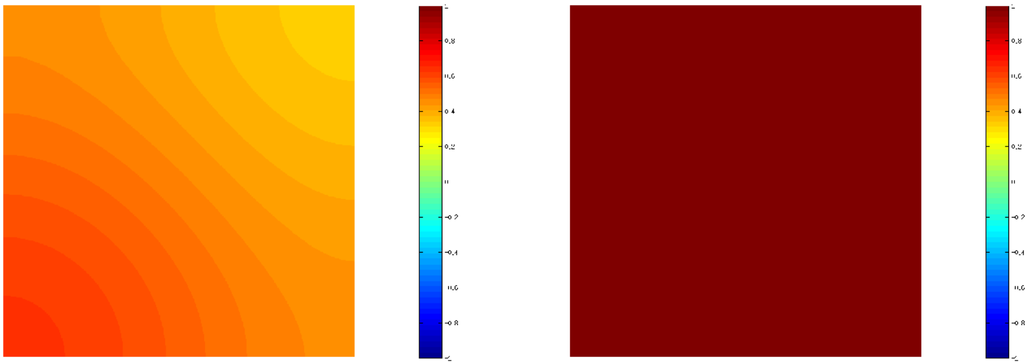
$$\begin{aligned} \partial_t u - \epsilon \Delta u &= u - u^3 \text{ in } \Omega \times [0,+\infty] \\ n \cdot \nabla u &= 0 \text{ on } \partial\Omega \times [0,+\infty] \end{aligned}$$

We choose the initial condition $u(x,y,0) = \cos(\pi x^2)\cos(\pi y^2)$. We consider the domain $\Omega = [0,1] \times [0,1]$ for VEM of order $k = 1$ with mesh size $h = 1/20$(voronoi mesh) and $\tau = 1/400$. For the discretization of this problem, we have used fully discrete linearized scheme demonstrated in Section 2.3 and the resulted linear system is solved by conjugate gradient method. As it is well known that for small values of ϵ, we have only two stable



(a) $t = 0.1$ **(b)** $t = 0.5$

Figure 2.6: Snapshots of the computed solution for $\epsilon = 0.1$



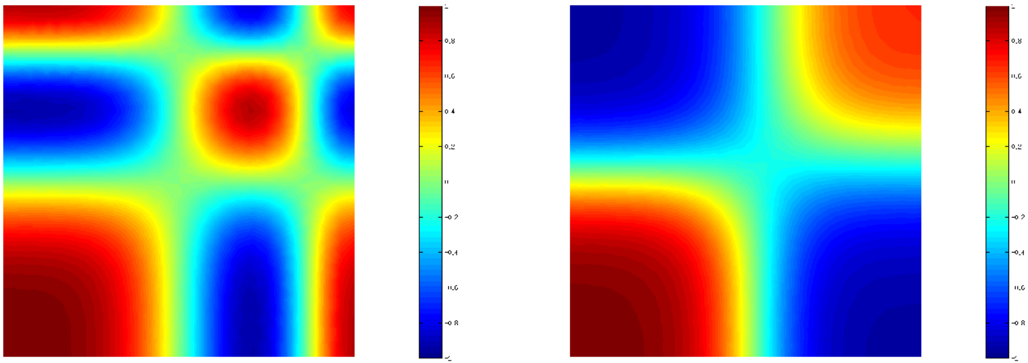
(a) $t = 5$ **(b)** $t = 10$

Figure 2.7: Snapshots of the computed solution for $\epsilon = 0.1$

equilibrium solutions $u = 1$ and $u = -1$. This behavior also depicted in the snapshots given in Figures 2.6- 2.9 for different time levels and ϵ.

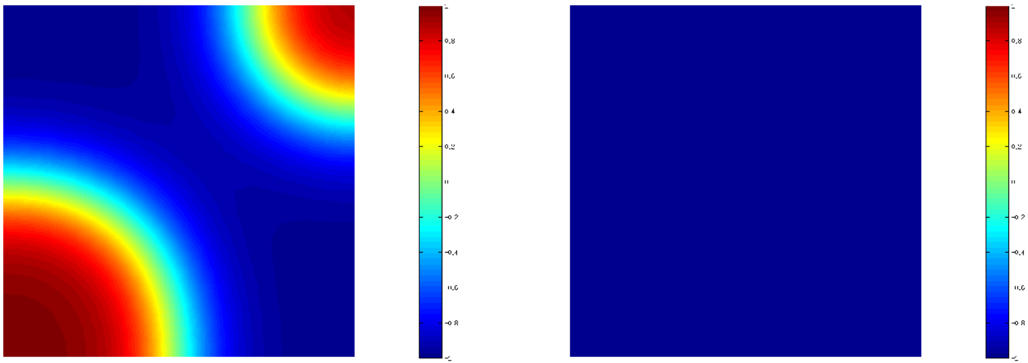
2.6 Discussion

In this chapter, virtual element methods are introduced for the numerical approximation of semi-linear parabolic problems. For the computation of nonlinear term which appears in the right hand side of the equation, a L^2-projection operator is used, and $\Pi^{\nabla}_{k,K}$ is used for the computation of local bilinear forms involved in discrete formulation. Optimal *a priori* error estimates are derived for semi discrete and fully discrete schemes. In view of [34] a linearized scheme is introduced and error estimates have been established, and a comparison study made for the computed solution associated with linearized and non linearized scheme. Several numerical experiments have been conducted in order to judge the efficiency and robustness of the proposed schemes. Future study includes the developments of nonconforming VEM for nonlinear elasticity problem and stabilized



(a) $t = 0.1$ **(b)** $t = 5$

Figure 2.8: Snapshots of the computed solution for $\epsilon = 0.01$



(a) $t = 20$ **(b)** $t = 25$

Figure 2.9: Snapshots of the computed solution for $\epsilon = 0.01$

VEM for unsteady fluid flow problem. In next chapter, we recast this idea for semi-linear hyperbolic problem in the context of virtual element methods.

Chapter 3

Virtual element methods for semi-linear hyperbolic equation

3.1 Introduction

This chapter is related with development of virtual element method to the approximation of semi-linear hyperbolic equations. We have exploited L^2 projection operator $\Pi^0_{k,K}$ in order to approximate non-linear problems which may possess a unique or infinitely many solution; however, one requires a special numerical treatment while looking for numerical solution of these problems, as the discrete formulation is nonlinear. In this direction, several finite element methods (FEM) have been proposed for solving nonlinear problems, see [35, 37]. On the other hand, employing FEM on polygonal mesh for solving nonlinear problems would invite two main difficulties: construction of basis functions is cumbersome and evaluation of its integrals which uses Gauss-Lobatto quadrature is also expensive. In view of these issues associated with numerical approximation of nonlinear problems on polygonal meshes, the basic purpose of this contribution is to propose a robust and efficient scheme for the solvability of nonlinear hyperbolic equations which would provide more accurate solution and also easy to implement with less computational cost in comparison with other numerical schemes. This newly introduced method is characterized by the capability of dealing with polygonal meshes and to possibility of easily implementing on polygonal mesh by avoiding the explicit construction of the local basis functions.

In literature, VEM has been studied for the approximation of linear problems; however, there are only very few contributions dealing with nonlinear equations, for instance, Antoneitti *et al.* in [30] have analyzed fourth order non-linear Cahn-Hilliard equation by using C^1- VEM with certain assumption on the nonlinear term. We would like to mention that VEM have been discussed for linear hyperbolic problem in [28] and convergence analysis has been carried out. In this work, an attempt has been made to extend this analysis to semilinear hyperbolic problems on polygonal domain. We recall that in VEM, we do have local polynomial basis functions for the finite dimensional space, and hence computation of nonlinear term would be difficult or in other words the integral term involving nonlinear functions may not be directly computable with the help of degrees of freedom. Therefore, one needs to devise a scheme such that the term corresponding to nonlinear force function is exactly computable. For the accomplishment of this, we have modified the approximation of right hand term (containing nonlinear term) with the help of orthogonal L^2 projection operator, and we have shown that with this modifications, this term is computable and still optimal order of convergence can be achieved. Moreover, the error estimates of the proposed scheme will depends on the regularity of the exact solution u and source term $f(u)$. In general, while dealing convergence analysis of numerical schemes applied to semilinear problems, the error estimates may depend on certain bound of unknown solution u_h; and these issues have been reported in [35], where FEM is used for the approximation of semilinear problems.

The contents of this chapter are arranged in the following manner. In Section 3.2, we recall model problem with its continuous weak formulation. Section 3.3 deals with the discrete formulation of the underlying problems by employing a combination of virtual element method and Newmark scheme. In this section we also discuss the basic properties of the proposed schemes and computability of integral terms. Optimal convergence rate for both semi and fully discrete in suitable norms are established in Section 3.4. Finally Section 3.5 collects several numerical examples in order to confirm theoretically estimated rate of convergence obtained in Section 3.4. Finally, based on theoretical results, we have made some conclusion in Section 3.6.

3.2 Continuous problem and weak formulation

For simplicity, we consider the following second order hyperbolic problem: Find $u(x, y)$ which satisfy

$$\begin{cases} D_t^2 u - \Delta u = f(u,t) & \text{in} \quad \Omega, \quad \text{for } t \in (0,T), \\ u = 0 & \text{on } \Gamma = \partial\Omega, \quad \text{for } t \in (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \\ D_t u(\cdot,0) = \omega_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $D_t^2 u$ and $D_t u$ denote second order and first order time derivative of u, respectively. As demanded by our analysis, we assume certain regularity assumption on the given data: For a given $t \in [0, T]$, the nonlinear external force function $f(u)(\cdot, t), \in H^{k+1}(\Omega)$, and also u_0, ω_0 belongs to $H^{k+1}(\Omega)$. Moreover, $f(u)$ satisfies globally Lipschitz continuity condition with respect to u, i.e., there exists a positive constant C such that

$$|f(u) - f(w)| \leq C|u - w|,$$

for all $u, w \in H^{k+1}(\Omega)$. Now by multiplying (3.1) with test function $v \in H_0^1(\Omega)$ and applying Green's theorem, we obtain semi-discrete formulation, which reads as follows: Given u_0 and ω_0 in $H_0^1(\Omega)$, find $u(\cdot,t) : [0,T] \to H_0^1(\Omega)$ satisfying

$$\begin{aligned} &(D_t^2 u, v) + a(u,v) = (f(u), v) \, \forall \, v \in H_0^1(\Omega) \;\; t \in [0,T], \\ &u(\cdot,0) = u_0, \; D_t u(\cdot,0) = \omega_0 \text{ in } \Omega, \end{aligned} \tag{3.2}$$

where $(\cdot,\cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is L^2 scalar-product and $a(u,v) := (\nabla u, \nabla v)$. The equation (3.2) represents a second order nonlinear differential equation with respect to time t, and the global Lipschitz continuity of $f(u)$ guarantees to ensure the unique solvability of this equation.

3.3 The virtual element framework

In order to discretize model problem (3.1) employing VEM, we recollect virtual element space $\mathcal{Z}^k(K)$(local) defined in Chapter 1(subsection 1.2.1, Theoretical Development). Global virtual element space $\mathcal{Z}_h^k$ is also defined in the same Chapter. Furthermore, we stress that the local virtual element space $\mathcal{Z}^k(K)$ is unisolvent w.r.t. degrees of freedom defined in $(D_1) - (D_3)$. However, in order to depict degrees of freedom, we exhibit mesh decomposition in Figure 3.1 .

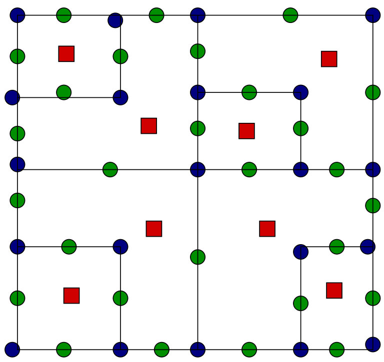


Figure 3.1: Degrees of freedom of polygonal elements for for k=2; $(\mathbf{D_1})$ and $(\mathbf{D_2})$ degrees of freedom are marked by blue circle and green circle respectively; cell moments are marked by a red square.

3.3.1 Semidiscrete schemes

Since our admissible virtual element space consists of polynomial and non-polynomial functions, we need to introduce the following projection operator to handle the nonlinear part and accordingly we define the load term which are computable (an integral term said to be computable if it is a polynomial or can be computed with the help of degrees of freedom defined for that space). Let $\Pi^0_{k,K} : H^1(K) \to \mathbb{P}_k(K)$ be the local projection operator and we recollect the definition as

$$(\Pi^0_{k,K} q - q, p_k)_{0,K} = 0 \quad \forall\, p_k \in \mathbb{P}_k. \tag{3.3}$$

With the help of $\Pi^{\nabla}_{k,K}$ (also known as elliptic projection operator, define in (1.3)) and L^2 projection operator $\Pi^0_{k,K}$, we define $a^K_h(\cdot,\cdot) : \mathcal{Z}^k(K) \times \mathcal{Z}^k(K) \to \mathbb{R}$ and $m^K_h(\cdot,\cdot) : \mathcal{Z}^k(K) \times \mathcal{Z}^k(K) \to \mathbb{R}$ local bilinear forms in the following manner (for more detail we refer to [11])

$$a^K_h(u_h, v_h) := a^K(\Pi^{\nabla}_{k,K}(u_h), \Pi^{\nabla}_{k,K}(v_h)) + s^K_a((I-\Pi^{\nabla}_{k,K})u_h, (I-\Pi^{\nabla}_{k,K})v_h) \;\forall\, u_h, v_h \in \mathcal{Z}^k(K), \tag{3.4}$$

and

$$m^K_h(u_h, v_h) := m^K(\Pi^0_{k,K} u_h, \Pi^0_{k,K} v_h) + s^K_m((I-\Pi^0_{k,K})u_h, (I-\Pi^0_{k,K})v_h) \;\forall\, u_h, v_h \in \mathcal{Z}^k(K), \tag{3.5}$$

where $m^K(u,v) := \int_K u\, v \;\forall\, u, v \in L^2(K)$. Here, the stabilization terms $s^K_a(\cdot,\cdot)$ and $s^K_m(\cdot,\cdot)$ are symmetric bilinear forms whose matrix representation in the canonical basis function $\{\phi_i\}$ of $\mathcal{Z}^k(K)$ can be taken as identity matrix and the identity matrix multiplied by h^2_K, respectively. We note that the construction of $a^K_h(\cdot,\cdot)$ and $m^K_h(\cdot,\cdot)$ satisfies the following usual consistency(with respect to polynomials $\mathbb{P}^k(K)$) and stability properties revealed in Chapter-2(equations 2.10 and 2.11).

Now the corresponding global discrete bilinear forms $a_h(\cdot,\cdot) : \mathcal{Z}^k_h \times \mathcal{Z}^k_h \to \mathbb{R}$, and $m^K_h(\cdot,\cdot) : \mathcal{Z}^k_h \times \mathcal{Z}^k_h \to \mathbb{R}$ are defined as

$$a_h(u_h, v_h) := \sum_K a^K_h(u_h, v_h), \tag{3.6}$$

and

$$m_h(u_h, v_h) := \sum_K m^K_h(u_h, v_h), \tag{3.7}$$

and in the sequel, we describe VEM the formulation: Find $u_h \in \mathcal{Z}_h^k \subset H_0^1(\Omega)$ such that

$$\begin{cases} m_h(D_t^2 u_h, v_h) + a_h(u_h(t), v_h) = \Big(f_h(u_h(t), t), v_h\Big) \ \forall\, v_h \in \mathcal{Z}_h^k, \text{ for a.e. } t \in (0,T), \\ u_h(0) = u_{h,0}, \\ \omega_{h,0} = \omega_{h,0}. \end{cases} \tag{3.8}$$

Here, $u_{h,0}$ and $\omega_{h,0}$ are approximation of the $u(0)$ and $D_t u(0)$ respectively ; and $f_h(u_h, t)$ is discrete load function (also can be think of an approximation of $f(u_h, t)$), and the construction of which is described below.

3.3.2 Fully discrete schemes

In view of applicability of Newmark schemes generally used for time discretization in the context of linear hyperbolic problems (see [38]), we also employ Newmark's schemes for temporal variable together with VEM discretization for space variable. An interesting feature of the Newmark method is that it is a single-implicit scheme that means while the computation of the displacement is implicit, the computation of the velocity is explicit; which requires only a substitution or a function evaluation. Moreover, the scheme is capable of dealing with engineering complication appeared in structural dynamics. Apart from structural dynamics a various type of dynamics shocks, impact, vibration created due to earthquake or blast from a atomic explosion can also be examined. Let N be a positive integer and let time step $\Delta t = \tau = T/N$. Let $u_h^n = u_h(t_n)$, where $t_n = n\,\Delta t$ then our two-step Newmark method fully discrete scheme corresponding to semi-discrete scheme (3.8) read as follows: Find $u_h^n \in \mathcal{Z}_h^k$ such that

$$\begin{aligned} & m_h\left(\frac{u_h^{n+2} - 2u_h^{n+1} + u_h^n}{\tau^2},\ v_h\right) + a_h\left(\beta u_h^{n+2} + (1/2 - 2\beta + \gamma)u_h^{n+1} + (1/2 + \beta - \gamma)u_h^n,\ v_h\right) \\ & = \left(\beta F_h^{n+2} + (1/2 - 2\beta + \gamma)F_h^{n+1} + (1/2 + \beta - \gamma)F_h^n,\ v_h\right) \\ & m_h\left(\frac{u_h^1 - u_{h,0} - \tau\omega_{h,0}}{\tau^2}, v_h\right) + a_h\left(\beta u_h^1 + (1/2 - \beta)u_{h,0},\ v_h\right) \\ & = \left(\beta F_h^1 + (1/2 - \beta)F_h^0,\ v_h\right), \end{aligned} \tag{3.9}$$

where $F_h^n := f_h(u_h(t_n))$, $\omega_{h,0}$ is approximation of ω_0 and β and γ are arbitrary parameters. It has been shown that for $\beta = 1/4$ and $\gamma = 1/2$, the scheme is unconditionally stable and also rate of convergence of temporal order is 2. Therefore, we also consider

$\beta = 1/4$ and $\gamma = 1/2$ in (3.9). Moreover, for simplicity in the notation, we will define the following

$$\begin{aligned} \chi_{n+1/2} &:= 1/2(\chi_{n+1} + \chi_n), \quad \partial_t \chi_{n+1/2} := \Big(\frac{\chi_{n+1} - \chi_n}{\Delta t}\Big), \\ \partial_t^2 \chi_n &:= \frac{\chi_{n+2} - 2\,\chi_{n+1} + \chi_n}{\Delta t^2}, \quad \delta_t^2 \chi_n := \frac{\chi_{n+2} + 2\chi_{n+1} + \chi_n}{4}, \end{aligned} \tag{3.10}$$

where χ is a continuous function in space and time variable and $\chi_n = \chi(t_n)$.

3.3.3 Construction of the nonlinear load term

In order to approximate the nonlinear load term $(f_h(u_h), v_h)$ for $v_h \in \Gamma_h^k$, we define force function $f_h(u_h)$ on each element $K \in \mathcal{T}_h$ using the L^2 projection $\Pi^0_{k,K}$ as

$$f_h(u_h)|_K := \Pi^0_{k,K} f(\Pi^0_{k,K} u_h), \tag{3.11}$$

and globally is understood as $f_h(u_h) := \Pi^0_k f(\Pi^0_k u_h)$. We notice that this load term is exactly computable for any values of k, also an application of L^2 orthogonal projection operator yields

$$(f_h(u_h), v_h) = \sum_K (f_h(u_h), v_h)_K = \sum_K (f(\Pi^0_{k,K} u_h), \Pi^0_{k,K} v_h)_K. \tag{3.12}$$

In particular, if we choose discrete function $v_h = \phi_i$, the nonlinear load term reduces to $(f_h(u_h), \phi_i)_K = \sum_K (f(\sum_{j=1}^{N^{dof}} u_j\ \Pi^0_{k,K}\phi_j), \Pi^0_{k,K}\phi_i)_K$, where u_j denotes j^{th} degree of freedom of u_h . It follows evidently that $(f_h(u_h), v_h)$ can be computed with the help of degrees of freedom, since $\Pi^0_{k,K}\phi_i$ is computable for $1 \leq i \leq N^{dof}$, where N^{dof} represents total number of degrees of freedom (locally).

3.4 Convergence Analysis

In this section, we will establish the error estimates in H^1 semi-norm and L^2 norm for both semi and fully discrete schemes discussed in the previous section. We note that since f depend on u, it is natural to think that the desired error estimates may depend on the bound of computed solution u_h. However, our emphasis is to acquire these estimates in terms of exact solution u instead of the computed solution u_h; and we believe that this can not be achieved by proceeding in conventional manner. Therefore, some advanced tools are required to achieve this purpose. First we attempt to derive the error estimates for semi discrete scheme and later we extend this analysis to fully discrete case. In connection with this, as usual, we decompose the error as follows

$$u - u_h = u - R_h u + R_h u - u_h, \quad \text{with} \quad \rho := u - R_h u, \ \theta := u_h - R_h u.$$

Where $R^h : H_0^1(\Omega) \to \mathcal{Z}_h^k$ is an elliptic projection defined by

$$a_h(R^h u, v_h) = a(u, v_h) \ \ \forall \, v_h \in \mathcal{Z}_h^k.$$

By proceeding in the standard way, below we state error estimates for R_h in L^2 and H^1 norm which will be used in the subsequent analysis. For a proof we refer to [11].

Lemma 14. *Let $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ and the domain Ω is convex. Then there exists a generic constant C independent of h such that*

$$|R^h u - u|_1 \leq C h^k |u|_{k+1}, \qquad \|R^h u - u\|_0 \leq C h^{k+1} |u|_{k+1}.$$

In what follows, we use the similar arguments used in [39] for deriving our error estimates.

3.4.1 Optimal L^2 error estimates

Theorem 15. *Let u be the solution of (3.2) and u_h be the discrete solution of (3.8), and assume that nonlinear force function $f(u) \in L^2(H^{k+1}(\Omega))$, $u_0 \in H^{k+1}(\Omega)$ and $\omega_0 \in H^{k+1}(\Omega)$. Additionally, let $u_h(\cdot, 0) = I_h(u_0)$ and $u_{h,t}(0) = I_h(\omega_0)$, where I_h is the interpolation operator defined in [13]. Then the following estimation hold*

$$\begin{aligned} \|(u - u_h)(t)\| \leq C \Big(\|u_{h,0} - u_0\| + \|D_t(u - u_h)(0)\| \Big) + C \, h^{k+1} \Big(\|u\|_{L^2(0,T,H^{k+1}(\Omega))} \\ + |u_0|_{k+1} + \|D_t u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t^2 u\|_{L^2(0,T,H^{k+1}(\Omega))} \\ + \|f(u)\|_{L^2(0,T,H^{k+1}(\Omega))} \Big). \end{aligned} \tag{3.13}$$

Proof. Since the estimation of $\|\rho\|$ is readily available from lemma(14), we proceed to estimate $\theta(t)$. An application of (3.2) together with (3.8) yields

$$\begin{aligned} m_h(D_t^2 \theta(t), v_h) + a_h(\theta(t), v_h) &= (f_h(u_h), v_h) - m_h(D_t^2 R^h u(t), v_h) - a_h(R^h u(t), v_h) \\ &= (f_h(u_h), v_h) - m_h(R^h D_t^2 u(t), v_h) - a(u(t), v_h) \\ &= (f_h(u_h) - f(u), v_h) + (D_t^2 u(t), v_h) - m_h(R^h D_t^2 u(t), v_h). \end{aligned} \tag{3.14}$$

Now, (3.14) can be rewritten as follows:

$$
\begin{aligned}
-m_h(D_t\theta(t), D_t v_h) + a_h(\theta(t), v_h) = {} & \frac{d}{dt} m_h(D_t(u-u_h), v_h) + \frac{d}{dt}(G_1(t), v_h) \\
& - m_h(D_t^2\rho, v_h) - m_h(D_t\rho, D_t v_h) - (G_1(t), \frac{d}{dt}v_h) \\
& + \frac{d}{dt}(G_2(t), v_h) - (G_2(t), \frac{d}{dt}v_h) \\
& - \frac{d}{dt} m_h(G_3(t), v_h) + m_h(G_3(t), \frac{d}{dt}v_h),
\end{aligned} \tag{3.15}
$$

where $G_1(t)$, $G_2(t)$ and $G_3(t)$ are given by

$$
\begin{aligned}
& G_1(t) := \int_0^t (f_h(u_h) - f(u))(s)ds; \quad G_2(t) := \int_0^t D_t^2 u(s)ds; \\
& G_3(t) := \int_0^t R^h D_t^2 u(s)ds.
\end{aligned} \tag{3.16}
$$

For any $\xi \in (0, T]$, let us define $\hat{\theta}(t) := \int_t^\xi \theta(s)ds$. Plugging $v_h = \hat{\theta}(t)$ into (3.15), recast the equation in the following form

$$
\begin{aligned}
m_h(D_t\theta(t), \theta(t)) + a_h(\theta(t), \hat{\theta}(t)) = {} & \frac{d}{dt} m_h(D_t(u-u_h), \hat{\theta}(t)) + \frac{d}{dt}(G_1(t), \hat{\theta}(t)) \\
& - m_h(D_t^2\rho, \hat{\theta}(t)) - m_h(D_t\rho, D_t\hat{\theta}(t)) - (G_1(t), \frac{d}{dt}\hat{\theta}(t)) \\
& + \frac{d}{dt}(G_2(t), \hat{\theta}(t)) - (G_2(t), \frac{d}{dt}\hat{\theta}(t)) \\
& - \frac{d}{dt} m_h(G_3(t), \hat{\theta}(t)) + m_h(G_3(t), \frac{d}{dt}\hat{\theta}(t)).
\end{aligned} \tag{3.17}
$$

By introducing $\|v\|_h = m_h(v, v)$ for all $v \in \mathcal{Z}_h^k$ and using the fact that the time derivative commutes with discrete bilinear forms $m_h(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$, we infer that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_h^2 - \frac{1}{2}\frac{d}{dt} a_h(\hat{\theta}(t), \hat{\theta}(t)) = {} & \frac{d}{dt} m_h(D_t(u-u_h), \hat{\theta}(t)) + \frac{d}{dt}(G_1(t), \hat{\theta}(t)) \\
& - m_h(D_t^2\rho, \hat{\theta}(t)) + m_h(D_t\rho, \theta) + (G_1(t), \theta(t)) + \frac{d}{dt}(G_2(t), \hat{\theta}(t)) \\
& + (G_2(t), \theta(t)) - \frac{d}{dt} m_h(G_3(t), \hat{\theta}(t)) - m_h(G_3(t), \theta(t)).
\end{aligned} \tag{3.18}
$$

Integrating (3.18) w.r.t. t from 0 to ξ and using $a_h(\hat{\theta}(0), \hat{\theta}(0)) > 0$, we arrive at

$$
\begin{aligned}
\|\theta(\xi)\|_h^2 \leq \|\theta(0)\|_h^2 &- 2m_h(D_t(u-u_h)(0), \hat{\theta}(0)) - 2\int_0^{\xi} m_h(D_t^2\rho, \hat{\theta}(t)) \\
&+ 2\int_0^{\xi} m_h(D_t\rho, \theta(t)) + 2\int_0^{\xi} (G_1(t), \theta(t)) + 2\int_0^{\xi} (G_2(t), \theta(t)) \\
&- 2\int_0^{\xi} m_h(G_3(t), \theta(t)).
\end{aligned} \tag{3.19}
$$

In view of continuity property of discrete bilinear form $m_h(\cdot,\cdot)$ and Young's inequality, we deduce that

$$
m_h(D_t(u-u_h)(0), \hat{\theta}(0)) \leq C\, \|D_t(u-u_h)(0)\|^2 + \; C\, \int_0^{\xi} \|\theta(t)\|^2 dt. \tag{3.20}
$$

and

$$
\begin{aligned}
\int_0^{\xi} m_h(D_t^2\rho(t), \hat{\theta}(t)) &\leq C\, \int_0^{\xi} \|D_t^2\rho(t)\|\, \|\hat{\theta}(t)\| \\
&\leq \; C\, h^{2k+2}\, \|D_t^2 u\|^2_{L^2(0,T;H^{k+1}(\Omega))} + C\, T^2 \int_0^{\xi} \|\theta(t)\|^2 dt.
\end{aligned} \tag{3.21}
$$

Again with the help of continuity of $m_h(\cdot,\cdot)$ and Cauchy-Schwarz inequality and approximation property of elliptic projection operator R^h, we obtain

$$
\begin{aligned}
\int_0^{\xi} m_h(D_t\rho, \theta(t)) &\leq C\, \int_0^{\xi} \|D_t\rho\|\, \|\theta(t)\| \\
&\leq C\, h^{2k+2} \|D_t u\|^2_{L^2(0,T;H^{k+1}(\Omega))} + C\, \int_0^{\xi} \|\theta(t)\|^2 dt.
\end{aligned} \tag{3.22}
$$

The estimates for right hand side is bit tricky, first we split the right-hand in the following manner

$$
\begin{aligned}
f_h(u_h) - f(u) = \Pi^0_{k,K} f(\Pi^0_{k,K} u_h) &- \Pi^0_{k,K} f(\Pi^0_{k,K} u) + \Pi^0_{k,K} f(\Pi^0_{k,K} u) \\
&- \Pi^0_{k,K} f(u) + \Pi^0_{k,K} f(u) - f(u).
\end{aligned} \tag{3.23}
$$

Then the following holds by employing the regularity of u, $f(u)$, triangle inequality, Lipschitz continuity for f, Cauchy-Schwarz inequality together with approximation property of $\Pi^0_{k,K}$;

$$\begin{aligned}\|f_h(u_h) - f(u)\| \le \sum_K \Big(&\|\Pi^0_{k,K} f(\Pi^0_{k,K} u_h) - \Pi^0_{k,K} f(\Pi^0_{k,K} u)\|_{0,K} \\ &+ \|\Pi^0_{k,K} f(\Pi^0_{k,K} u)\|_{0,K} + \|\Pi^0_{k,K} f(u) - f(u)\|_{0,K} \Big) \\ \le\ & C\ \|u_h - u\| + C\ h^{k+1}\ |u|_{k+1} + C\ h^{k+1}\ |f(u)|_{k+1}. \end{aligned} \tag{3.24}$$

Hence, we obtain the following bound for $G_1(t)$

$$\begin{aligned}&\int_0^\xi (G_1(t), \theta(t)) \le C\ \int_0^\xi \Big(\int_0^t \|f_h(u_h) - f(u)\| \Big)\ \|\theta(t)\|\ dt \\ &\le C\ h^{2k+2}\ \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + C\ h^{2k+2}\ \|f(u)\|^2_{L^2(0,T,H^{k+1}(\Omega))} + C\ \int_0^\xi\ \|\theta(t)\|^2\ dt. \end{aligned} \tag{3.25}$$

Definition of $G_2(t)$ and $G_3(t)$ implies that

$$\begin{aligned}\int_0^\xi \Big((G_2(t), \theta(t)) - m_h(G_3(t), \theta(t)) \Big) = \int_0^\xi \Big(&\Big(\int_0^t D_t^2 u(s)\ ds, \theta(t) \Big) \\ &- m_h \Big(\int_0^t R^h D_t^2 u(s) ds, \theta(t) \Big) \Big). \end{aligned}$$

Now, thanks to polynomial consistency property of $m_h^K(\cdot, \cdot)$ (2.10) for helping in finding the following estimates

$$\begin{aligned}\Big(\int_0^t D_t^2 u(s)\ ds, \theta(t) \Big)_K &- m_h^K \Big(\int_0^t R^h D_t^2 u(s) ds, \theta(t) \Big) \\ &= \Big(\int_0^t D_t^2 u(s) - \Pi^0_{k,K} D_t^2 u(s)\ ds, \theta(t) \Big)_K \\ &\quad + m_h^K \Big(\int_0^t \Pi^0_{k,K} D_t^2 u(s) - R^h D_t^2 u(s) ds, \theta(t) \Big). \end{aligned}$$

Using standard approximation properties $\Pi^0_{k,K}$ (see (3.3)) and R^h from lemma (14), we obtain

$$\int_0^\xi \Big((G_2(t), \theta(t)) - m_h(G_3(t), \theta(t)) \Big) \le C\ \Big(h^{2k+2}\ \|D_t^2 u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \int_0^\xi\ \|\theta(t)\|^2 \Big). \tag{3.26}$$

Substituting all the estimates obtained in (3.20),(3.21),(3.22),(3.25) and (3.26) into (3.19), using Grownwall's inequality [34], and stability properties of bilinear form $m_h(\cdot, \cdot)$ we have

$$\|\theta(t)\|^2 \le \|\theta(0)\|^2 + C\,\|D_t(u-u_h)(0)\|^2 + C\,h^{2k+2}\Big(\|D_t^2 u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|^2_{L^2(0,T,H^{k+1}(\Omega))}\Big). \tag{3.27}$$

Again approximation property of elliptic operator R^h given in lemma (14), enable us to write

$$\|\theta(0)\|^2 \le C\,\Big(\|u_h(0)-u_0\|^2 + h^{2k+2}\|u_0\|^2_{k+1}\Big), \tag{3.28}$$

and

$$\|\rho(t)\| = \|(u-R^h u)(t)\| \le C\,h^{k+1}\Big(|u_0|_{k+1} + |D_t u|_{L^1(0,T;H^{k+1}(\Omega))}\Big). \tag{3.29}$$

Using (3.27),(3.28) and (3.29), we obtain the desired result

$$\begin{aligned}&\|(u-u_h)(t)\| \le \|\rho(t)\| + \|\theta(t)\|\\ &\le C\,\Big(\|u_h(0)-u(0)\| + \|D_t(u-u_h)(0)\|\Big) + C\,h^{k+1}\,\Big(|u_0|_{k+1} + \|u\|_{L^2(0,T;H^{k+1}(\Omega))}\\ &\quad + \|D_t u\|_{L^2(0,T;H^{k+1}(\Omega))} + \|D_t^2 u\|_{L^2(0,T;H^{k+1}(\Omega))} + \|f(u)\|_{L^2(0,T;H^{k+1}(\Omega))}\Big).\end{aligned}$$

■

3.4.2 Optimal H^1 error estimates

Theorem 16. *Let u be the solution of (3.2) and u_h the discrete solution of (3.8). Further assume that all the assumption of theorem (15) holds. Then, there exists a positive constant C, independent of mesh size h, but may depend on regularity of u and $f(u)$, such that the following holds*

$$\begin{aligned}|u(t)-u_h(t)|_1 \le{}& C\,\Big(\|D_t(u-u_h)(0)\| + |u_0-u_{h,0}|_1\Big) + C\,h^k\Big(|u_0|_{k+1}\\ &+ \|D_t u\|_{L^2(0,T,H^{k+1}(\Omega))}\Big) + C\,h^{k+1}\,\Big(|D_t u(0)|_{k+1} + \|u\|_{L^2(0,T,H^{k+1}(\Omega))}\\ &+ \|D_t^2 u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|_{L^2(0,T,H^{k+1}(\Omega))}\Big).\end{aligned} \tag{3.30}$$

Proof. Proceeding analogously as in the proof of above theorem, as usual, we split $u-u_h$ as

$$\begin{aligned}u(t)-u_h(t) &= u(t) - R^h u(t) + R^h u(t) - u_h(t)\\ &=: \rho(t) - \theta(t),\end{aligned} \tag{3.31}$$

The estimates for $|\rho(t)|_1$ is readily available from lemma 14; and in order to estimate $|\theta|_1$, we put θ in discrete formulation (3.8)

$$\begin{aligned}
m_h(D_t^2\theta(t), v_h) + a_h(\theta(t), v_h) &= (f_h(u_h), v_h) - m_h\Big(\frac{d^2}{dt^2}R^h u(t), v_t\Big) - a_h(R^h u(t), v_h(t)) \\
&= (f_h(u_h, t), v_h) - m_h(R^h D_t^2 u, v_h) - a(u(t), v_h) \\
&= (f_h(u_h(t), t) - (f(u,t), v_h) + (D_t^2 u(t), v_h) - m_h(R^h D_t^2 u, v_h) \\
&=: (\phi(t), v_h) + (\eta(t), v_h).
\end{aligned} \tag{3.32}$$

It follows by substituting, $v_h = D_t\theta(t)$ in (3.32) and using the fact that the time derivative commute with $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$

$$\frac{1}{2}\frac{d}{dt}m_h(D_t\theta, D_t\theta) + \frac{1}{2}\frac{d}{dt}a_h(\theta(t), \theta(t)) = (\phi(t), D_t\theta(t)) + (\eta(t), D_t\theta(t)). \tag{3.33}$$

The right hand side terms can be estimated piece-wise on each polygon $K \in \mathcal{T}_h$. We first consider the term $(\eta(t), D_t\theta(t))$. In view of polynomial approximation property of discrete bilinear form $m_h(\cdot,\cdot)$(2.10), Cauchy-Schwarz inequality and standard approximation property of L^2 projection operator $\Pi^0_{k,K}$ and R^h operator, we acquire

$$\begin{aligned}
&(\eta(t), D_t\theta(t)) = (D_t^2 u(t), D_t\theta(t)) - m_h(R^h D_t^2 u(t), D_t\theta(t)) \\
&= \sum_{K\in\mathcal{T}_h}\Big((D_t^2 u(t) - \Pi^0_{k,K}D_t^2 u(t), D_t\theta(t))_K - m_h^K(R^h D_t^2 u(t) - \Pi^0_{k,K}D_t^2 u(t), D_t\theta(t))\Big) \\
&\le Ch^{k+1}\|D_t^2 u(t)\|_{k+1}\,\|D_t\theta(t)\|.
\end{aligned} \tag{3.34}$$

Another term of the estimation (3.33) can be obtained locally as

$$\begin{aligned}
(\phi(t), D_t\theta) &= (f_h(u_h) - f(u), D_t\theta(t)) \\
&= \sum_K \Big(\Pi^0_{k,K} f(\Pi^0_{k,K}u_h) - f(u), D_t\theta(t)\Big)_K \\
&\le \sum_K \|\Pi^0_{k,K} f(\Pi^0_{k,K}u_h) - f(u)\|_K\,\|D_t\theta(t)\|_K.
\end{aligned}$$

With the help of estimations borrowing from (3.23) and (3.24), we get

$$(\phi(t), D_t\theta) \le C\left(\|u_h - u\| + h^{k+1}\,|u|_{k+1} + h^{k+1}\,|f(u)|_{k+1}\right)\|D_t\theta\|. \tag{3.35}$$

Plugging (3.34) and (3.35) in (3.33), we obtain

$$m_h(D_t\theta(t), D_t\theta(t)) + a_h(\theta(t), \theta(t)) \le m_h(D_t\theta(0), D_t\theta(0)) + a_h(\theta(0), \theta(0)) + C\,h^{k+1} \int_0^t |D_t^2 u(t)|_{k+1}\, \|D_t\theta\| + \; C \int_0^t \Big(h^{k+1}\, |u|_{k+1} + h^{k+1}\, |f(u)|_{k+1} + \|u - u_h\|\Big)\, \|D_t\theta\|.$$

Since the time derivative commute with R^h operator and utilizing standard approximation property of R^h operator, we can bound the term $m_h(D_t\theta(0), D_t\theta(0))$ as

$$m_h(D_t\theta(0), D_t\theta(0)) \le C\,\Big(\|D_t(u - u_h)(0)\|^2 + h^{2k+2}\, |D_t u(0)|_{k+1}^2\Big).$$

In a similar fashion, we obtain

$$a_h(\theta(0), \theta(0)) \le C\,\Big(|u_{h,0} - u_0|_1^2 + h^{2k}\, |u_0|_{k+1}^2\Big).$$

Using estimation of $\|u - u_h\|$ from Theorem(15) and applying Gronwall's inequality and then neglecting $\|D_t\theta(t)\|^2$, we get

$$|\theta(t)|_1^2 \le C\,\Big(\|D_t(u - u_h)(0)\|^2 + |u_{h,0} - u_0|_1^2\Big) + C\,h^{2k}\, |u_0|_{k+1}^2 + C\,h^{2k+2}\Big(\|u_0\|_{k+1}^2 + \|D_t u(0)\|_{k+1}^2 + \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t^2 u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|^2_{L^2(0,T,H^{k+1}(\Omega))}\Big). \tag{3.36}$$

Plugging (3.36) and $|\rho(t)|_1 \le C\,h^k\,\Big(|u_0|_{k+1} + |D_t u|_{L^1(0,T,H^{k+1}(\Omega))}\Big)$ into (3.31), we obtain

$$|u_h(t) - u(t)|_1 \le C\,\Big(\|D_t(u - u_h)(0)\| + |u_{h,0} - u_0|_1\Big) + C\,h^k\Big(|u_0|_{k+1} + |D_t u|_{L^2(0,T,H^{k+1}(\Omega))}\Big) + C\,h^{k+1}\,\Big(\|D_t u(0)\|_{k+1} + \|u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t^2 u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|_{L^2(0,T,H^{k+1}(\Omega))}\Big).$$

■

3.4.3 Estimates for fully discrete scheme

Next, we proceed to estimate error estimation for fully discrete scheme proposed in section 3.3.2. Most of the ideas used in the derivation are borrowed from [40].

Theorem 17. *Let u be the solution of (3.2) and further assume that* $\|\partial_t\theta_{1/2}\| + \|\theta_{1/2}\|_1 =$

$O(\tau^2+h^{k+1})$, $D_t^4 u \in L^2(0,T,L^2(\Omega))$, $u \in L^2(0,T,H^{k+1}(\Omega))$, $D_t u \in L^1(0,T,H^{k+1}(\Omega))$, $D_t^2 u \in L^2(0,T,H^{k+1}(\Omega))$ and $U_n = u_h(t_n)$. Also, we consider $\{U_n\}_{n=0}^N$ be a sequence generated by (3.9). Then there exist a constant C independent of Δt and h may be dependent on regularity of u and $f(u)$ such that the following estimation holds

$$\begin{aligned}\|u(t_n) - U_n\|_0 \leq C\,\Big(\|\partial_t \theta_{1/2}\| + \|\theta_{1/2}\|_1\Big) &+ C\,\Delta t^2\,\|D_t^4 u\|_{L^2(0,T,L^2(\Omega))} \\ &+\ C\,h^{k+1}\,\Big(|u_0|_{k+1} + \|u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,T,H^{k+1}(\Omega))} \\ &+ \|D_t^2 u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|_{L^2(0,T,H^{k+1}(\Omega))}\Big).\end{aligned}$$

Proof. Using the same idea as Theorem (15), we decompose the error $u(t_n) - U_n$ as follows $u(t_n) - U_n = \rho_n - \theta_n$. With the help of approximation property of R^h operator, we can easily bound $\|\rho_n\|$. In order to estimate $\|\theta_n\|$, we plug θ_n in (3.9) that yields

$$m_h\Big(\partial_t^2 \theta_n, v_h\Big) + a_h\Big(\delta_t^2 \theta_n, v_h\Big) = \Big(\delta_t^2 F_h^n, v_h\Big) - m_h\Big(\partial_t^2\, R^h u_n, v_h\Big) - a_h\Big(\delta_t^2\, R^h u_n, v_h\Big).$$

An application of definition of R^h operator and utilizing the continuous bilinear form (3.2) at nth level, we have

$$\begin{aligned}m_h\Big(\partial_t^2 \theta_n, v_h\Big) + a_h\Big(\delta_t^2 \theta_n, v_h\Big) &= \Big(\delta_t^2 F_h^n, v_h\Big) - \Big(\delta_t^2 f_n, v_h\Big) - m_h\Big(\partial_t^2 R^h u_n, v_h\Big) \\ &\quad + \Big(\delta_t^2 D_t^2 u_n, v_h\Big) \\ &= \Big(\delta_t^2 F_h^n, v_h\Big) - \Big(\delta_t^2 f_n, v_h\Big) - m_h\Big(\partial_t^2 R^h u_n, v_h\Big) \\ &\quad + \Big(\partial_t^2 u_n, v_h\Big) - \Big(\partial_t^2 u_n, v_h\Big) + \Big(\delta_t^2 D_t^2 u_n, v_h\Big).\end{aligned} \tag{3.37}$$

Using Taylor's theorem, we can estimate

$$\Big(\partial_t^2 u_n, v_h\Big) - \Big(\partial_t^2 D_t^2 u_n, v_h\Big) = O(\Delta t^2).$$

Again using polynomial consistency property of discrete bilinear form $m_h(\cdot,\cdot)$, lemma(14), standard approximation property of L^2 projection operator Π_k^0 and finally Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}-m_h\Big(\partial_t^2 R^h u_n, v_h\Big) + \Big(\partial_t^2 u_n, v_h\Big) &= -m_h\Big(\partial_t^2 R^h u_n, v_h\Big) + \Big(\Pi_k^0\Big(\partial_t^2 u_n\Big), v_h\Big) \\ &\quad - \Big(\Pi_k^0\Big(\partial_t^2 u_n\Big), v_h\Big) + \Big(\partial_t^2 u_n, v_h\Big) \\ &\leq C\,h^{k+1}\,\Big|\partial_t^2 u_n\Big|_{k+1}\,\|v_h\|^2.\end{aligned}$$

It is easy to notice that

$$\partial_t^2 u_n = (1/\Delta t^2) \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|)\, D_t^2 u(t_{n+1} + \tau)\, d\tau.$$

Before presenting further detailed discussion, we split the non-linear term $f_h(u_h) - f(u)$, in the following manner

$$(f_h(u_h) - f(u), v_h)_K = \underbrace{(f(\Pi^0_{k,K} u_h), \Pi^0_{k,K} v_h)_K - (f(u_h), \Pi^0_{k,K} v_h))_K}_{T_1} \\ + \underbrace{(\Pi^0_{k,K} f(u_h), v_h)_K - (f(u_h), v_h)_K}_{T_2} + \underbrace{(f(u_h), v_h)_K - (f(u), v_h)_K}_{T3}. \tag{3.38}$$

Again we split T_1 in the following fashion

$$T_1 = (f(\Pi^0_{k,K} u_h), \Pi^0_{k,K} v_h)_K - (f(\Pi^0_{k,K} u), \Pi^0_{k,K} v_h)_K + (f(\Pi^0_{k,K} u), \Pi^0_{k,K} v_h)_K \\ - (f(u), \Pi^0_{k,K} v_h)_K + (f(u), \Pi^0_{k,K} v_h)_K - (f(u_h), \Pi^0_{k,K} v_h)_K.$$

Analogously, we rewrite other terms in the same fashion

$$T_2 = (\Pi^0_{k,K} f(u_h), v_h)_K - (\Pi^0_{k,K} f(u), v_h)_K + (\Pi^0_{k,K} f(u), v_h)_K - (f(u), v_h)_K \\ + (f(u), v_h)_K - (f(u_h), v_h)_K.$$

And

$$T_3 = (f(u_h), v_h) - (f(u), v_h)_K.$$

With the help of approximation property of projection operator $\Pi^0_{k,K}$, global Lipschitz continuity of source function f and Cauchy-schwarz inequality, we obtain

$$\|T_1\| \leq C\, \|u - u_h\|\, \|v_h\| + C\, h^{k+1}\, |u|_{k+1}\, \|v_h\|. \tag{3.39}$$

Other two terms $T2$ and $T3$ can be bound analogously

$$\|T_2\| \leq C\, \|u - u_h\|\, \|v_h\| + C\, h^{k+1}\, |f(u)|_{k+1}\, \|v_h\|, \tag{3.40}$$

and

$$\|T_3\| \leq\ C\, \|u - u_h\|\, \|v_h\|. \tag{3.41}$$

Substituting all the estimates (3.39),(3.40) and (3.41) in (3.38) and summing over K, we obtain desired estimation

$$(f_h(u_h) - f(u), v_h) \leq C\ h^{k+1} \Big(|u|_{k+1} + |f(u)|_{k+1} \Big)\ \|v_h\| + C\ \|u - u_h\|\ \|v_h\|. \quad (3.42)$$

Putting $v_h = \frac{\theta_{n+2}-\theta_n}{2\Delta t}$ in (3.37) and with the help of result (3.42) at n-th level, we have

$$\begin{aligned}
&(1/2\Delta t)\Big(\Big\| \frac{\theta_{n+2} - \theta_{n+1}}{\Delta t} \Big\|^2 - \Big\| \frac{\theta_{n+1} - \theta_n}{\Delta t} \Big\|^2 + \Big| \frac{\theta_{n+2} + \theta_{n+1}}{2} \Big|_1^2 - \Big| \frac{\theta_{n+1} + \theta_n}{2} \Big|_1^2 \Big) \\
&\leq C\ (\Delta t)^4\ D_t^4 u(t_{n+1})^2 + C\ h^{2k+2} \Big(\|D_t^2 u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \ |u|^2_{L^2(0,T,H^{k+1}(\Omega))} \\
&\quad + |f(u)|^2_{L^2(0,T,H^{k+1}(\Omega))} \Big) + C\ \Big(\|\theta_{n+2}\|^2 + \|\theta_{n+1}\|^2 + \|\theta_n\|^2 \Big) \\
&\quad + C\ \Big(\Big\| \frac{\theta_{n+2} - \theta_{n+1}}{\Delta t} \Big\|^2 + \Big\| \frac{\theta_{n+1} - \theta_n}{\Delta t} \Big\|^2 \Big).
\end{aligned}$$

An application of discrete Grownwall's inequality and varying iteration from 0 to $n-2$, we have

$$\begin{aligned}
&\Big(\|\partial_t \theta_{n-1/2}\|^2 + \|\theta_{n-1/2}\|_1^2 \Big) \leq \Big(\|\partial_t \theta_{1/2}\|^2 + \|\theta_{1/2}\|_1^2 \Big) + C\ \Delta t^4\ \|D_t^4 u\|^2_{L^2(0,T,L^2(\Omega))} \\
&\quad + C\ h^{2k+2}\ \Big(\|D_t^2 u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|^2_{L^2(0,T,H^{k+1}(\Omega))} \Big) \\
&\quad + C\ \Delta t\ \sum_{j=0}^{n} \|\theta(t_j)\|_0^2,
\end{aligned} \quad (3.43)$$

where $\partial_t \theta_{n-1/2}$ and $\theta_{n-1/2}$ have same definition as (3.10). Again, some simple manipulation yields

$$\theta_n = \theta_{n-1/2} + (\Delta t/2)\ \partial_t \theta_{n-1/2}. \quad (3.44)$$

With the help of(3.44), (3.43) can be written as

$$\begin{aligned}
\|\theta_n\|^2 &\leq C\ \Big(\|\partial_t \theta_{1/2}\|^2 + \|\theta_{1/2}\|_1^2 \Big) + C\ \Delta t^4\ \|D_t^4 u\|^2_{L^2(0,T,L^2(\Omega))} \\
&\quad + C\ h^{2k+2}\ \Big(\|D_t^2 u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} \\
&\quad + \|f(u)\|^2_{L^2(0,T,H^{k+1}(\Omega))} \Big) + C\ \Delta t\ \sum_{j=0}^{n} \|\theta(t_j)\|_0^2.
\end{aligned}$$

An application of discrete Grownwall inequality yields,

$$\begin{aligned}
\|\theta_n\| &\leq C\ \Big(\|\partial_t \theta_{1/2}\| + \|\theta_{1/2}\|_1 \Big) + C\ \Delta t^2\ \|D_t^4 u\|_{L^2(0,T,L^2(\Omega))} \\
&\quad + C\ h^{k+1}\ \Big(\|D_t^2 u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|u\|_{L^2(0,T,H^{k+1}(\Omega))} \\
&\quad + \|f(u)\|_{L^2(0,T,H^{k+1}(\Omega))} \Big).
\end{aligned}$$

Again, with the help of lemma(14) for R^h at time $t = t_n$, we have

$$\|\rho_n\| \leq C\ h^{k+1} \Big(|u_0|_{k+1} + \|D_t u\|_{L^1(0,T,H^{k+1}(\Omega))} \Big).$$

Now the estimation of θ_n and ρ_n deduce the final thesis

$$\begin{aligned}\|u(t_n) - U_n\| &\leq \|\theta_n\| + \|\rho_n\| \\ &\leq C \Big(\|\partial_t \theta_{1/2}\| + \|\theta_{1/2}\|_1 \Big) + C\ \Delta t^2\ \|D_t^4 u\|_{L^2(0,T,L^2(\Omega))} \\ &\quad + C\ h^{k+1} \Big(|u_0|_{k+1} + \|u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,T,H^{k+1}(\Omega))} \\ &\quad + \|D_t^2 u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|f(u)\|_{L^2(0,T,H^{k+1}(\Omega))} \Big).\end{aligned}$$

■

3.5 Numerical Experiment

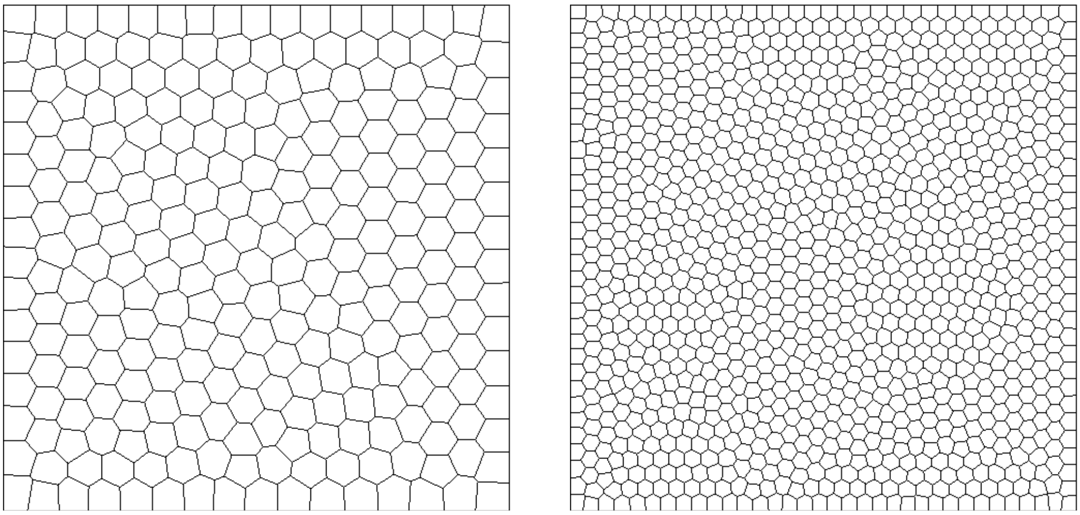
In this section, we provide two numerical examples in order to illustrate the performance of the proposed scheme. First example is concern with the theoretical rate of convergence derived in previous section, while the second is to demonstrate the computational efficiency and advantages of presented VEM formulation. For these examples, we have employed unconditionally stable Newmark schemes for time discretization and VEM for space variables. After spatial discretization, the discrete formulation(3.9) reduces to second order nonlinear ODE and this is solved with the help of Newton's methods. First example serve as an evidence to confirm theoretically predicted optimal order of convergence for error generated by spatial discretization.

3.5.1 Example 1: Convergence rate

Consider the following semilinear hyperbolic equation

$$D_t^2 u - \Delta u = u - u^2 + g(x, y, t) \text{ on } \quad \Omega \times I, \tag{3.45}$$

where $\Omega = [0, 1] \times [0, 1]$ and $I = [0, 1]$. The force function $g(x, y, t)$ is chosen such that $u(x, y, t) = (\sin(t) + \cos(t))\ \sin(\pi x)\ \sin(\pi y)$ satisfies the equation (3.45). We decompose the domain Ω into polygonal elements using polymesher [36]. The initial data u_0 and $D_t u(\cdot, 0)$ are chosen as a consequence of exact solution.The errors $e_{h,0}$ and $e_{h,1}$ are computed as



(a) $h = 1/10$ **(b)** $h = 1/20$

Figure 3.2: Voronoi meshes

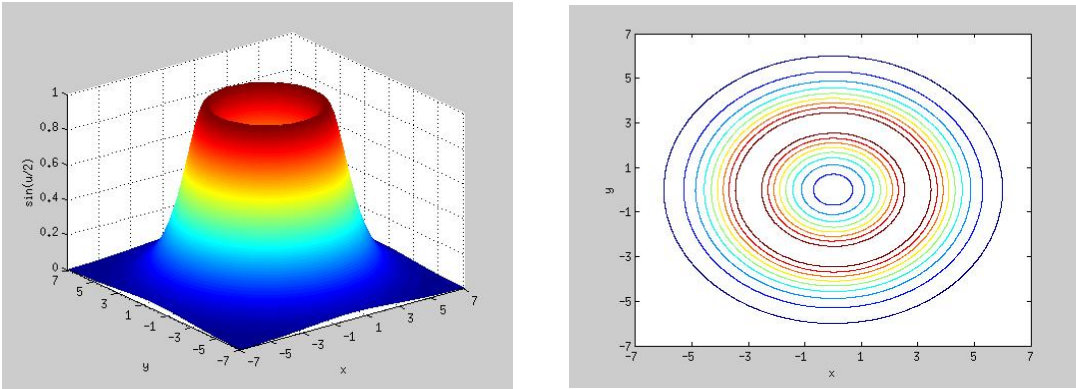
- L^2-norm error : $e_{h,0} = \sqrt{\sum_{K\in\mathcal{T}_h} \|u - \Pi^0_{k,K} u_h\|^2_{L^2(K)}}$
- H^1-norm error : $e_{h,1} = \sqrt{\sum_{K\in\mathcal{T}_h} |u - \Pi^\nabla_{k,K} u_h|^2_{H^1(K)}}$

at the final time T. Further, $r_{h,0}$ and $r_{h,1}$ denote rate of convergence in L^2 norm and H^1 semi-norm, respectively. In our implementation, we have chosen time step $\tau = O(h^k)$; and we have reported computational rate of convergence in Table-3.1. From this table, we clearly observe that rate of convergence evaluated in L^2 norm and H^1 semi-norm are matching with theoretical estimates revealed in section (3.4). However, we would like to remark that L^2 error converges with $O(h^{k+1} + \tau^2)$. Hence, for $k = 1$ and $k = 2$, $\tau = O(h)$ and $\tau = O(h^{3/2})$, respectively would be ideal choice for time step τ. Through

Table 3.1
Error table for non-linear scheme

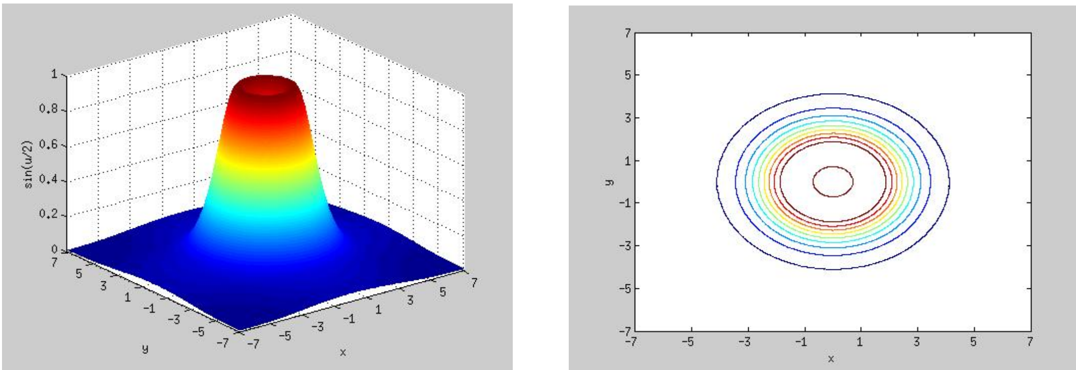
	h	$e_{h,0}$	$r_{h,0}$	$e_{h,1}$	$r_{h,1}$
$k = 1$	1/5	1.2072e-02	-	4.3891e-01	-
	1/10	2.6459e-03	2.18	2.1379e-01	1.03
	1/20	6.4981e-4	2.02	1.0776e-01	0.98
	1/40	1.4288e-04	2.18	5.3338e-02	1.01
$k = 2$	1/5	1.4917e-03	-	3.4499e-02	-
	1/10	1.9972e-04	2.90	8.0376e-03	2.10
	1/20	2.3947e-05	3.06	1.9503e-03	2.04
	1/40	3.1174e-06	2.94	4.9919e-04	1.96

our numerical experiment we perceived that using small time step τ, may reduce the global error. Considering smaller time step does not affect the computing time, since the number of iterations in Newton method reduces for small values of τ.



(a) **(b)**

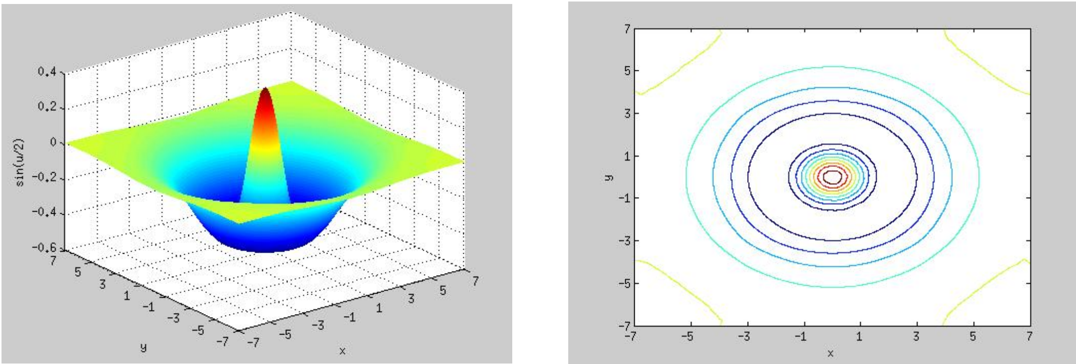
Figure 3.3: Numerical solutions, circular ring solitons at time t=0.



(a) **(b)**

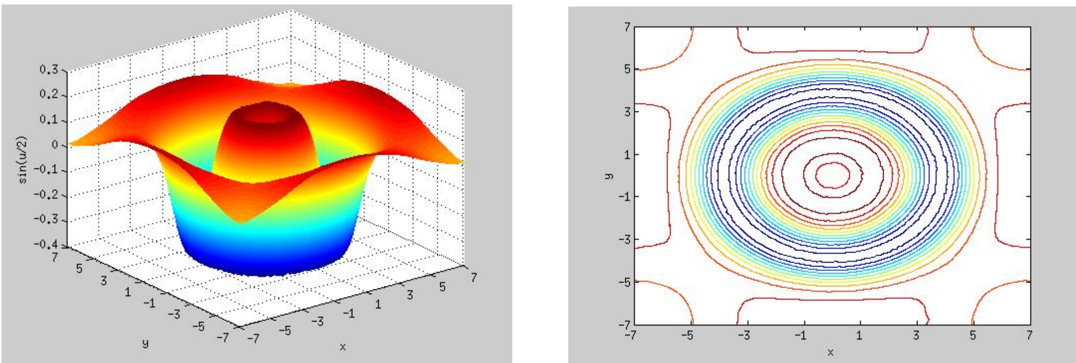
Figure 3.4: Numerical solutions, circular ring solitons at time t=2.74.

In the next example, we move to demonstrate delicate features of VEM for undamped sine-Gordon(SG) equations [41, 42]. SG equation is a nonlinear hyperbolic equation consisting with DAlembert operator with sine of unknown functions. This particular equation is used to model several physical phenomena, for instance, this describes relativistic field theory, Josephson junctions, mechanical transmission. Moreover, various problems of differential geometry can be solved with the help of SG equations. Additionally, SG equation leads to solitons which draw attention of many researchers to find analytic or accurate numerical solution of this equation. As far as numerical approximation of this problem is concern, there are several numerical techniques including FEM (see [43]) are available in literature. However, we believe that VEM would be more appropriate for seeking numerical solution of SG, since these methods also considered as generalization of FEM and also applicable for polygonal meshes.



(a) **(b)**

Figure 3.5: Numerical solutions, circular ring solitons at time t=3.88.



(a) **(b)**

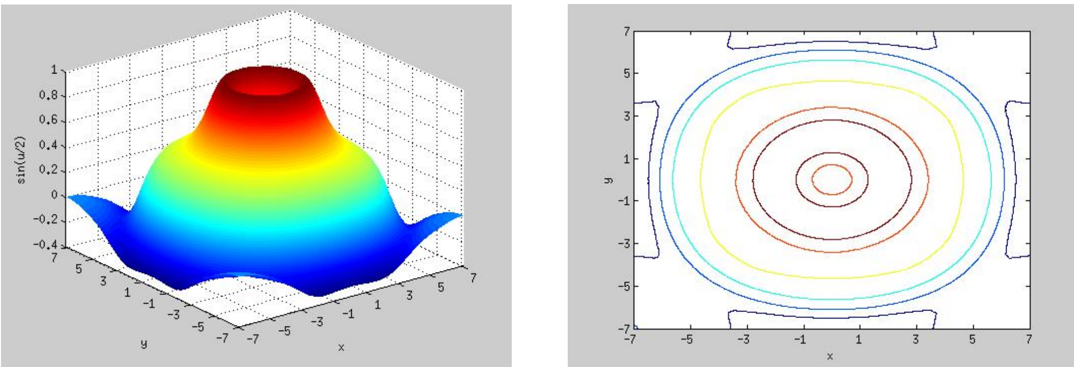
Figure 3.6: Numerical solutions, circular ring solitons at time t=8.62

3.5.2 Example-2: Sine-Gordon equation

We examine the following model problem which is given [41]

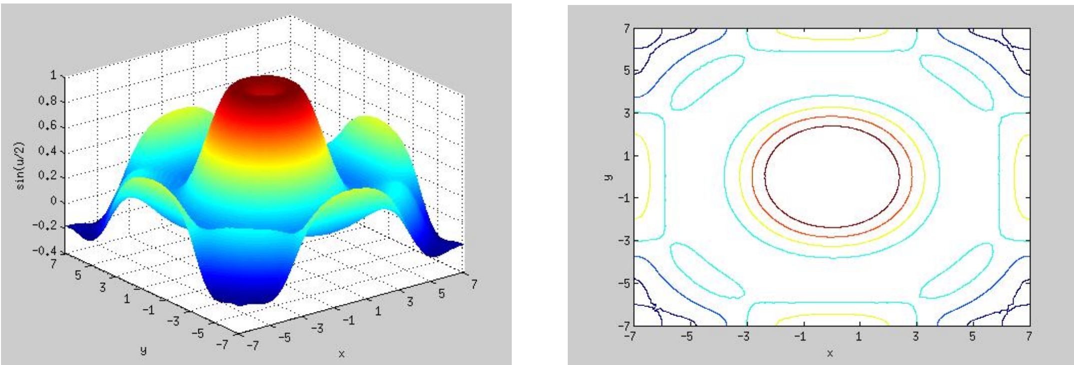
$$D_t^2 u - \Delta u + sin(u) = 0 \quad \text{on} \quad \Omega \times I \tag{3.46}$$

where computational domain $\Omega = [-7, 7] \times [-7, 7]$ and time interval $I = [0, 15]$. In accordance with model problem(3.46), we have assumed initial solution $u(x, y, 0) = 4\tan^{-1}(\exp(3 - \sqrt{x^2 + y^2}))$ and initial velocity $\omega_0(x, y) = 0$. We discretize the domain into voronoi mesh with mesh size $h = 0.25$. The solitons appears in all Figures 3.3-3.8 are homocentric. As expected from Figures 3.3-3.8, we observe that the numerical solution include shrinking and expanding phases. We also stress that our reported numerical experiments are in accordance with the numerical solution obtained in [41].



(a) **(b)**

Figure 3.7: Numerical solutions, circular ring solitons at time t=11.34



(a) **(b)**

Figure 3.8: Numerical solutions, circular ring solitons at time t=12.60

3.6 Discussion

The basic objective of this contribution is to design an efficient VEM for solving nonlinear second order initial value problem over polygonal mesh. As we have mentioned that direct computation of nonlinear part is not possible, we exploit splendid property of local L^2 projection $\Pi^0_{k,K}$ in order to compute nonlinear part. Discrete formulation is built with L^2 orthogonal projection operator and elliptic projection operator $\Pi^\nabla_{k,K}$ which are computable over polynomial subspace of virtual space. Exploiting the technique stated in in [1], we have modified the virtual element spaces where L^2 projection operator is fully computable as demanded by theory. A robust mathematical framework with accessible implementation procedure have made this scheme appreciable. To best of our knowledge, this is the first schemes which solve semilinear hyperbolic problems over polygonal mesh for arbitrary polynomial degrees k with optimal order of convergence. Moreover, the technique can be easily extended to non-conforming VEM. An exhaustive study of nonconforming-VEM for nonlinear convection dominated diffusion reaction equation will be considered as a future work.

Chapter 4

Convection dominated diffusion reaction equation

4.1 Introduction

Convection dominated diffusion reaction equation is a simple model problem for convection diffusion effects that arise in many physical applications (e.g. in various fluid flow problems). The main difficulty arises when one studies the problem numerically is that the solution possess interior and boundary layers which leads to spurious(nonphysical) oscillations. When this problem is studied by the classical finite element method using the Galerkin formulation, the discrete solution produces spurious oscillations. To encounter this problem various stabilization techniques have been developed. Christie et al [44], proposed a stabilization achieved using asymmetric test functions in a weighted residual formulation. Two dimensional upwind finite element discretization were derived by Heinrich et al. [45] and by Tabata [46]. Upwind finite element formulations were able to remove the spurious oscillations but with less accuracy. Further the schemes are inconsistent limiting to first order accuracy. Streamline upwind/Petrov-Galerkin (SUPG) method introduced by Brooks and Hughes [47] can overcome all the above mentioned difficulties. One can also attain stability without compromising accuracy and convergence which regards SUPG as the most prefered method in solving convection-dominated problems numerically.

In this chapter we introduce a new nonconforming element whose degrees of freedom are edge oriented which in turn reduce the overall computational cost. We propose a nonconforming space that uses piecewise quadratic and linear polynomial for which the degrees of polynomials is less compared to $P2$ nonconforming finite element space. We have carried out the discretization of bilinear forms in two ways. One without modifying the convection part and the other by splitting the convection term into symmetric and skew-symmetric parts. We also discuss the pros and cons of both these approaches. We would also like to explore SUPG like stabilization techniques in the context of Virtual element method for polygonal meshes. The major difficulty in doing this is to suitably define the stabilizers in terms of the local projection operators Π_k^0 and Π_k^∇ as the basis functions are defined only implicitly in VEM. In this context Benedetto et al. [18] proposed a order preserving SUPG formulation in the context of VEM. Error estimates in L^2

and energy norms are derived with optimal order of convergence and numerical examples are performed to validate the theoretical results.

Evolutionary convection diffusion equation model the transport and reaction of species. In [48] transient convection equation without diffusive and reactive terms was studied. It is observed that SUPG with backward Euler and Crank-Nicolson in time lead to classical error bound in L^2 norm and energy norm. The results are obtained with suitable regularity assumptions on SUPG stabilization paramter δ which depends only on the mesh size. In this chapter we study this problem in the context of VEM by using SUPG stabilizer along with Crank-Nicholson scheme for time discretization. The proposed scheme is discrete inf-sup stable for sufficiently small mesh size. We have proved the optimal error estimates in $|||\cdot|||$ and L^2 norm by suitably defining the elliptic projection operator R^h.

The outline of this chapter is as follows. In Section (4.2), we have studied a new finite element for stationary convection dominated diffusion reaction equation. Construction and basic properties of the new element are discussed in Section (4.3). Section (4.4) deals with error estimation and convergence analysis in mesh-dependent norm. In Section (4.5), we have conducted two numerical experiments in order to justify theoretical results. Furthermore, in Section (4.6), we extend our analysis for non-stationary convection dominated diffusion reaction equation in the context of virtual element methods. Afterwards, in Section (4.7), we discuss basic virtual element formulation for model problems and state some fundamental lemmas which will be utilized to derive theoretical estimations in next section. Section (4.8) deals with error estimation for semi-discrete and fully discrete case in L^2 and $|||\cdot|||$ norms. Section (4.10) is dedicated to justify theoretical convergence result with robust numerical evidences. Finally in Section (4.11), we have made some conclusion and future works.

4.2 SUPG with NC1-C2 element

In this section, we desire to introduce a new finite element and study for convection dominated diffusion reaction equation. We consider the convection-diffusion-reaction equation

$$\begin{cases} -\epsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where diffusion coefficient ϵ is a very small quantity ($\epsilon \ll 1$). $\mathbf{b}$ and c represent convection coefficient and reaction coefficient respectively. We assume that $\mathbf{b}$, c are $L^\infty(\Omega)$ function of space variable $\mathbf{x}$. Furthermore, we assume that the force function f is a $L^2(\Omega)$ function. Now, we start our discussion by construction of new element.

4.3 Construction and basic properties of NC1-C2 element

In order to define NC1-C2 element, first let us define the following nonconforming space which satisfies patch test of order 1.

$$V_h^1 = \left\{ v \in L^2(\Omega) : v|_K \text{ is linear } \forall\, K \in \mathcal{T}_h, \int_e [|v|] \cdot n_e \, q \, ds = 0 \;\forall\, q \in \mathbb{P}_0(e), \forall e \in \varepsilon_h \right\}, \tag{4.2}$$

where $\mathcal{T}_h$ denotes quasi-uniform triangulation of Ω and ε_h denotes the set of edges of $\mathcal{T}_h$. Now, let $\{\phi_1, \phi_2, \phi_3\}$ be basis functions which are defined as:

$$\phi_i = \hat{\phi}_i \circ F_K^{-1}.$$

Here, F_K is affine mapping from $\hat{K}$ to K, see [Figure:4.1] and $\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3$ are basis functions on reference triangle $\hat{K}$ corresponding to the vertices $\hat{b}_1, \hat{b}_2, \hat{b}_3$ respectively, which are defined by

$$\begin{aligned} \hat{\phi}_1 &= (-1 + 2x + 2y)(-1 + x + y), \\ \hat{\phi}_2 &= (2x - 1)x, \\ \hat{\phi}_3 &= (2y - 1)y. \end{aligned}$$

In view of the definition of $\hat{\phi}_i$, we note that $\phi_i, 1 \leq i \leq 3$ is continuous along edge on

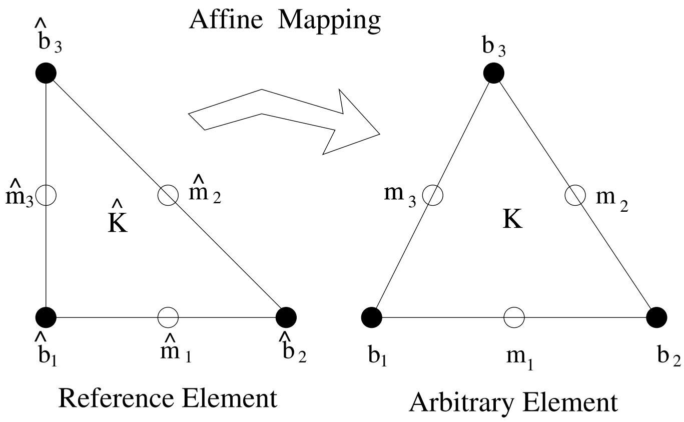


Figure 4.1

each element. If we define $D_K^2 := \text{span}\{\phi_1, \phi_2, \phi_3\}$, then our NC1-C2 element space is defined as

$$V_h := V_h^1 \oplus V_h^2, \quad V_h^2 := \{v_h \in L^2(\Omega) : v_h|_K \in D_K^2\}. \tag{4.3}$$

We stress that the above finite element space V_h consists of piecewise quadratic function which is discontinuous along edge of each triangle except at mid points of edges. Next we demonstrate the construction of a typical element $\omega \in V_h$:

$$\omega = \omega^1 + \omega^2 \text{ where } \omega^1 \in V_h^1 \text{ and } \omega^2 \in V_h^2.$$

The construction properties ensure that the newly defined NC1-C2 element satisfies patch test of order 1, and this can be explained as follows. Let $e \in \varepsilon_h^0$ be an interior edge which is shared by two triangles K^1 and K^2, and $\omega|_{K^1}$ and ω_{K^2} are restriction of ω on K^1 and K^2 respectively. Hence the jump of ω on this edge e is given by

$$\begin{aligned} [|\omega|] &= \omega|_{K^1} - \omega|_{K^2} \\ &= (\omega^1|_{K^1} + \omega^2|_{K^1}) - (\omega^1|_{K^2} + \omega^2|_{K^2}) \\ &= (\omega^1|_{K^1} - \omega^1|_{K^2}) + (\omega^2|_{K^1} - \omega^2|_{K^2}) \\ &= (\omega^1|_{K^1} - \omega^1|_{K^2}). \end{aligned}$$

We notice that this space contains the space of continuous piecewise quadratic and space of nonconforming piecewise linear function, since

$$V_h = V_h^1 + P_2.$$

where P_2 is the piecewise quadratic finite element space.

4.3.1 Discretization

We define the following discrete bilinear forms:

$$a_h^d(u, v) = \epsilon \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K,$$

$$b_h^1(u, v) = \sum_{K \in \mathcal{T}_h} (\mathbf{b} \cdot \nabla u, v)_K, \quad c_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K cu\, v dx,$$

$$b_h^2(u, v) = \sum_{K \in \mathcal{T}_h} \frac{1}{2}\{(\mathbf{b} \cdot \nabla u, v) - (\mathbf{b} \cdot \nabla v, u) - (\nabla \cdot \mathbf{b}, uv)\}.$$

To capture the non-physical oscillation occurred in sharp region, we need to add an additional diffusion term which is defined as:

$$S_h(u, v) = \sum_{K \in \mathcal{T}_h} (-\epsilon \Delta u + \mathbf{b} \cdot \nabla u + cu, \delta_K \mathbf{b} \cdot \nabla v)_K.$$

Now finite element formulation of the model problem (4.1) read as follows: find $u_h \in V_h$ such that

$$a_h^i(u_h, v_h) = f_h(v_h) \quad \forall\, v_h \in V_h,\, i = 1, 2, \tag{4.4}$$

where $a_h^i(\cdot, \cdot)$ is defined by

$$a_h^i(u_h, v_h) = a_h^d(u_h, v_h) + b_h^i(u_h, v_h) + c_h(u_h, v_h) + S_h(u_h, v_h)$$

and right hand side term $f_h(v_h)$ is defined as:

$$f_h(v_h) := \sum_{K \in \mathcal{T}_h} (f, v_h) + (f, \delta_K \mathbf{b} \cdot \nabla v_h)_K.$$

We observe that because of $b_h^1(u, v)$ and $b_h^2(u, v)$, we have two different formulation. Next, we show that the defined bilinear forms are coercive. In order to established the coercivity of the bilinear forms we define the following suitable domain dependent norm on V_h as

$$|||v||| = \left(\sum_{K \in \mathcal{T}_h} \{\epsilon |v|_{1,K}^2 + c_0 \|v\|_{0,K}^2 + \delta_K \|\mathbf{b} \cdot \nabla v\|_{0,K}^2 \} \right)^{1/2}, \tag{4.5}$$

where $c_0 := c - \frac{1}{2} \nabla \cdot \mathbf{b}$. Using standard arguments given in [49], we can show that there exist constants μ_1, μ_2 independent of h such that

$$\|\Delta v_h\|_{0,K} \le \mu_1 h_K^{-1} |v_h|_{1,K} \;\forall\, v_h \in V_h, K \in \mathcal{T}_h \tag{4.6}$$

$$|v_h|_{1,K} \le \mu_2 h_K^{-1} \|v_h\|_{0,K} \;\forall\, v_h \in V_h, K \in \mathcal{T}_h. \tag{4.7}$$

By choosing,

$$0 \le \delta_K \le \min \left\{ \frac{c_0}{2\|c\|_{0,\infty,K}^2}, \frac{h_K^2}{2\epsilon \mu_1^2} \right\}, \tag{4.8}$$

with the help of (4.6) and (4.7), it is not hard to see that

$$a_h^2(v_h, v_h) \ge \frac{1}{2} |||v_h|||^2 \quad \forall\, v_h \in V_h. \tag{4.9}$$

Now an application of Lax-milgram theorem guaranteed that the discrete problem (4.4) has a unique solution in V_h. We also note that if the weak solution of (4.4) satisfies $u \in H^2(\Omega)$ and $v_h \in V_h$, then the bilinear form can be written as

$$a_h^2(u, v_h) = f_h(v_h) + \epsilon \sum_{e \in \varepsilon_h} \int_e \frac{\partial u}{\partial n_e} [|v_h|] ds - \frac{1}{2} \sum_{e \in \varepsilon_h} \int_e (\mathbf{b} \cdot n_e) u [|v_h|] ds, \tag{4.10}$$

where n_e denotes the unit outer normal vector to the boundary of K.

4.4 Error analysis

4.4.1 Interpolation and its properties

For defining a suitable interpolation operator $u_I \in V_h$ which agrees with exact solution u at mid point of each edges of ε_h of triangulation $\mathcal{T}_h$, we proceed in the following manner. Let $\hat{K}$ be the reference triangle with vertices $\hat{b}_1, \hat{b}_2, \hat{b}_3$ whose coordinates are $(0,0), (1,0), (0,1)$, respectively and $\hat{m}_i$ be the midpoint of the side joining i and $i+1$ (modulo 3)vertices. Now we define the following interpolation functions based on $\hat{\phi}_i$

$$\hat{I}^1(\hat{u}) = \hat{u}(\hat{m}_1)\hat{\phi}_4 + \hat{u}(\hat{m}_2)\hat{\phi}_5 + \hat{u}(\hat{m}_3)\hat{\phi}_6,$$

$$\hat{I}^2(\hat{u}) = (\hat{u}(\hat{b}_1) - \hat{I}^1(\hat{u})(\hat{b}_1))\hat{\phi}_1 + (\hat{u}(\hat{b}_2) - \hat{I}^1(\hat{u})(\hat{b}_2))\hat{\phi}_2 + (\hat{u}(\hat{b}_3) - \hat{I}^1(\hat{u})(\hat{b}_3))\hat{\phi}_3.$$

Now we define interpolation operator as follows:

$$\hat{I}(\hat{u}) := \hat{I}^1(\hat{u}) + \hat{I}^2(\hat{u}) = \sum_{j=1}^{6} \hat{L}_j(\hat{u})\hat{\phi}_j,$$

where $\hat{L}_i$, $i = 1,2,3,4,5,6$ are continuous linear functionals.

We claim that $P_2(\hat{K})$ is unisolvent with respect to these functionals, i.e., for a typical polynomial $\hat{p} \in P_2(\hat{K})$ $\hat{L}_i(\hat{p}) = 0$ implies $\hat{p} = 0$. To see this, first we note that an arbitrary polynomial $\hat{p} \in P_2(\hat{K})$ can be written as linear combination of basis of $P_2(\hat{K})$, i.e., there exist $C_1, \ldots, C_6$ such that

$$\hat{p} = \sum_{i=1}^{6} C_i \phi_i.$$

We observed that

$$\begin{aligned} C_4 &= \hat{p}(\hat{m}_1) = \hat{L}_4(\hat{p}) = 0, \\ C_5 &= \hat{p}(\hat{m}_2) = \hat{L}_5(\hat{p}) = 0, \\ C_6 &= \hat{p}(\hat{m}_3) = \hat{L}_6(\hat{p}) = 0. \end{aligned}$$

Therefore, $\hat{p} = C_1\hat{\phi}_1 + C_2\hat{\phi}_2 + C_3\hat{\phi}_3$. Proceeding in the same fashion, we can also show that $C_1 = C_2 = C_3 = 0$, and hence $\hat{p} = 0$. This prove our assertion.

Using the properties of $\hat{I}^1$ and $\hat{I}^2$, it can be shown that for arbitrary $\hat{p} \in P_2(\hat{K})$, we have $\hat{I}(\hat{p}) = \hat{p}$. We define $(K, P_2(K), \Sigma)$ be an affine finite element of $(\hat{K}, P_2(\hat{K}), \hat{\Sigma})$, where $\Sigma = \{L_1, L_2, L_3, L_4, L_5, L_6\}$ and $\hat{\Sigma} = \{\hat{L}_1, \hat{L}_2, \hat{L}_3, \hat{L}_4, \hat{L}_5, \hat{L}_6\}$. Analogously we can define interpolation on arbitrary element K.

The interpolation satisfy the following approximation properties. The proof hinges on Bramble-Hilbert lemma and properties of affine transformation, for a detailed proof, we refer to [50, 49, 51].

Lemma 18. *Let $u \in H^{m+1}(K)$ be an arbitrary element. Then we have*

$$\|D^s(u - u_I)\|_{L^2(K)} \leq C h_K^{m+1-s} \|D^{m+1} u\|_{L^2(K)}, \tag{4.11}$$

where $s \leq m+1$ and $m = 0, 1, 2$ and C is a positive constant does not depend on h_K.

4.4.2 Convergence Analysis

For the accomplishment of our main result, we would require the following well known lemma. The proof can be found in [52].

Lemma 19. *For any edge $e \in \varepsilon_h$ and any integer $k \geq 0$, let Π_e^k be the projection operator from $L^2(e)$ onto $P_k(e)$ defined by*

$$\int_e q \Pi_e^k v ds = \int_e q v ds \; \forall \, q \in P_k(e), v \in L^2(e).$$

Then there exists a constant C independent of e and h such that

$$|\int_e \phi(v - \Pi_e^k v) ds| \leq C h_e^{k+1} |\phi|_{1,K} |v|_{k+1,K} \tag{4.12}$$

for all $K \in \mathcal{T}_h$, $e \subset \partial K$, $\Phi \in H^1(K)$ and $v \in H^{k+1}(K)$.

First we consider the discrete formulation (4.4) with $i = 1$. John et al [53] has studied analogous bilinear form in the non-conforming finite element framework with additional jump term. We stress that even after neglecting additional jump term, it is hard to show that the bilinear form $a_h^1(\cdot, \cdot)$ satisfy the coercivity property with respect to $|||\cdot|||$ (defined in (4.5), and therefore, one can not directly appeal to Lax-Miligram lemma in order to ensure the unique solvability of the discrete formulation corresponding to this bilinear form. However, existence and uniqueness of solution can be shown by using different arguments, for instance, if the bilinear form posses certain type of inf-sup condition then the well-possedness of the corresponding discrete formulation can be established. By following the definition of the bilinear form $a_h^1(\cdot, \cdot)$ and $|||\cdot|||$, it is easy to show that this bilinear form satisfy the following inequality:

$$a_h^1(v_h, v_h) \geq \frac{1}{2}|||v_h|||^2 + \frac{1}{2}\sum_{e\in\varepsilon_h}\int_e (\mathbf{b}\cdot\mathbf{n}_e)[|v_h^2|]_e ds \;\forall\; v_h \in V_h.$$

Now we prove main results of this paper under the assumption that space V_h satisfy patch-test of order one i.e.

$$\int_K [|v_h|]_e q ds = 0 \quad \forall v_h \in V_h,\; q \in P_0(e),\; e \in \varepsilon_h. \tag{4.13}$$

Theorem 20. *Let the assumption (4.8) holds and $\mathcal{T}_h$ is a quasi-uniformly triangular partition of domain Ω, also $u \in H^3(\Omega)$ be the solution of (4.1), u_h be the solution of (4.4) with $i = 1$ and $\mathbf{b} \in W^{1,\infty}(\Omega)^2$. Then the following estimation holds*

$$\begin{aligned} |||u - u_I||| + \alpha_h |||u_I - u_h||| \;&\leq\; C\Bigg[h^2\left(\sum_{K\in\tau_h}\gamma_K |u|_{3,K}^2\right)^{1/2} + h^2\left(\sum_{e\in\varepsilon_h}\gamma_e \|u\|_{3,se}^2\right)^{1/2} \\ &+\; h\epsilon^{1/2}|u|_{2,\Omega}\Bigg], \end{aligned} \tag{4.14}$$

where,

$$\alpha_h = \mathop{inf}_{m_h\in V_h}\mathop{sup}_{v_h\in V_h}\frac{a_h^1(m_h, v_h)}{|||v_h|||\;|||m_h|||},$$

$$\gamma_K := \epsilon + h_K^2 + \delta_K + \frac{h_K^2}{\epsilon} + \frac{h_K^2}{\delta_K} \qquad \gamma_e := min\{\frac{h_K^2}{\epsilon}, 1\},$$

and se is the union of the elements adjacent to e.

Proof. We spilt the error as $u - u_h = u - u_I + u_I - u_h = \eta + \eta_h$, where $\eta := u - u_I$ and $\eta_h := u_I - u_h$. Now since the estimates of η is known, we proceed to estimate η_h. Let us first recall that the bilinear form $a_h^1(\cdot,\cdot)$ was defined as:

$$a_h^1(u_h, v_h) = a_h^d(u_h, v_h) + b_h^1(u_h, v_h) + c_h(u_h, v_h) + S_h(u_h, v_h),$$

In view of above definition η satisfy the following:

$$a_h(\eta, v_h) := a_h^d(\eta, v_h) - b_h^1(v_h, \eta) + ((c - \nabla\cdot\mathbf{b}), \eta v_h) + S_h(\eta, v_h).$$

Using definition of α_h it immediately follows that

$$\alpha_h|||\eta_h||| \leq \sup_{v_h\in V_h}\frac{a_h(\eta, v_h)}{|||v_h|||} + 2\sup_{v_h\in V_h}\frac{n_h^s(\eta, v_h)}{|||v_h|||} + \sup_{v_h\in V_h}\frac{r_h^d(u, v_h)}{|||v_h|||},$$

where

$$r_h^d(u, v_h) = \epsilon \sum_{e \in \varepsilon_h} \int_e \frac{\partial u}{\partial n_e} [|v_h|]_e ds,$$

and

$$n_h^s(\eta, v_h) = \frac{1}{2} \sum_{e \in \varepsilon_h} \int_e (\mathbf{b} \cdot n_e) \eta [|v_h|].$$

An application of the patch-test (4.13) of NC1-C2 element and together with lemma(19), yield

$$r_h^d(u, v_h) \leq Ch\epsilon^{1/2} |u|_{2,\Omega} |||v_h|||. \tag{4.15}$$

In the light of Lemma (18), we have

$$n_h^s(\eta, v_h) \leq Ch^2 (\sum_{e \in \varepsilon_h} \gamma_e \|u\|_{3,se}^2)^{1/2} |||v_h|||. \tag{4.16}$$

Now, by imitating the arguments used in [54] and using the estimates derived in (4.15), (4.16), we obtain desired result. ■

Remark 2. *From the above result we observe that u_h converges to u for any values of ϵ (referred as ϵ-uniform convergence) with at least convergence rate of O(h); however, in case of convection dominated diffusion case, i.e., $\epsilon \leq h$, we get $h^{3/2}$ rate of convergence.*

We prove our next main result for $i = 2$, i.e., $a_h^2(\cdot, \cdot)$ is used instead of $a_h^1(\cdot, \cdot)$.

Theorem 21. *Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω and u_h be the solution of (4.4) with $i = 2$. Then under the assumption (4.8), $u \in H^3(\Omega)$, $\mathbf{b} \in W^{1,\infty}(\Omega)^2$, there exists a positive constant independent of h such that*

$$\begin{aligned} |||u - u_h||| &\leq C\Bigg[h^2 \left(\sum_{K \in \mathcal{T}_h} \gamma_K |u|_{3,K}^2\right)^{1/2} + h\epsilon^{1/2} |u|_{2,\Omega} \\ &+ \left(\sum_{e \in \varepsilon_h} \gamma_e \|u\|_{1,se}^2\right)^{1/2} + h^2 \left(\sum_{e \in \varepsilon_h} \gamma_e \|u\|_{3,se}^2\right)^{1/2}\Bigg], \end{aligned} \tag{4.17}$$

where γ_K, γ_e are same constant defined in Theorem 20.

Proof. Let $\eta = u_I - u$ and $\eta_h = u_I - u_h \in V_h$. Now, using (4.10), we arrive at

$$a_h^2(\eta_h, v_h) = a_h^2(\eta, v_h) + r_h^d(u, v_h) + r_h^s(u, v_h),$$

where,

$$r_h^d(u, v_h) = \epsilon \sum_{e \in \varepsilon_h} \int_e \frac{\partial u}{\partial n_e} [|v_h|]_e ds,$$

$$r_h^s(u, v_h) = -\frac{1}{2} \sum_{e \in \varepsilon_h} \int_e (\mathbf{b} \cdot n_e) u [|v_h|]_e ds.$$

We note that the bilinear form $b_h^2(\cdot, \cdot)$ corresponding to the convective term can be written in the following fashion:

$$b_h^2(\eta, v_h) = -b_h^1(v_h, \eta) - (\nabla \cdot \mathbf{b}, \eta v_h) + n_h^s(\eta, v_h).$$

Here,

$$n_h^s(\eta, v_h) = \frac{1}{2} \sum_{e \in \varepsilon_h} \int_e (\mathbf{b} \cdot n_e) \eta [|v_h|].$$

If we defined a new bilinear $a_h(\cdot, \cdot)$ form as:

$$a_h(\eta, v_h) = a_h^d(\eta, v_h) - b_h^1(v_h, \eta) + ((c - \nabla \cdot \mathbf{b}), \eta v_h) + S_h(\eta, v_h).$$

Then with the help of above definition, $a_h^2(\cdot, \cdot)$ have the following representation

$$a_h^2(\eta, v_h) = a_h(\eta, v_h) + n_h^s(\eta, v_h).$$

It is easy to see that r_h and n_h satisfy the following relation

$$\begin{aligned} \frac{1}{2}|||u - u_h||| \quad &\leq \quad \frac{1}{2}|||\eta||| + \sup_{v_h \in V_h} \frac{a_h(\eta, v_h)}{|||v_h|||} + \sup_{v_h \in V_h} \frac{n_h^s(\eta, v_h)}{|||v_h|||} \\ &+ \quad \sup_{v_h \in V_h} \frac{r_h^d(u, v_h)}{|||v_h|||} + \sup_{v_h \in V_h} \frac{r_h^s(u, v_h)}{|||v_h|||}. \end{aligned} \tag{4.18}$$

Now we proceed to find estimates of η and r_h and n_h. By following [54, 53, 55], we obtain the following estimate for η

$$\frac{1}{2}|||\eta||| + \sup_{v_h \in V_h} \frac{a_h(\eta, v_h)}{|||v_h|||} \leq Ch^2 (\sum_{K \in \mathcal{T}_h} \gamma_K |u|_{3,K}^2)^{1/2}. \tag{4.19}$$

The following estimate for r_h^d follows from patch-test of NC1-C2 element and Lemma(19)

$$r_h^d(u, v_h) \leq Ch\epsilon^{1/2} |u|_{2,\Omega} |||v_h|||. \tag{4.20}$$

An application standard inverse inequality given in [49, 50] and (4.6), (4.7), we deduce

$$r_h^s(u, v_h) \leq C (\sum_{e \in \varepsilon_h} \gamma_e \|u\|_{1,se}^2)^{\frac{1}{2}} |||v_h|||. \tag{4.21}$$

Table 4.1
Errors for the P_1 nonconforming element for Example 1

h	ϵ	$\|\cdot\|_{0,\Omega}$	$\|\cdot\|_{1,h}$
0.0707	$2.50e-5$	11.8902	37.8126
0.0354	$1.56e-6$	12.4633	40.2896
0.0177	$9.77e-8$	12.6249	40.9892
0.0088	$6.10e-9$	12.6667	41.1702

By employing Lemma (18), we obtain

$$n_h^s(\eta, v_h) \leq Ch^2(\sum_{e\in\varepsilon_h} \gamma_e \|u\|_{3,se}^2)^{1/2}|||v_h|||. \tag{4.22}$$

Collecting the estimates obtained in (4.19),(4.20),(4.21) and (4.22) and putting in (4.18), we complete the rest of the proof. ■

Remark 3. *As we could see that in the proof of above theorem the constant γ_e depends on ϵ, therefore, if the values of ϵ is very small then $\gamma_e = 1$. Hence, we can not predict $\epsilon-$ uniform convergence of u_h to u as $h \to 0$. We also conclude that the estimation(4.14) suggest that the bilinear form $a_h^1(\cdot,\cdot)$ performs better than $a_h^2(\cdot,\cdot)$.*

4.5 Numerical Results

In this section, we shall present two numerical experiments to illustrate the performance of our NC1-C2 method and also to verify our theoretical results. In both of the experiments, we have taken the domain $\Omega = [0,1] \times [0,1]$ with zero Dirichlet boundary condition. Numerical errors were calculated using the following norms $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{1,h}$ and $|||\cdot|||$. We have taken the stabilization parameter $\delta_K = h_K$ for the bilinear form $a_h^1(\cdot,\cdot)$ and $\delta_K = h_K/4$ for the bilinear form $a_h^2(\cdot,\cdot)$ respectively.

Example 1

This example is taken from [56]. We have chosen $\mathbf{b} = (3,2)$, $c = 2$. The right hand side f is chosen such that the following smooth polynomial function

$$u(x,y) = 100\, x^2\, (1-x)^2\, y\, (1-y)\, (1-2y)$$

satifies the equation (4.1).

In Table 4.1 and Table 4.2, we want to demonstrate the difficulty faced with the stan-

Table 4.2
Errors for the P_2 conforming element for Example 1

h	ϵ	$\|\cdot\|_{0,\Omega}$	$|\cdot|_{1,h}$
0.0707	$2.50e-5$	$2.22e+2$	$1.50e+2$
0.0354	$1.56e-6$	$3.76e+2$	$5.01e+2$
0.0177	$9.77e-8$	$1.09e+2$	$8.55e+1$
0.0088	$6.10e-9$	$1.74e+2$	$1.82e+2$

Table 4.3
Errors and rate of convergence for $a_h^1(\cdot,\cdot)$ using $NC1-C2$ element for Example 1

		$|||\cdot|||$		$\|\cdot\|_{1,h}$		$\|\cdot\|_{0,\Omega}$	
h	ϵ	Error	Rate	Error	Rate	Error	Rate
0.0707	$2.50e-5$	0.11210	-	5.27432	-	0.06377	-
0.0354	$1.56e-6$	0.04048	1.4724	4.54551	0.2150	0.02789	1.1956
0.0177	$9.77e-8$	0.01425	1.5064	3.39485	0.4211	0.01047	1.4129
0.0088	$6.10e-9$	0.00499	1.5111	2.43087	0.4811	0.00375	1.4764

Table 4.4
Errors and rate of convergence for $a_h^2(\cdot,\cdot)$ using $NC1-C2$ element for Example 1

		$|||\cdot|||$		$\|\cdot\|_{1,h}$		$\|\cdot\|_{0,\Omega}$	
h	ϵ	Error	Rate	Error	Rate	Error	Rate
0.0707	$2.50e-5$	0.58226	-	$3.94e+1$	-	0.380947	-
0.0354	$1.56e-6$	0.54819	0.0872	$8.87e+1$	-1.1703	0.417416	-0.1322
0.0177	$9.77e-8$	0.51554	0.0886	$1.88e+2$	-1.0852	0.438904	-0.0724
0.0088	$6.10e-9$	0.49315	0.0640	$3.88e+2$	-1.0425	0.451048	-0.0393

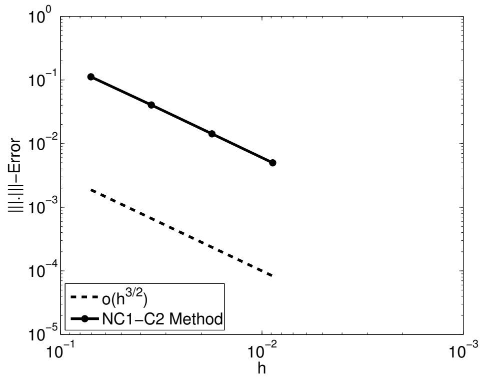


Figure 4.2: Log-log plot of the numerical error for the bilinear form $a_h^1(\cdot,\cdot)$ for Example 1

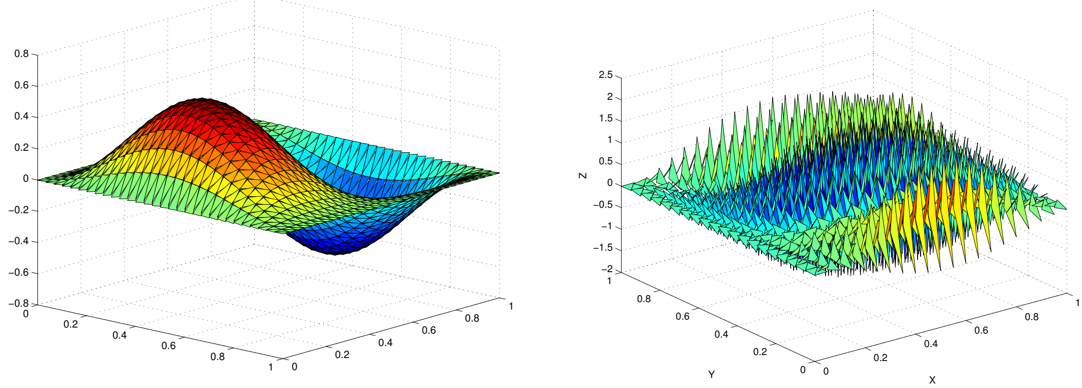
dard P_1 nonconforming method and standard P_2 conforming method for convection dominated problem respectively. In order to observe the convergence uniform with respect to ϵ we have varied the values of ϵ for decreasing values of h. We observe that the error increases with decreasing values of h so the method diverges finally. Table 4.3 shows the numerical error and the rate of convergence for the bilinear form $a_h^1(\cdot,\cdot)$. From Table 4.3 we observe that the rate of convergence with respect to $|||\cdot|||$ norm satisfies with our theoretical estimate (Eq. 4.14). The log-log plot of the error from the Figure 4.2 with respect to $|||\cdot|||$ norm also confirms the expected rate of convergence. We have also calculated the errors using H^1 norm and L^2 norm to show the behaviour of convergence.

From Table 4.4 we observe that the numerical error for the bilinear form $a_h^2(\cdot,\cdot)$ with respect to $|||\cdot|||$ norm does not converge as the error remains almost constant even for small values of h. We can also observe this behaviour with repect to L^2 and H^1 norm with negative rate of convergence. From the theoretical estimate (Eq. 4.17) we cannot predict the convergence behaviour for the bilinear form $a_h^2(\cdot,\cdot)$ which is explained in Table 4.4. The solution on right in Figure 4.3 shows the oscillations in the numerical solutions with respect to the bilinear form $a_h^2(\cdot,\cdot)$. In the next example we will discuss the convergence in the case of solution with circular internal layers.

Example 2

This example is taken from [53]. Let $\mathbf{b} = (2,3)$ and $c = 2$. The right hand side and the boundary conditions are chosen such that

$$u(x,y) = 16\,x\,(1-x)\,y\,(1-y)\left\{\frac{1}{2} + \frac{\arctan[200(r_0^2-(x-x_0)^2-(y-y_0)^2)]}{\pi}\right\}$$



(a) Exact solution

(b) Numerical solution

Figure 4.3: Exact solution and numerical solution for the bilinear form $a_h^2(\cdot,\cdot)$ for Example 1

Table 4.5

Errors for the P_1 nonconforming element for Example 2

h	ϵ	$\|\cdot\|_{0,\Omega}$	$\lvert\cdot\rvert_{1,h}$
0.0707	$1e-11$	0.677367	4.0234
0.0354	$1e-11$	0.673782	3.9806
0.0177	$1e-11$	0.672883	3.9722
0.0088	$1e-11$	0.672879	3.9720

is the exact solution of (4.1) with $x_0 = y_0 = 0.5$ and $r_0 = 0.25$. We have computed the numerical error for $\epsilon = 10^{-5}$. From Table 4.7, we observe that the rate of convergence with respect to $|||\cdot|||$ norm matches with our theoretical estimate. We have shown the log-log plot of the numerical error in Figure 4.4 with the expected rate of convergence. Errors calculated with respect to $H1$ norm and L^2 norm also decreases with reducing mesh size h. As the method converges for the bilinear form $a_h^1(\cdot,\cdot)$ we have not shown the Figure for the sake of brevity.

Numerical errors calculated for the bilinear form $a_h^2(\cdot,\cdot)$ were given in the Table 4.8. For decreasing values of h the error almost remains constant with respect to $|||\cdot|||$ norm and error increases with respect to H^1 and L^2 norm with negative rate of convergence. We can also observe this behaviour from the oscillations present in Figure 4.5.

We have observed that SUPG discretization with the proposed nonconforming element performs better by capturing non-physical oscillation and also maintains the accuracy. As pointed out in the introduction we would like to develop the SUPG formulation in the context of VEM by suitably defining the stabilizers with the help of projection operators Π_k^0 and Π_k^∇. In the following sections we introduce the evolutionary convection diffusion equation and perform the numerical analysis using SUPG stabilizers along with

Table 4.6
Errors for the P_2 conforming element for Example 2

h	ϵ	$\|\cdot\|_{0,\Omega}$	$\lvert\cdot\rvert_{1,h}$
0.0707	$1e-11$	0.677312	4.02011
0.0354	$1e-11$	0.673772	3.98098
0.0177	$1e-11$	0.672879	3.97193
0.0088	$1e-11$	0.672878	3.97192

Table 4.7
Errors for $a_h^1(\cdot,\cdot)$ using $NC1-C2$ element for Example 2

	$\lvert\lvert\lvert\cdot\rvert\rvert\rvert$		$\|\cdot\|_{1,h}$		$\|\cdot\|_{0,\Omega}$	
h	Error	Rate	Error	Rate	Error	Rate
0.0707	1.88919	-	$1.04e+2$	-	1.2618	-
0.0354	1.06827	0.8242	$1.47e+2$	-0.4977	0.8534	0.5658
0.0177	0.46228	1.2084	$1.45e+2$	0.0143	0.4051	1.0747
0.0088	0.14818	1.6387	$1.00e+2$	0.5435	0.1354	1.5779

Table 4.8
Errors for $a_h^2(\cdot,\cdot)$ using $NC1-C2$ element for Example 2

	$\lvert\lvert\lvert\cdot\rvert\rvert\rvert$		$\|\cdot\|_{1,h}$		$\|\cdot\|_{0,\Omega}$	
h	Error	Rate	Error	Rate	Error	Rate
0.0707	0.9670	-	$4.67e+1$	-	0.4496	-
0.0354	0.8648	0.1614	$1.12e+2$	-1.2685	0.5315	-0.2419
0.0177	0.7871	0.1358	$2.38e+2$	-1.0833	0.5583	-0.0709
0.0088	0.7221	0.1241	$4.93e+2$	-1.0468	0.5749	-0.0423

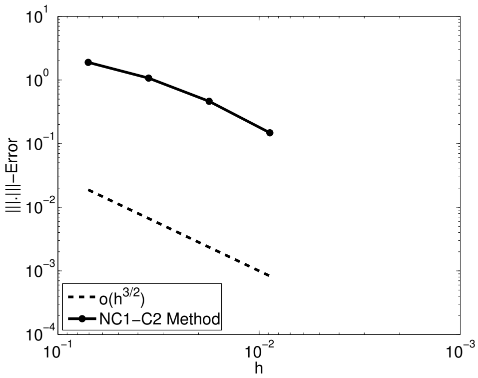
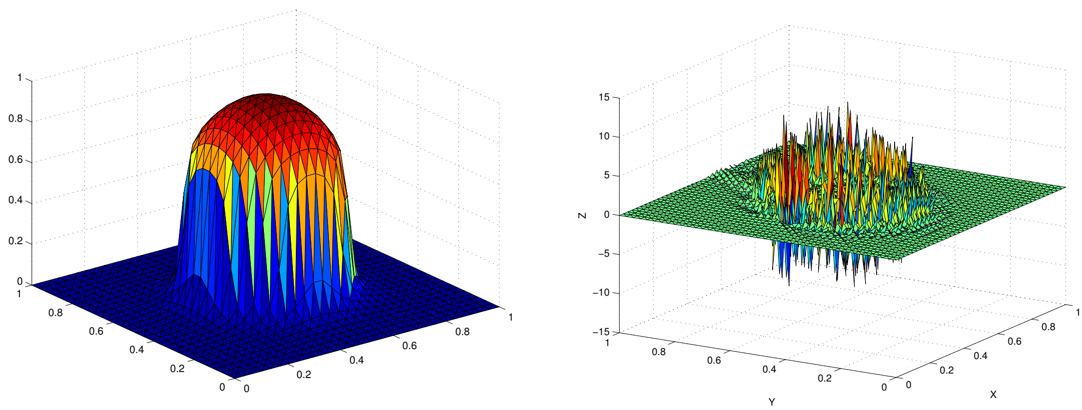


Figure 4.4: Log-log plot of the numerical error for the bilinear form $a_h^1(\cdot,\cdot)$ for Example 2



(a) Exact solution

(b) Numerical solution

Figure 4.5: Exact solution and oscillation of numerical solution for Example 2

Crank-Nicholson scheme for temporal discretization. We provide the error estimates and supporting numerical experiments to validate the theoretical results.

4.6 Continuous problem and weak formulation

We consider the following time-dependent convection diffusion reaction equation

$$\begin{cases} u_t - \nabla\cdot(\kappa(\mathbf{x})\nabla u) + \mathbf{b}(\mathbf{x})\cdot\nabla u + c(\mathbf{x})\,u = f(\mathbf{x},t) & \text{in} \quad \Omega, \quad \text{for } t\in(0,T), \\ u = 0 & \text{on } \Gamma=\partial\Omega, \quad \text{for } t\in(0,T), \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{4.23}$$

where $\kappa(\mathbf{x})$,$\mathbf{b}(\mathbf{x})$,$c(\mathbf{x})$ are $L^\infty(\Omega)$ functions of $\mathbf{x}$. Furthermore, we assume that $\nabla\cdot\mathbf{b}=0$ and all coefficients $\kappa(\mathbf{x})$,$\mathbf{b}(\mathbf{x})$, $c(\mathbf{x})$ are independent of temporal variable t . Also we consider $\inf_{\mathbf{x}\in\Omega} \kappa(\mathbf{x}) \geq \hat{K} > 0$, $\sup_{\mathbf{x}\in\Omega} \kappa(\mathbf{x}) \leq K_0,\ \mathcal{B} = \sup_{\mathbf{x}} \|\mathbf{b}(\mathbf{x})\|_{(L^2(\Omega))^2},\ \nu_0 = \sup_{\mathbf{x}\in\Omega} c(\mathbf{x}),\ \hat{\nu} = \inf_{\mathbf{x}\in\Omega} c(\mathbf{x}) > 0$. Multiplying the Equation (4.23) by test function v and exploiting Green's theorem, we obtain the continuous variational formulation. In order to reduce cumbersome notation, we introduce following notation. We represent the following bilinear forms

$$\begin{cases} A_1(u,v) := (\kappa\nabla u, \nabla v) + (cu, v) + \sum_K \delta_K(\mathbf{b}\cdot\nabla u, \mathbf{b}\cdot\nabla v)_K, \\ A_2(u,v) := (\mathbf{b}\cdot\nabla u, v), \\ A_3(u,v) := \sum_K \delta_K(-\nabla\cdot(\kappa\nabla u) + cu, \mathbf{b}\cdot\nabla v)_K. \end{cases}$$

Summing $A_1(\cdot,\cdot)$, $A_2(\cdot,\cdot)$ and $A_3(\cdot,\cdot)$, we define $\mathcal{A}_{\text{supg}}(u,v) = A_1(u,v) + A_2(u,v) + A_3(u,v)$. Moreover, the right hand side load term is defined as $F(v) := (f,v) + \sum_K \delta_K(f, \mathbf{b}\cdot\nabla v)_K$. Therefore the continuous formulation is defined as

$$(u_t, v) + \sum_K \delta_K(u_t, \mathbf{b}\cdot\nabla v)_K + \mathcal{A}_{\text{supg}}(u,v) = F(v). \tag{4.24}$$

4.7 The virtual element framework

Following (1.16) and (1.17), we first recollect the definition of local and global virtual element space $\mathcal{Z}^k(K)$ and $\mathcal{Z}_h^k$ respectively, where k denotes order of virtual element space and K represents polygon respectively. We have already mentioned that in VEM, we discretize the bilinear form using two projection operators $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ which are defined in equations(1.5) and (1.3) respectively in chapter 1. In previous two chapters, we approximate the bilinear form $(\nabla u, \nabla v)_K$ employing projection operator $\Pi^\nabla_{k,K}$. However, the same bilinear form can be approximated using vector valued external L^2 projection operator $\Pi^0_{k-1,K}$. We define the projection operator $\Pi^0_{k-1,K}$ (we exploit same notation as scalar valued L^2 function.) component-wise, such that

$$\Big(\Pi^0_{k-1,K}\nabla\phi_i, \mathbf{m}_\alpha\Big)_K = \int_K \nabla\phi_i\cdot\mathbf{m}_\alpha = \int_{\partial K}(\mathbf{m}_\alpha\cdot\mathbf{n})\phi_i\,ds - \Big(\phi_i, \nabla\cdot\mathbf{m}_\alpha\Big)_K \tag{4.25}$$

holds. We decompose the polygonal domain Ω satisfying Assumption 1 which is described in detail in Chapter 2.

4.7.1 Discrete virtual element formulation

On the virtual element space $\mathcal{Z}^k(K)$, we represent the following bilinear forms

$$\begin{cases} A_{1,h}(u_h, v_h) := \sum_K \Big(a_h^K(u_h, v_h) + b_h^K(u_h, v_h) + d_h^K(u_h, v_h) \Big). \\ A_{2,h}(u_h, v_h) := \sum_K (\mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla u_h), \Pi^0_{k-1,K}(v_h))_K. \\ A_{3,h}(u_h, v_h) := \sum_K \delta_K \Big(-\nabla \cdot (\kappa \Pi^0_{k-1,K} \nabla u_h) + c\Pi^0_{k,K}(u_h), \mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla v_h) \Big)_K, \end{cases} \tag{4.26}$$

where symmetric discrete bilinear forms are defined as

$$\begin{cases} a_h^K(u_h, v_h) := (\kappa \Pi^0_{k-1,K} \nabla u_h, \Pi^0_{k-1,K} \nabla v_h)_K + K_0 \Big(\nabla (I - \Pi^\nabla_{k,K} u_h), \nabla (I - \Pi^\nabla_{k,K} v_h) \Big)_K, \\ b_h^K(u_h, v_h) := \delta_K (\mathbf{b} \cdot \Pi^0_{K,k-1}(\nabla u_h), \mathbf{b} \cdot \Pi^0_{K,k-1}(\nabla v_h)) \\ \qquad\qquad + \delta_K \mathcal{B}^2 \Big(\nabla (I - \Pi^\nabla_{k,K} u_h), \nabla (I - \Pi^\nabla_{k,K} v_h) \Big)_K, \\ d_h^K(u_h, v_h) := (c\Pi^0_{k,K} u_h, \Pi^0_{k,K} v_h)_K + |K| \nu_0 \Big((I - \Pi^0_{k,K}) u_h, (I - \Pi^0_{k,K}) v_h \Big). \end{cases}$$

The global forms are defined by summing local contributions on each polygon K such as

$$a_h(u_h, v_h) := \sum_K a_h^K(u_h, v_h);\ b_h(u_h, v_h) := \sum_K b_h^K(u_h, v_h);\ d_h(u_h, v_h) := \sum_K d_h^K(u_h, v_h).$$

Furthermore, the discrete load term is defined as $F_h(v_h) := (f, \Pi^0_k v_h) + \sum_K \delta_K (f, \mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla v_h))_K$, where Π^0_k is designed as $\Pi^0_k|_K(u_h) := \Pi^0_{k,K}(u_h)$.

In order to make the notation simple, we represent $\mathcal{A}_{\text{supg},h}(u_h, v_h) := A_{1,h}(u_h, v_h) + A_{2,h}(u_h, v_h) + A_{3,h}(u_h, v_h)$.

Discrete virtual element formulation for the model problem (4.23) is defined as

$$m_h(u_{h,t}, v_h) + \mathcal{A}_{\text{supg},h}(u_h, v_h) + \sum_K \delta_K (\Pi^0_{k,K} u_{h,t}, \mathbf{b} \cdot \Pi_{k-1,K} \nabla v_h) = F_h(v_h). \tag{4.27}$$

Furthermore, we define mesh dependent SUPG norm as

$$|||v|||^2 = \Big(\|\sqrt{\kappa} \nabla v\|^2 + \sum_K \delta_K \|\mathbf{b} \cdot \nabla v\|_K^2 + \|\sqrt{c}\, v\|^2 \Big), \tag{4.28}$$

where δ_K represents stabilization parameter which will be defined in next section.

4.7.2 Discrete inf-sup condition

In order to prove discrete inf-sup condition, we first prove the following result.

Lemma 22. *Let $\omega \in H_0^1(\Omega)$ be an arbitrary element. Then there exists $\hat{\omega} \in \mathcal{Z}_h^k$ such that*

$$A_{1,h}(\hat{\omega}, z_h) = A_1(\omega, z_h) \quad \forall\, z_h \in \mathcal{Z}_h^k.$$

Moreover, there exist positive constants C_0 and C_1, such that

$$|||\hat{\omega}||| \leq \ C_0\, |||\omega|||,$$

and

$$\|\omega - \hat{\omega}\| \leq \ C_1\, h\, |||\omega|||.$$

Proof. In order to prove the above result, most of tools are borrowed form (Lemma 5 [18], Lemma 5.6 [19]). ■

Lemma 23. *For all $v_h \in \mathcal{Z}_h^k$ and for sufficiently finer mesh, the following estimation hold*

$$\alpha_0\, |||v_h||| \leq \sup_{z_h \in \mathcal{Z}_h^k} \frac{\mathcal{A}_{supg,h}(v_h, z_h)}{|||z_h|||}$$

for some positive constant α_0.

Proof. Proof follows same technique as [18] (Theorem 1) ■

Next, we shall prove the following two inverse estimations which are accomplished as the main tools in order to prove Theorem 28 considering mesh regularity Assumption 1. Moreover, we know that discrete virtual element space $\mathcal{Z}^k(K)$ is not associated with particular shaped element. Hence, in contrast with FEM, we can not employ standard scaling argument to prove inverse estimation. Following idea form [27], making use of generalize scaling argument, we can justify the following result.

Lemma 24. *Let v_h be an element of $\mathcal{Z}^k(K)$. Then there exists a constant c_K^{inv} independent of h_K such that*

$$|v_h|_{1,K} \leq \ c_K^{inv}\, h_K^{-1}\, \|v_h\|_{0,K} \qquad \forall K \in \mathcal{T}_h. \tag{4.29}$$

Proof. See in detail Lemma 4.5 [27]. ■

Furthermore, exploiting the characteristic of local virtual element space $\mathcal{Z}^k(K)$ that $\Delta q \in \mathbb{P}_k(K)$ for all $q \in H^1(K)$ and considering the mesh regularity Assumption 1, we can exhibit the result.

Lemma 25. *Let v_h be an arbitrary element of $\mathcal{Z}^k(K)$ and $K \in \mathcal{T}_h$ be a polygonal element. Then there exists a constant $c_{inv,K}$ independent of mesh size h_K, such that*

$$\|\nabla \cdot (\kappa(\mathbf{x})\nabla v_h)\|_K^2 \leq \; c_{inv,K}^2 \; h_K^{-2} \; \|\kappa(\mathbf{x}) \; \nabla v_h\|_K^2 \tag{4.30}$$

Proof. See in detail Lemma 10[26]. ■

Moreover, the symmetric bilinear form $A_{1,h}(u_h, v_h)$ associated with the discrete bilinear form is coercive, i.e., there exists a positive constant c_α such that the following thesis holds.

Lemma 26. *Let u_h be an element of $\mathcal{Z}_h^k$ and $A_{1,h}(\cdot,\cdot)$ be the symmetric bilinear form defined in (4.26). Then there exists a constant c_α depending on regularity of coefficients $\kappa(\mathbf{x}), \mathbf{b}(\mathbf{x}), c(\mathbf{x})$ such that*

$$A_{1,h}(u_h, u_h) \geq \; c_\alpha \; |||u_h|||^2. \tag{4.31}$$

Here, we leave the proof since it follows directly from [18] with minor modification.

4.8 Convergence analysis

In order to accomplish convergence analysis, we first introduce elliptic projection operator $R^h : V^0 := \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\} \to \mathcal{Z}_h^k$ associated with stationary bilinear form $\mathcal{A}_{\text{supg},h}(\cdot,\cdot)$. The projection operator is defined by

$$\mathcal{A}_{\text{supg},h}(R^h u, v_h) = \mathcal{A}_{\text{supg}}(u, v_h) \quad \forall \, v_h \in \mathcal{Z}_h^k. \tag{4.32}$$

An application of discrete inf-sup condition of $\mathcal{A}_{\text{supg},h}(\cdot,\cdot)$ together with continuity of $\mathcal{A}_{\text{supg}}(u,\cdot)$ ensures existence and uniqueness of $R^h u$.

4.8.1 Semi-discrete error estimation in SUPG norm

Lemma 27. *Let u be an element of V^0. Then there exist an element $R^h u$ in discrete space $\mathcal{Z}_h^k$ such that the following estimation holds*

$$|||u - R^h u||| \leq \; C \; h^k \; |u|_{k+1}. \tag{4.33}$$

Proof. We split the term $u - R^h u$ using u_I as

$$u - R^h u = u - u_I + u_I - R^h u.$$

Let $\rho = R^h u - u_I$. Exploiting standard approximation property of interpolation operator u_I [9], we can estimate

$$|||u - u_I||| \leq C\, h^{2k}.$$

In order to estimate $|||\rho|||$, we proceed as follows. since $\rho \in \mathcal{Z}_h^k$, there exists an element $w_h \in \mathcal{Z}_h^k$ such that

$$\begin{aligned} |||\rho|||\, |||w_h||| &\leq C\, \mathcal{A}_{\text{supg},h}(R^h u - u_I, w_h) \\ &= \mathcal{A}_{\text{supg},h}(R^h u, w_h) - \mathcal{A}_{\text{supg},h}(u_I, w_h) \\ &= \mathcal{A}_{\text{supg}}(u, w_h) - \mathcal{A}_{\text{supg},h}(u_I, w_h). \end{aligned} \tag{4.34}$$

An application of orthogonal property of projection operator $\Pi_{k,K}^0$ helps us to write

$$\begin{aligned} \mathcal{A}_{\text{supg}}(u, w_h) - \mathcal{A}_{\text{supg},h}(u_I, w_h) &= \sum_K \mathcal{A}_{\text{supg}}^K(u, w_h) - \mathcal{A}_{\text{supg},h}^K(u_I, w_h) \\ = \sum_K \Big(&- \mathcal{A}_{\text{supg},h}^K(u_I - \Pi_{k,K}^0 u, w_h) - \mathcal{A}_{\text{supg}}^K(\Pi_{k,K}^0 u - u, w_h) \\ &+ \mathcal{A}_{\text{supg}}^K(\Pi_{k,K}^0 u, w_h) - \mathcal{A}_{\text{supg},h}^K(\Pi_{k,K}^0 u, w_h)\Big). \end{aligned} \tag{4.35}$$

Since the discrete virtual element formulation is not polynomial consistency, error associated with discrete formulation can be bounded as

$$\mathcal{A}_{\text{supg}}^K(\Pi_{k,K}^0 u, w_h) - \mathcal{A}_{\text{supg},h}^K(\Pi_{k,K}^0 u, w_h) = \sum_{i=1}^{3} \Big(A_i^K(\Pi_{k,K}^0 u, w_h) - A_{i,h}^K(\Pi_{k,K}^0 u, w_h)\Big). \tag{4.36}$$

In order to estimate $A_1^K(\Pi_{k,K}^0 u, w_h) - A_{1,h}^K(\Pi_{k,K}^0 u, w_h)$, we write as

$$\begin{aligned} A_1^K(\Pi_{k,K}^0 u, w_h) - A_{1,h}^K(\Pi_{k,K}^0 u, w_h) &= \Big(a^K(\Pi_{k,K}^0 u, w_h) - a_h^K(\Pi_{k,K}^0 u, w_h)\Big) \\ + \Big(b^K(\Pi_{k,K}^0 u, w_h) - b_h^K(\Pi_{k,K}^0 u, w_h)\Big) &+ \Big(d^K(\Pi_{k,K}^0 u, w_h) - d_h^K(\Pi_{k,K}^0 u, w_h)\Big). \end{aligned} \tag{4.37}$$

An application of standard property of projection operator $\Pi_{k,K}^0$ and using Cauchy-Schwarz inequality yields

$$a^K(\Pi_{k,K}^0 u, w_h) - a_h^K(\Pi_{k,K}^0 u, w_h) \leq C \frac{h_K^k}{\sqrt{\hat{K}}}\, |u|_{k+1,K}\, \|\sqrt{\kappa}\nabla w_h\|_{0,K}. \tag{4.38}$$

An similar argument indicate

$$b^K(\Pi^0_{k,K}u, w_h) - b^K_h(\Pi^0_{k,K}u, w_h) \leq C \; \frac{b_E^2}{\sqrt{\hat{K}}} \; h^k_K \; |u|_{k+1,K} \; \|\sqrt{\kappa}\nabla w_h\|_{0,K}. \tag{4.39}$$

Proceeding analogously, we can bound

$$d^K(\Pi^0_{k,K}u, \rho) - d^K_h(\Pi^0_{k,K}u, \rho) \leq C \; h^k_K \; |u|_{k+1,K} \; \|w_h\|_{0,K}. \tag{4.40}$$

Summing estimations (4.38), (4.39), (4.40) element-wise and an application of Cauchy-schwarz inequality, yields

$$A_1(\Pi^0_{k,K}u, w_h) - A_{1,h}(\Pi^0_{k,K}u, w_h) \leq \; C \; h^k \; |u|_{k+1} \; |||w_h|||. \tag{4.41}$$

Skew-symmetric part is not polynomial consistency but error decreases with optimal order with $O(h^k)$. Using boundedness of convective coefficients $\mathbf{b}$, approximation property and orthogonal property of L^2 projection operator $\Pi^0_{k,K}$ and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} &A^K_2(\Pi^0_{k,K}u, w_h) - A^K_{2,h}(\Pi^0_{k,K}u, w_h) \leq \|\mathbf{b}\cdot\nabla\Pi^0_{k,K}u - \Pi^0_{k,K}(\mathbf{b}\cdot\nabla u)\| \; \|w_h\| \\ &\leq \Big(\|\mathbf{b}\cdot\nabla\Pi^0_{k,K}u - \mathbf{b}\cdot\nabla u\| + \|\mathbf{b}\cdot\nabla u - \Pi^0_{k,K}(\mathbf{b}\cdot\nabla u)\|\Big) \; \|w_h\| \leq C \; h^k \; |u|_{k+1} \; \|w_h\|. \end{aligned} \tag{4.42}$$

Hence summing over all polygon K and using Cauchy-Schwarz inequality, we have

$$\sum_K \Big(A^K_2(\Pi^0_{k,K}u, w_h) - A^K_{h,2}(\Pi^0_{k,K}u, w_h)\Big) \leq C \; h^k \; |u|_{k+1} \; \|w_h\|. \tag{4.43}$$

Now, the error generated by additional stabilizer term can be bounded as follows

$$\begin{aligned} A^K_3(\Pi^0_{k,K}u, w_h) - A^K_{h,3}(\Pi^0_{k,K}u, w_h) &= \delta_K \; \Big(-\nabla\cdot(\kappa(\mathbf{x})\nabla\Pi^0_{k,K}u) + c(\mathbf{x}) \; \Pi^0_{k,K}u, \mathbf{b}\cdot\nabla\rho\Big) \\ &\quad - \delta_K \; \Big(-\nabla\cdot(\kappa(\mathbf{x}) \; \nabla\Pi^0_{k,K}u) + c(\mathbf{x}) \; \Pi^0_{k,K}u, \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla w_h)\Big). \end{aligned} \tag{4.44}$$

An approximation property of local L^2 projection operator $\Pi^0_{k,K}$ and exploiting regularity of convective coefficient $\mathbf{b}(\mathbf{x})$ and reaction coefficient $c(\mathbf{x})$, we have

$$
\begin{aligned}
&(c\,\Pi^0_{k,K}u, \mathbf{b}\cdot\nabla w_h) - (c\,\Pi^0_{k,K}u, \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla w_h))\\
&= (c\,\Pi^0_{k,K}u, \mathbf{b}\cdot\nabla w_h - \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla w_h))\\
&= \sum_{i=1}^{2}(c\,\Pi^0_{k,K}u, b_i w_{h,i} - b_i\Pi^0_{k-1,K}w_{h,i}) \le \sum_{i=1}^{2}\|(I-\Pi^0_{k-1,K})(c\,b_i\,\Pi^0_{k,K}u)\|\,\|w_{h,i}\|\\
&\le \sum_{i=1}^{2}\Big(\|c\,b_i\,\Pi^0_{k,K}u - c\,b_i\,u\| + \|c\,b_i\,u - \Pi^0_{k-1,K}(c\,b_i\,u)\|\Big)\,\|w_{h,i}\|\\
&\le C\,h^k\,|u|_{k+1,K}\,\|\sqrt{\kappa(x)}\nabla w_h\|,
\end{aligned}
$$

where $w_{h,i}$ represents derivative of w_h w.r.t. x_i and b_i denotes component of convective coefficient $\mathbf{b}$ in i-th direction. Using orthogonality property of projection operator $\Pi^0_{k-1,K}$, the other term can be represented as

$$
\begin{aligned}
&\delta_K\,\Big(\nabla\cdot(\kappa(\mathbf{x})\nabla\Pi^0_{k,K}u), \mathbf{b}(\mathbf{x})\cdot\nabla w_h\Big)\\
&-\delta_K\Big(\nabla\cdot(\kappa(\mathbf{x})\,\Pi^0_{k-1,K}(\nabla\Pi^0_{k,K}u)), \mathbf{b}(\mathbf{x})\cdot\Pi^0_{k-1,K}(\nabla w_h)\Big)\\
&=\delta_K\,\Big(\nabla\cdot(\kappa(\mathbf{x})\nabla\Pi^0_{k,K}u), \mathbf{b}(\mathbf{x})\cdot\nabla w_h - \mathbf{b}(\mathbf{x})\cdot\Pi^0_{k-1,K}(\nabla w_h)\Big).
\end{aligned}
$$

An application of arguments given in [18](Lemma 2), helps us to estimate

$$
\begin{aligned}
&\delta_K\,\Big|\Big(\nabla\cdot(\kappa(\mathbf{x})\nabla\Pi^0_{k,K}u), \mathbf{b}(\mathbf{x})\cdot\nabla w_h - \mathbf{b}(\mathbf{x})\cdot\Pi^0_{k-1,K}(\nabla w_h)\Big)\Big|\\
&\le C\,h^k\,|u|_{k+1}\,\|\sqrt{\kappa(\mathbf{x})}\nabla w_h\|.
\end{aligned}
$$

Hence summing over all element K, we have

$$
\sum_K\Big(A_3^K(\Pi^0_{k,K}u, w_h) - A_{3,h}^K(\Pi^0_{k,K}u, w_h)\Big) \le C\,h^k\,|u|_{k+1}\,|||w_h|||. \tag{4.45}
$$

In order to estimate $\mathcal{A}_{\text{supg},h}(u_I - \Pi^0_{k,K}u, w_h)$, we reveal the bilinear form into constituents as

$$
\begin{aligned}
\mathcal{A}_{\text{supg},h}(u_I - \Pi^0_{k,K}u, w_h) &= A_{1,h}(u_I - \Pi^0_{k,K}u, w_h) + A_{2,h}(u_I - \Pi^0_{k,K}u, w_h)\\
&\quad + A_{3,h}(u_I - \Pi^0_{k,K}u, w_h).
\end{aligned} \tag{4.46}
$$

Symmetric bilinear form $A_{1,h}(u_I - \Pi^0_{k,K}u, w_h)$ can be approximated as

$$
\begin{aligned}
&A_{1,h}(u_I - \Pi^0_{k,K}u, w_h) = \sum_K A_{1h}^K(u_I - \Pi^0_{k,K}u, w_h)\\
&= a_h^K(u_I - \Pi^0_{k,K}u, w_h) + b_h^K(u_I - \Pi^0_{k,K}u, w_h) + d_h^K(u_I - \Pi^0_{k,K}u, w_h).
\end{aligned}
$$

An application of Cauchy-Schwarz inequality and stability of discrete bilinear form yields

$$
\begin{aligned}
a_h^K(u_I - \Pi^0_{k,K}u, w_h) &\le \Big(a_h^K(u_I - \Pi^0_{k,K}u, u_I - \Pi^0_{k,K}u)\Big)^{1/2} \, (a_h^K(w_h, w_h))^{1/2} \\
&\le C\, a^K(u_I - \Pi^0_{k,K}u, u_I - \Pi^0_{k,K}u)^{1/2}\, a^K(w_h, w_h)^{1/2} \le C\, h_K^k\, |u|_{k+1,K}\, \|\sqrt{\kappa(\mathbf{x})}\nabla w_h\|_K.
\end{aligned} \tag{4.47}
$$

Using analogous argument as (4.47), we can bound

$$
b_h^K(u_I - \Pi^0_{k,K}u, w_h) \le C\, h_K^k\, |u|_{k+1,K}\, \|\sqrt{\kappa(\mathbf{x})}\nabla w_h\|_K. \tag{4.48}
$$

and

$$
d_h^K(u_I - \Pi^0_{k,K}u, \rho) \le C\, h_K^{k+1}\, |u|_{k+1,K}\, \|c(\mathbf{x}) w_h\|. \tag{4.49}
$$

In view of (4.47),(4.48), (4.49) and summing over all element K, we have

$$
\sum_K A_{1,h}^K(u_I - \Pi^0_{k,K}u, w_h) \le C\, h_K^k\, |u|_{k+1}\, |||w_h|||. \tag{4.50}
$$

Skew-symmetric term can be exhibited as

$$
\sum_K A_{2,h}^K(u_I - \Pi^0_{k,K}u, w_h) = \sum_K (\mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla(u_I - \Pi^0_{k,K}u)), \Pi^0_{k-1,K} w_h)_K.
$$

Using standard approximation property and boundedness of L^2 projection operator $\Pi^0_{k-1,K}$ and Cauchy-schwarz inequality, we have

$$
(\mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla(u_I - \Pi^0_{k,K}u)), \Pi^0_{k,K}(w_h)) \le C\, h_K^k\, |u|_{k+1,K}\, \|w_h\|_K. \tag{4.51}
$$

Hence, summing over all element K, we have

$$
A_{2,h}(u_I - \Pi^0_{k,K}u, w_h) \le C\, h^k\, |u|_{k+1}\, |||w_h|||. \tag{4.52}
$$

The last term of (4.46) can be estimated as

$$
\begin{aligned}
\sum_K A_{3,h}^K(u_I - \Pi^0_{k,K}u, w_h) = \sum_K \delta_K \Big(&- \nabla \cdot (\kappa(\mathbf{x}) \Pi^0_{k-1,K}(\nabla(u_I - \Pi^0_{k,K}u))) \\
&+ c(\mathbf{x})\, \Pi^0_{k-1,K}(u_I - \Pi^0_{k,K}u), \mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla w_h)\Big).
\end{aligned} \tag{4.53}
$$

Using the boundedness property of projection operator $\Pi^0_{k-1,K}$, regularity of convective and reaction coefficient $\mathbf{b}(\mathbf{x})$, $c(\mathbf{x})$ and Cauchy-schwarz inequality, we can approximate

$$\Big(c\,\Pi^0_{k-1,K}(u_I - \Pi^0_{k,K}u), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla w_h)\Big) \leq C\,h^{k+1}\,|u|_{k+1,K}\|\sqrt{\kappa(\mathbf{x})}\nabla w_h\|_K. \tag{4.54}$$

Using $\delta_K \leq C\,h_K$, and inverse estimation Lemma 25, we can bound the term as

$$\begin{aligned}\delta_K\,\Big(-\nabla\cdot(\kappa(\mathbf{x})\Pi^0_{k-1,K}(\nabla(u_I - \Pi^0_{k,K}u))), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla w_h)\Big)\\ \leq\; C\,h_K^k\,|u|_{k+1,K}\,\|\sqrt{\kappa(\mathbf{x})}\nabla w_h\|_K.\end{aligned}$$

Exploiting the estimations (4.54) and (4.8.1) and summing over all polygon K, we have

$$A_{3,h}(u_I - \Pi^0_{k,K}u, w_h) \leq\; C\,h^k\,|u|_{k+1}\,|||w_h|||. \tag{4.55}$$

Collecting (4.50),(4.52) and (4.55) and putting in(4.46) we have

$$\mathcal{A}_{\text{supg},h}(u_I - \Pi^0_{k,K}u, w_h) \leq\; C\,h^k\,|u|_{k+1}\,|||w_h|||. \tag{4.56}$$

Using analogous argument as (4.56), we can estimates

$$\mathcal{A}_{\text{supg}}(\Pi^0_{k,K}u - u, w_h) \leq\; C\,h^k\,|u|_{k+1}\,|||w_h|||. \tag{4.57}$$

Collecting(4.45),(4.56) and (4.57) and putting in(4.35), we get

$$|||\rho||| \leq\; C\,h^k\,|u|_{k+1}. \tag{4.58}$$

Hence, we have final thesis

$$|||u - R^h u||| \leq\; C\,h^k\,|u|_{k+1}. \tag{4.59}$$

■

Let u be exact solution of the model problem (4.24), and u_h be the discrete solution of (4.27). With the help of elliptic projection operator R^h, we can split the error in this fashion

$$u - u_h = u - R^h u + R^h u - u_h.$$

Using the approximation property of projection operator R^h(define in 4.32), we can easily bound the term $u - R^h u$. Since $R^h u$ and u_h both are elements of $\mathcal{Z}^k_h$ hence, $R^h u - u_h$

is an element of $\mathcal{Z}_h^k$. Define $\theta(t) := u_h(t) - R^h u(t)$. Moreover, the boundedness of $\theta(t)$ depends on stabilization parameter δ which depend on regularity of $\kappa(\mathbf{x})$,$\mathbf{b}(\mathbf{x})$ and $c(\mathbf{x})$. In contrast with stationary case [18], we assume the following Assumption

Assumption 2.

- *We consider uniform polygonal mesh with local mesh size h.*
- *The stabilization parameter $\delta_K = \delta$ for all element K.*

Stabilization parameter δ can be chosen as

$$\delta = \min\left\{1, \min\left\{\frac{\hat{\nu}\, c_\alpha}{2\,\nu_0^2}, \frac{\hat{K}\,\min\{c_\alpha\,\alpha_*, c_\alpha\}}{48\,\mathcal{B}^2}, \frac{\hat{K}\,h\,c_\alpha}{6\,K_0\,c_{\text{inv}}\,\mathcal{B}}, \frac{\hat{K}\,h^2\,c_\alpha}{3\,K_0^2\,c_{\text{inv}}^2}, \frac{h\,\sqrt{\alpha_*}}{\sqrt{6}\,\mathcal{B}\,c^{\text{inv}}}, \frac{h\,\alpha_*}{6\,\mathcal{B}}\right\}\right\}. \tag{4.60}$$

Here, we have considered $c_{\text{inv}} = \sup_{K\in\mathcal{T}_h}(c_{\text{inv},K})$ and $c^{\text{inv}} = \sup_{K\in\mathcal{T}_h}(c_K^{\text{inv}})$.

Theorem 28. *Let $u \in V^0$ be a solution of (4.24) and u_h be the discrete solution of (4.27). Also assume that $u_h(\cdot,0) = u(\cdot,0)$. Then for sufficiently small h, the following error estimation hold*

$$\begin{aligned}\|u(t) - u_h(t)\| + \|u - u_h\|_{L^2(0,t,|||\cdot|||)} \leq\ & C\,h^k\,\Big(\|f\|_{L^2(0,T,H^k(\Omega))} + |f_t|^2_{L^2(0,T,H^k(\Omega))} \\ & + \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|u_t\|_{L^(0,t,H^k(\Omega))} + \|u_{tt}\|^2_{L^2(0,T,H^k(\Omega))}\Big).\end{aligned}$$

Proof. $|||u - R^h u|||$ can be estimated from lemma(27). In order to estimate the other term $|||\theta(t)|||$, we proceed as follows: replacing u_h by θ, the semi-discrete bilinear form reduces to the following form

$$\begin{aligned}& m_h(\theta_t, v_h) + \mathcal{A}_{\text{supg},h}(\theta, v_h) + \sum_K \delta\,(\Pi^0_{k,K}\theta_t, \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla v_h))_K = F_h(v_h) \\ & - m_h(R^h u_t, v_h) - \mathcal{A}_{\text{supg},h}(R^h u(t), v_h) - \sum_K \delta(\Pi^0_{k,K}R^h u_t, \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla v_h).\end{aligned} \tag{4.61}$$

An application of projection operator R^h and using the continuous bilinear form (4.24), we can write

$$\begin{aligned}& m_h(\theta_t, v_h) + \mathcal{A}_{\text{supg},h}(\theta, v_h) + \sum_K \delta\,(\Pi^0_{k-1,K}\theta_t, \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla v_h))_K \\ & = F_h(v_h) - F(v_h) - m_h(R^h u_t, v_h) + (u_t, v_h) \\ & \quad - \sum_K \delta(\Pi^0_{k,K}R^h u_t, \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla v_h) + \sum_K \delta\,(u_t, \mathbf{b}\cdot\nabla v_h).\end{aligned} \tag{4.62}$$

Inserting first $v_h = \theta$ in (4.62) then $v_h = \delta\partial_t\theta$ exploiting Lemma 26 and adding both equations yields

$$
\begin{aligned}
&\frac{1}{2}\,\partial_t\,\|\theta\|_h^2 + c_\alpha\left(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\,\theta\|^2 + \sum_K \delta\,\|\mathbf{b}\cdot\nabla\theta\|^2\right) + \delta\,\|\partial_t\theta\|_h^2 \\
&\quad + c_\alpha\,\delta\,\frac{1}{2}\,\partial_t\Big(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\,\theta\|^2 + \sum_K \delta\,\|\mathbf{b}\cdot\nabla\theta\|^2\Big) \\
&+ \sum_K (\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta), \Pi^0_{k-1,K}(\theta + \delta\,\partial_t\,\theta))_K \\
&\quad + \sum_K \delta\Big(-\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta) + c\Pi^0_{k,K}(\theta), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta + \nabla\,\delta\,\partial_t\,\theta)\Big)_K \\
&+ \sum_K \delta\Big(\Pi^0_{k,K}\partial_t\theta, \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta + \nabla(\delta\,\partial_t\theta))\Big) \\
&\le F_h(\theta + \delta\,\partial_t\,\theta) - F(\theta + \delta\,\partial_t\,\theta) - m_h(R^h u_t, \theta + \delta\,\partial_t\,\theta) + (u_t, \theta + \delta\,\partial_t\,\theta) \\
&\quad - \sum_K \delta(\Pi^0_{k,K}R^h u_t, \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla\theta + \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla(\delta\,\partial_t\,\theta))_K \\
&+ \sum_K \delta\,(u_t, \mathbf{b}\cdot\nabla\theta + \mathbf{b}\cdot\nabla(\delta\,\partial_t\,\theta))_K.
\end{aligned}
\tag{4.63}
$$

With the help of the assumption $\nabla\cdot\mathbf{b} = 0$, the following term can be bounded as follows

$$
\begin{aligned}
\sum_K (\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta), \Pi^0_{k-1,K}\theta)_K &= \sum_K \Big((\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta), \Pi^0_{k-1,K}\theta)_K - (\mathbf{b}\cdot\nabla\theta, \theta)_K\Big) \\
&= \sum_K \Big((\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta), \Pi^0_{k-1,K}\theta - \theta)_K + (\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta) - \mathbf{b}\cdot\nabla\theta, \theta)_K\Big) \\
&\le 2\,\frac{\hat{B}}{\hat{K}}\,h\,\|\sqrt{\kappa}\,\nabla\theta\|^2,
\end{aligned}
\tag{4.64}
$$

where $\hat{B} := \|\mathbf{b}(\mathbf{x})\|_{W^1_\infty(\Omega)}$. This term can absorbed by choosing mesh size sufficiently small.

Using Cauchy-schwarz inequality and exploiting control parameter δ (4.60), the term associated with $\delta\,\partial_t\,\theta$, can be approximated as

$$
\begin{aligned}
\sum_K (\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta), \Pi^0_{k-1,K}(\delta\,\partial_t\theta)) &\le \sum_K \mathcal{B}\,\|\nabla\theta\|_{0,K}\,\delta\,\|\delta_t\theta\|_{0,K} \\
&\le \sum_K \frac{\delta^{1/2}\,\mathcal{B}}{\sqrt{\hat{K}}}\,\|\sqrt{\kappa}\,\nabla\theta\|_{0,K}\,\delta^{1/2}\,\|\partial_t\,\theta\| \le \frac{c_\alpha}{16}\,\|\sqrt{\kappa}\nabla\theta\|^2 + \frac{\delta\,\alpha_*}{10}\,\|\partial_t\theta\|^2.
\end{aligned}
\tag{4.65}
$$

The inconsistency term associated with discrete bilinear form can be bound as follows

$$\sum_K \delta\Big(-\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta) + c\Pi^0_{k,K}(\theta), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta + \nabla\,\delta\,\partial_t\,\theta)\Big)_K. \tag{4.66}$$

Using inverse inequality stated in lemma-(25) and an application of upper bound of δ, we can bound

$$\begin{aligned}
&\sum_K \delta\,\Big(-\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta)\Big)_K \\
&\quad\le \sum_K \delta\,\|\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta)\|\,\|\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta)\| \\
&\quad\le \sum_K \delta\, c_{\mathrm{inv},K}\, h^{-1}\,(K_0/\hat{K})\,\mathcal{B}\,\|\sqrt{\kappa}\,\nabla\theta\|^2 \le \frac{c_\alpha}{6}\,\|\sqrt{\kappa}\,\nabla\theta\|^2.
\end{aligned} \tag{4.67}$$

The term associated with reaction can be estimated as

$$\begin{aligned}
\sum_K \delta\,\Big(c\,\Pi^0_{k,K}(\theta), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta)\Big)_K &\le \sum_K \delta\,\|c\,\Pi^0_{k,K}(\theta)\|\,\|\mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\theta)\| \\
&\le \sum_K \delta\,(\nu_0/\sqrt{\hat{\nu}})\,\|\sqrt{c}\theta\|\,(\mathcal{B}/\sqrt{\hat{K}})\,\|\sqrt{\kappa}\nabla\theta\| \le \frac{c_\alpha}{96}\,\|\sqrt{\kappa}\nabla\theta\|^2 + \frac{c_\alpha}{4}\,\|\sqrt{c}\theta\|^2.
\end{aligned} \tag{4.68}$$

Another term associated with $\delta\ \nabla\partial_t\theta$ can be estimated as

$$\sum_K \delta\,\Big(-\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta) + c\,\Pi^0_{k,K}(\theta), \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\delta\,\partial_t\theta)\Big)_K. \tag{4.69}$$

An application of lemma(24), lemma(25),boundedness of convection and diffusion coefficients and approximation property of projection operator $\Pi^0_{k-1,K}$, we have

$$\begin{aligned}
&\sum_K \delta\,\Big(-\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta, \mathbf{b}\cdot\Pi^0_{k-1,K}(\nabla\delta\,\partial_t\theta)\Big)_K \\
&\le \sum_K \delta^2\,\|\nabla\cdot(\kappa\Pi^0_{k-1,K}\nabla\theta)\|_0\,\mathcal{B}\,\|\nabla\theta_t\|_0 \\
&\le \sum_K \delta^{1/2}\,h^{-1}\,c_{inv,K}\,(K_0/\sqrt{\hat{K}})\,\|\sqrt{\kappa(x)}\,\nabla\theta\|_0\,\delta\,\mathcal{B}\,c_K^{\mathrm{inv}}\,h^{-1}\,\delta^{1/2}\,\|\partial_t\,\theta\| \\
&\le \frac{c_\alpha}{6}\,\|\sqrt{\kappa}\nabla\theta\|^2 + \frac{\alpha_*}{12}\,\delta\,\|\partial_t\,\theta\|^2.
\end{aligned} \tag{4.70}$$

The other term can be estimated as

$$\sum_K \delta \left(c\Pi^0_{k,K}(\theta), \delta\, \mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla \partial_t \theta)\right) \leq \sum_K \delta \,\|c\Pi^0_{k,K}(\theta)\|\, \delta\mathcal{B}\, \|\nabla \partial_t \theta\|$$
$$\leq \sum_K \delta\, (\nu_0/\sqrt{\hat{\nu}})\, \|\sqrt{c}\,\theta\|\, \delta\, \mathcal{B}\, h^{-1}\, c_K^{\text{inv}} \|\partial_t\, \theta\|$$
$$\leq \delta\, (\nu_0^2/\hat{\nu})\, \|\sqrt{c}\,\theta\|^2 + \delta^2\, \mathcal{B}^2\, h^{-2}\, (c^{\text{inv}})^2\, \delta\, \|\partial_t\, \theta\|^2 \leq \frac{c_\alpha}{4}\, \|\sqrt{c}\,\theta\|^2 + \frac{\alpha_*}{12}\, \delta\, \|\partial_t\, \theta\|^2. \tag{4.71}$$

Again with the help of boundedness of L^2 projection operators $\Pi^0_{k,K}$ and $\Pi^0_{k-1,K}$, we have

$$\sum_K \delta \left(\Pi^0_{k,K}\partial_t\theta, \mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla\theta)\right)_K \leq \sum_K \delta\, \|\partial_t\, \theta\|\, \mathcal{B}\, \|\nabla\theta\|$$
$$\leq \sum_K\, (\delta^{1/2}\, \mathcal{B}/\sqrt{\hat{K}})\, \delta^{1/2}\, \|\theta_t\|\, \|\sqrt{\kappa}\nabla\theta\| \leq \frac{c_\alpha}{16}\, \|\sqrt{\kappa}\nabla\theta\|^2 + \frac{\alpha_*}{10}\, \delta\, \|\partial_t\, \theta\|^2. \tag{4.72}$$

An application of inverse inequality(provided in lemma(24))yields

$$\begin{aligned}\sum_K \delta \left(\Pi^0_{k,K}\partial_t\theta, \mathbf{b} \cdot \Pi^0_{k-1,K}(\nabla\delta\; \partial_t\theta)\right)_K &\leq \sum_K \delta\, \|\partial_t\theta\|\, \mathcal{B}\, \|\nabla(\delta\partial_t\theta)\| \\ \leq \sum_K\, \delta\, \|\partial_t\theta\|\, \mathcal{B}\, h^{-1}\, \delta\, \|\partial_t\theta\| &\leq \frac{\alpha_*}{6}\, \delta\, \|\partial_t\, \theta\|^2.\end{aligned} \tag{4.73}$$

Hence, with the choice of δ mentioned in (4.60), the equation (4.63) reduces to the following equation

$$\begin{aligned}&\frac{1}{2}\, \partial_t\, \|\theta\|_h^2 + C_1 \left(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\,\theta\|^2 + \sum_K \delta\, \|\mathbf{b}\cdot\nabla\theta\|^2\right) + C_2\, \delta\, \|\partial_t\theta\|_0^2 \\ &\quad + c_\alpha\, \frac{\delta}{2}\, \partial_t \left(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\,\theta\|^2 + \sum_K \delta\, \|\mathbf{b}\cdot\nabla\theta\|^2\right) \\ &\leq (F_h - F)(\theta + \delta\, \partial_t\, \theta) - \sum_K\, \delta(\Pi^0_{k,K} R^h u_t, \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla\theta + \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla(\delta\, \partial_t\, \theta))_K \\ &- m_h(R^h u_t, \theta + \delta\, \partial_t\, \theta) + (u_t, \theta + \delta\, \partial_t\, \theta) + \sum_K \delta\, (u_t, \mathbf{b}\cdot\nabla\theta + \mathbf{b}\cdot\nabla(\delta\, \partial_t\, \theta))_K.\end{aligned} \tag{4.74}$$

Moreover, following idea from [20], the right-hand side can be approximated as

$$\begin{aligned} F_h(\theta + \delta\, \partial_t\, \theta) - F(\theta + \delta\, \partial_t\, \theta) &= \Big((f_h, \theta) - (f, \theta) \Big) \\ &+ \sum_K \delta \Big(f, \mathbf{b} \cdot \Pi^0_{k-1,K} \nabla(\delta\, \partial_t \theta) - \mathbf{b} \cdot \nabla(\delta\, \partial_t \theta) \Big)_K. \end{aligned} \tag{4.75}$$

The first term in right hand side can be estimated as

$$(f_h, \theta) - (f, \theta) = \sum_K (\Pi^0_{k,K} f - f, \theta)_{0,K} \leq C\, h^k\, |f|_k\, \|\theta\|. \tag{4.76}$$

The another term associated with load term can be estimated with the help of orthogonality property of projection operator $\Pi^0_{k-1,K}$, we have

$$\begin{aligned} \sum_K \delta \Big(f, \mathbf{b} \cdot \Pi^0_{k-1,K} \nabla(\delta\, \partial_t \theta) - \mathbf{b} \cdot \nabla(\delta\, \partial_t \theta) \Big)_K &= \sum_K \delta \Big((\Pi^0_{k-1,K} - I)(\mathbf{b} f), (\nabla \delta\, \partial_t \theta) \Big) \\ = \sum_K \delta\, \partial_t \Big((\Pi^0_{k-1,K} - I)(\mathbf{b} f), (\delta \nabla \theta) \Big) &- \sum_K \delta \Big((\Pi^0_{k-1,K} - I)(\mathbf{b} f_t), (\delta\, \nabla \theta) \Big) \\ \leq\; C\, h^k\, |f_t|_k\, \|\sqrt{\kappa} \nabla \theta\| + \sum_K \delta\, \partial_t &\Big((\Pi^0_{k-1,K} - I)(\mathbf{b} f), (\delta \nabla \theta) \Big)_K. \end{aligned} \tag{4.77}$$

An application of orthogonality property of L^2 projection operator $\Pi^0_{k,K}$ and stability of discrete bilinear form $m_h(\cdot, \cdot)$ enable us to write(the technique is provided in [11])

$$\begin{aligned} & m_h(-R^h u_t, \theta) + (u_t, \theta) = m_h(-R^h u_t + \Pi^0_k u_t, \theta) + (u_t - \Pi^0_k u_t, \theta) \\ & \leq\; C \sum_K \Big(\| - R^h u_t + \Pi^0_{k,K} u_t \|\, \|\theta\| + \|u_t - \Pi^0_{k,K} u_t\|\, \|\theta\| \Big)_K \\ & \leq\; C \sum_K \Big((\|R^h u_t - u_t\| + \|\Pi^0_{k,K} u_t - u_t\|)\, \|\theta\| + \|u_t - \Pi^0_{k,K} u_t\|\, \|\theta\| \Big)_K \\ & \leq\; C\, h^k\, |u_t|_k\, \|\theta\|, \end{aligned} \tag{4.78}$$

where we denote $u_t = \partial_t u$ and C is generic constant. The notation Π^0_k signifies the L^2 projection in global form which is defined in (1.6). Another term associated with test function $v_h = \delta\, \partial_t \theta$, can be estimated as

$$\begin{aligned} m_h(-R^h u_t, \delta\, \partial_t \theta) + (u_t, \delta\, \partial_t \theta) &= m_h(-R^h u_t + \Pi^0_k u_t, \delta\, \partial_t \theta) + (u_t - \Pi^0_k u_t, \delta\, \partial_t \theta) \\ &\leq\; C\, h^k\, |u_t|_k\, \delta\, \|\partial_t \theta\|. \end{aligned} \tag{4.79}$$

We split the term as follows

$$
\begin{aligned}
&\sum_K \delta \Big(-(\Pi^0_{k,K} R^h u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}(\nabla\theta))_K + (u_t, \mathbf{b}\cdot\nabla\theta)_K \Big) \\
&= \sum_K \delta \underbrace{\Big((-\Pi^0_{k,K} R^h u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}(\nabla\theta))_K + (\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1}(\nabla\theta))_K \Big)}_{T_1} \\
&\quad + \sum_K \delta \underbrace{\Big((-\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}(\nabla\theta))_K + (u_t, \mathbf{b}\cdot\nabla\theta)_K \Big)}_{T_2}.
\end{aligned} \tag{4.80}
$$

T_1 can be estimated by exploiting the approximation property of projection operator R^h(Lemma 27), boundedness of projection operators $\Pi^0_{k,K}$ and $\Pi^0_{k-1,K}$

$$
\begin{aligned}
T_1 = \Big(\Pi^0_{k,K}(R^h u_t - u_t), \mathbf{b}\cdot \Pi^0_{k-1,K}(\nabla\theta)\Big) &\leq C\, \|R^h u_t - u_t\|_{0,K}\, \mathcal{B}\, \|\nabla\theta\| \\
&\leq C\, h^k\, |u_t|_k\, (\mathcal{B}/\sqrt{\hat{K}})\, \|\sqrt{\kappa}\nabla\theta\|.
\end{aligned} \tag{4.81}
$$

Moreover, adding and subtracting the term $(\Pi^0_{k,K} u_t, \mathbf{b}\cdot\nabla\theta)_K$, we get

$$
\begin{aligned}
T_2 = &\Big((-\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}(\nabla\theta))_K + (\Pi^0_{k,K} u_t, \mathbf{b}\cdot\nabla\theta)_K \Big) \\
&+ \Big((-\Pi^0_{k,K} u_t, \mathbf{b}\cdot\nabla\theta)_K + (u_t, \mathbf{b}\cdot\nabla\theta)_K \Big).
\end{aligned} \tag{4.82}
$$

The first term of (4.82) can be estimated as

$$
\begin{aligned}
\Big((-\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}(\nabla\theta))_K + (\Pi^0_{k,K} u_t, \mathbf{b}\cdot\nabla\theta)_K \Big) &= \underbrace{\Big((I - \Pi^0_{k-1,K})(\mathbf{b}\Pi^0_{k,K} u_t), \nabla\theta \Big)}_{\text{orthogonality of } \Pi^0_{k-1,K}} \\
\leq \underbrace{\|\mathbf{b}\Pi^0_{k,K} u_t - \Pi^0_{k-1,K}(\mathbf{b}u_t)\|}_{\text{best-approximation property}}\, \|\nabla\theta\| &\leq C\, h^k\, |u_t|_k\, \|\sqrt{\kappa}\nabla\theta\|.
\end{aligned} \tag{4.83}
$$

Exploiting Cauchy-Schwarz inequality the second term of (4.82)can be estimated as

$$
(-\Pi^0_{k,K} u_t, \mathbf{b}\cdot\nabla\theta)_K + (u_t, \mathbf{b}\cdot\nabla\theta) \leq C\, h^k\, |u_t|_k\, (\mathcal{B}/\sqrt{\hat{K}})\, \|\sqrt{\kappa}\nabla\theta\|. \tag{4.84}
$$

We have the term

$$
\sum_K \delta \underbrace{\Big(-(\Pi^0_{k,K} R^h u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\,\partial_t\theta))_K + (u_t, \mathbf{b}\cdot\nabla(\delta\,\partial_t\theta))_K \Big)}_{T_3}. \tag{4.85}
$$

In order to bound the term (4.85), we rewrite the T_3 in the following way

$$T_3 = \underbrace{\Big(-(\Pi^0_{k,K} R^h u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\, \partial_t\theta))_K + (\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\, \partial_t\theta))_K \Big)}_{T_4}$$
$$+ \underbrace{\Big(-(\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\, \partial_t\theta))_K + (u_t, \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K \Big)}_{T_5}. \tag{4.86}$$

An application of lemma (27), inverse estimation (demonstrated in lemma(24)), boundedness of projection operator $\Pi^0_{k,K}$ and convective coefficient ($\mathbf{b}(\mathbf{x})$) exploiting $\delta \leq\ C\,h$ yields

$$\delta\, T_4 \leq\ C\, h^k\, \|u_t\|_k\, \delta^{1/2}\, \|\partial_t\theta\|. \tag{4.87}$$

Analogously, the other term of T_3 can be bounded. We represent T_5 as

$$\begin{aligned} T_5 &= \Big(-(\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\, \partial_t\theta))_K + (u_t, \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K \Big) \\ &= \Big(-(\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\, \partial_t\theta))_K + (\Pi^0_{k,K} u_t, \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K \Big) \\ &\quad + \Big(-(\Pi^0_{k,K} u_t, \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K + (u_t, \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K \Big). \end{aligned} \tag{4.88}$$

Now, an application of approximation property of projection operator $\Pi^0_{k,K}$,Cauchy Schwartz inequality and Lemma (24) estimate as

$$\delta\,\Big(-(\Pi^0_{k,K} u_t,\ \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K + (u_t,\ \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K \Big) \leq Ch^k\, \|u_t\|_k\, \delta^{1/2}\|\partial_t\theta\|. \tag{4.89}$$

Exploiting same argument as(4.77), another term of T_5 can be estimated as

$$\begin{aligned} &\sum_K \delta\, \Big(-(\Pi^0_{k,K} u_t, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla(\delta\, \partial_t\theta))_K + (\Pi^0_{k,K} u_t, \mathbf{b}\cdot \nabla(\delta\, \partial_t\theta))_K \Big) \\ &\leq\ C\, h^k\, |u_{tt}|_k\, \|\sqrt{\kappa}\nabla\theta\| + \sum_K\ \delta\, \partial_t\Big(\Pi^0_{k,K} u_t, \mathbf{b}\cdot (I - \Pi^0_{k-1,K})\nabla\delta\theta\Big)_K. \end{aligned} \tag{4.90}$$

In view of right-hand side estimations, equation(4.74) reduces to the following form

$$\begin{aligned} &\frac{1}{2}\,\partial_t\,\|\theta\|_h^2 + C_1\,\Big(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\,\theta\|^2 + \sum_K \delta\,\|\mathbf{b}\cdot\nabla\theta\|^2\Big) + C_2\,\delta\,\|\partial_t\theta\|_0^2 \\ &\quad + c_\alpha\,\frac{\delta}{2}\,\partial_t\,\Big(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\,\theta\|^2 + \sum_K \delta\,\|\mathbf{b}\cdot\nabla\theta\|^2\Big) \\ &\leq\, C\,h^{2k}\,\Big(|f|_k^2 + |f_t|_k^2 + |u_t|_k^2 + |u_{tt}|_k^2\Big) + \sum_K\,\delta\,\partial_t\Big((\Pi^0_{k-1,K} - I)(\mathbf{b}f), \delta\nabla\theta\Big) \\ &\quad + \sum_K\,\delta\,\partial_t\Big((I - \Pi^0_{k-1,K})(\mathbf{b}\Pi^0_{k,K}u_t), \delta\nabla\theta\Big). \end{aligned} \tag{4.91}$$

Taking integration of (4.91) from 0 to t and considering $\theta(0) = 0$, we have

$$\begin{aligned} &\|\theta(t)\|_0^2 + C_1\,\|\theta\|^2_{L^2(0,t,|||\cdot|||)} + C_2\,\delta\|\partial_t\theta\|^2_{L^2(0,t,\|\cdot\|)} \\ &\quad + C_3\,\delta\,\Big(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\theta\|^2 + \sum_K\,\delta\,\|\mathbf{b}\cdot\nabla\theta\|^2\Big) \\ &\leq\, C\,h^k\,\Big(|f|^2_{L^2(0,T,H^k(\Omega))} + |f_t|^2_{L^2(0,T,H^k(\Omega))} + |u_t|^2_{L^2(0,T,H^k(\Omega))} + |u_{tt}|_{L^2(0,T,H^k(\Omega))}\Big). \end{aligned} \tag{4.92}$$

Since the terms $\|\partial_t\theta\|^2_{L^2(0,t,\|\cdot\|)}$ and $\Big(\|\sqrt{\kappa}\nabla\theta\|^2 + \|\sqrt{c}\theta\|^2 + \sum_K\,\delta\,\|\mathbf{b}\cdot\nabla\theta\|^2\Big)$ are positive, hence we have

$$\begin{aligned} \|\theta(t)\|_0^2 + C_1\,\|\theta\|^2_{L^2(0,t,|||\cdot|||)} \leq\, C\,h^{2k}\Big(&|f|^2_{L^2(0,T,H^k(\Omega))} + |f_t|^2_{L^2(0,T,H^k(\Omega))} \\ &+ |u_t|^2_{L^2(0,T,H^k(\Omega))} + |u_{tt}|_{L^2(0,T,H^k(\Omega))}\Big). \end{aligned} \tag{4.93}$$

An application of lemma(27) and estimation (4.93) yields final thesis

$$\begin{aligned} &\|u(t) - u_h(t)\| + \|u - u_h\|_{L^2(0,t,|||\cdot|||)} \leq\, C\,h^k\,\Big(\|f\|_{L^2(0,T,H^k(\Omega))} \\ &\quad + |f_t|^2_{L^2(0,T,H^k(\Omega))} + \|u\|^2_{L^2(0,T,H^{k+1}(\Omega))} + \|u_t\|_{L^(0,t,H^k(\Omega))} + \|u_{tt}\|^2_{L^2(0,T,H^k(\Omega))}\Big). \end{aligned} \tag{4.94}$$

■

In order to estimate $\|u - u_h\|$, where u is exact solution and u_h is discrete solution, we rewrite the error as $\|u - u_h\| \leq \|u - R^h u\| + \|R^h u - u_h\|$. Estimation of $\|R^h u - u_h\|$ follows same idea as (4.93). In order to bound first term $\|u - R^h u\|$, we prove the following result.

4.8.2 Time continuous case in L^2 norm

Lemma 29. *Let u be an element of V^0 and let $R^h u \in \mathcal{Z}_h^k$ be the element defined in (4.32). Then the following estimation holds*

$$\|u - R^h u\|_0 \leq \; C \, h^{k+1} \, |u|_{k+1},$$

where C is a positive generic constant.

Proof. We define the dual problem

$$-\nabla \cdot (\kappa \, \nabla \eta) - (\mathbf{b} \cdot \nabla \eta) + c\eta = u - R^h u. \tag{4.95}$$

Multiplying by v_h with both side of (4.95) and using green's theorem, we get an equation $B(v_h, \eta) = (u - R^h u, v_h)$, where $B(v_h, \eta)$ is defined as

$$B(v_h, \eta) := \int \kappa(\mathbf{x}) \nabla v_h \cdot \nabla \eta + \int \mathbf{b} \cdot \nabla v_h \, \eta + \int c(\mathbf{x}) \, v_h \, \eta. \tag{4.96}$$

We know from elliptic regularity theory [57]

$$h_K |\eta - \eta_I|_{1,K} + \|\eta - \eta_I\|_{0,K} \leq C \, h_K^2 \, |\eta|_2 \tag{4.97}$$

and

$$\|\eta\|_2 \leq \|u - R^h u\|. \tag{4.98}$$

With the help of (4.95) and using definition of R^h operator (4.32), we can write

$$\begin{aligned}
\|u - R^h u\|^2 &= (u - R^h u, u - R^h u) \\
&= B(u - R^h u, \eta) \\
&= B(u - R^h u, \eta - \eta_I) + \mathcal{A}_{\text{supg}}(u - R^h u, \eta_I) - \chi(u - R^h u, \eta_I) \\
&= B(u - R^h u, \eta - \eta_I) + \mathcal{A}_{\text{supg}}(u, \eta_I) - \mathcal{A}_{\text{supg}}(R^h u, \eta_I) + \mathcal{A}_{\text{supg},h}(R^h u, \eta_I) \\
&\quad - \mathcal{A}_{\text{supg},h}(R^h u, \eta_I) - \chi(u - R^h u, \eta_I) \\
&= \underbrace{B(u - R^h u, \eta - \eta_I)}_{Q_1} + \underbrace{\mathcal{A}_{\text{supg},h}(R^h u, \eta_I) - \mathcal{A}_{\text{supg}}(R^h u, \eta_I)}_{Q_2} \\
&\quad - \underbrace{\chi(u - R^h u, \eta_I)}_{Q_3},
\end{aligned} \tag{4.99}$$

where the additional stabilizer term $\chi(u - R^h u, \eta_I)$ is defined by

$$\chi(u-R^h u,\eta_I) := \sum_{K\in\mathcal{T}_h}\Big(-\nabla\cdot(\kappa\,\nabla(u-R^h u))+\mathbf{b}\cdot\nabla(u-R^h u)+c(u-R^h u),\mathbf{b}\cdot\nabla\eta_I\Big)_K. \tag{4.100}$$

$$\begin{aligned} Q_1 &= B(u-R^h u,\eta-\eta_I)\\ &= \int \kappa(\mathbf{x})\;\nabla(u-R^h u)\cdot\nabla(\eta-\eta_I)+\int c(\mathbf{x})\;(u-R^h u)\;(\eta-\eta_I)\\ &\quad+\int(\mathbf{b}(\mathbf{x})\cdot\nabla(u-R^h u))\;(\eta-\eta_I). \end{aligned} \tag{4.101}$$

Using $|u-P_h u|_1$ form estimation (27), (4.97), (4.98), Cauchy-Schwarz inequality and standard approximation property of orthogonal L^2 projection operator $\Pi^0_{k,K}$, we can have

$$|Q_1| \leq C\;h^{k+1}\;|u|_{k+1}\;\|u-P_h u\|. \tag{4.102}$$

Exploiting the idea stated in [20], we split the term Q_2 element wise as follows

$$\begin{aligned} &\mathcal{A}^K_{\text{supg},h}(R^h u,\eta_I)-\mathcal{A}^K_{\text{supg}}(R^h u,\eta_I)=\mathcal{A}^K_{\text{supg},h}(R^h u-\Pi^0_{k,K}u,\eta_I-\Pi^0_{1,K}\eta)\\ &\quad-\mathcal{A}^K_{\text{supg}}(R^h u-\Pi^0_{k,K}u,\eta_I-\Pi^0_{1,K}\eta)+\mathcal{A}^K_{\text{supg},h}(\Pi^0_{k,K}u,\eta_I)-\mathcal{A}^K_{\text{supg}}(\Pi^0_{k,K}u,\eta_I)\\ &\quad+\mathcal{A}^K_{\text{supg},h}(R^h u,\Pi^0_{1,K}\eta)-\mathcal{A}^K_{\text{supg}}(R^h u,\Pi^0_{1,K}\eta)\\ &\quad+\mathcal{A}^K_{\text{supg}}(\Pi^0_{k,K}u,\Pi^0_{1,K}\eta)-\mathcal{A}^K_{\text{supg},h}(\Pi^0_{k,K}u,\Pi^0_{1,K}\eta). \end{aligned} \tag{4.103}$$

Since $\Pi^0_{k,K}$ and $\Pi^0_{1,K}$ both are polynomials on polygon K, and $\Pi^0_{k,K}$ is orthogonal L^2 projection on $\mathbb{P}_k(K)$

$$A^K_1(\Pi^0_{k,K}u,\Pi^0_1\eta)-A^K_{1,h}(\Pi^0_{k,K}u,\Pi^0_1\eta)=0,$$

$$A^K_3(\Pi^0_{k,K}u,\Pi^0_1\eta)-A^K_{3,h}(\Pi^0_{k,K}u,\Pi^0_1\eta)=0$$

and

$$|A^K_2(\Pi^0_{k,K}u,\Pi^0_1\eta)-A^K_{2,h}(\Pi^0_{k,K}u,\Pi^0_1\eta)|\leq C\;h^{k+1}\;|u|_{k+1}\;\|\eta\|_2.$$

Collecting above three estimates, we deduce that

$$\mathcal{A}^K_{\text{supg}}(\Pi^0_{k,K}u,\Pi^0_{1,K}\eta)-\mathcal{A}^K_{\text{supg},h}(\Pi^0_{k,K}u,\Pi^0_{1,K}\eta)\leq C\;h^{k+1}\;|u|_{k+1}\;\|\eta\|_2. \tag{4.104}$$

Using approximation property of $\Pi^0_{k,K}$ operator and R^h operator and technique described in [20], we can evaluate

$$|\mathcal{A}^K_{\text{supg},h}(R^h u - \Pi^0_{k,K}u, \eta_I - \Pi^0_{1,K}\eta)| \leq C\ h_K^{k+1}\ |u|_{k+1,K}\ \|u - R^h u\|_{0,K}. \tag{4.105}$$

Similarly, other term can be approximated in same fashion

$$|\mathcal{A}^K_{\text{supg}}(R^h u - \Pi^0_{k,K}u, \eta_I - \Pi^0_{1,K}\eta)| \leq C\ h_K^{k+1}\ |u|_{k+1}\ \|u - R^h u\|_{0,K} \tag{4.106}$$

and

$$|\mathcal{A}^K_{\text{supg},h}(\Pi^0_{k,K}u, \eta_I) - \mathcal{A}^K_{\text{supg}}(\Pi^0_{k,K}u, \eta_I)| \leq C\ h_K^{k+1}\ |u|_{k+1,K}\ \|u - R^h u\|_{0,K} \tag{4.107}$$

and

$$|\mathcal{A}^K_{\text{supg},h}(R^h u, \Pi^0_1\eta) - A^K(R^h u, \Pi^0_1\eta)| \leq C\ h_K^{k+1}\ |u|_{k+1,K}\ \|u - R^h u\|_{0,K}. \tag{4.108}$$

Accumulating all estimations (4.104),(4.105),(4.106),(4.107) and (4.108) in (4.103) and exploiting Cauchy-Schwarz inequality, we have

$$|\mathcal{A}_{\text{supg},h}(R^h u, \eta_I) - \mathcal{A}_{\text{supg}}(R^h u, \eta_I)| \leq C\ h^{k+1}\ |u|_{k+1}\ \|u - R^h u\|_0. \tag{4.109}$$

Proceeding same as [18], we can estimate

$$|\chi(u - R^h u, \eta_I)| \leq C\ h^{k+1}\ |u|_{k+1}\ \|u - R^h u\|. \tag{4.110}$$

Collecting all the estimation (4.102),(4.109) and (4.110) and with the help of (4.99), we can estimate

$$\|u - R^h u\| \leq C\ h^{k+1}\ |u|_{k+1}.$$

■

Theorem 30. *Let u be enough regular solution (assumed in Theorem (28)) and u_h be discrete solution satisfying (4.27). Then there exist a positive generic constant C independent of h, may be dependent on u such that the following estimation holds*

$$\begin{aligned}\|u(t) - u_h(t)\| \leq\ & C\ h^{k+1} \Big(\|f\|_{L^2(0,T,H^{k+1}(\Omega))} + \|f_t\|_{L^2(0,T,H^{k+1}(\Omega))} \\ & + \|u\|_{L^2(0,T,H^{k+1}(\Omega))} + \|u_t\|_{L^2(0,T,H^{k+1}(\Omega))} + \|u_{tt}\|_{L^2(0,T,H^{k+1}(\Omega))}\Big),\end{aligned} \tag{4.111}$$

where k denotes maximum degree of polynomial belongs to $\mathcal{Z}^k(K)$.

Proof. In order to estimate $\|(u-u_h)(t)\|$, we split the term $(u-u_h)(t)$ with the help of projection operator R^h that gives $(u-u_h) = u - R^h u + R^h u - u_h$. Estimation of $\|u - R^h u\|$ is directly follows from lemma 29. Proceeding same as (4.93) and considering $\delta = O(h)$ we obtain

$$\begin{aligned}\|\theta(t)\| \leq \; C\, h^{k+1} \Big(&\|f\|_{L^2(0,T,H^{k+1}(\Omega))} + \|f_t\|_{L^2(0,T,H^{k+1}(\Omega))} \\ &+ \|u_t\|_{L^2(0,T,H^{k+1}(\Omega))} + \|u_{tt}\|_{L^2(0,T,H^{k+1}(\Omega))}\Big).\end{aligned} \tag{4.112}$$

Consequently, we have final thesis

$$\|u(t) - u_h(t)\| \leq \; C\, h^{k+1}. \tag{4.113}$$

■

4.9 Fully-Discrete Scheme

In view of applicability of $Crank - Nicolson/Q_1 - FEM$ scheme for simulation of Calcium Carbonate precipitation [58], we employ Crank-Nicolson/VEM scheme for time dependent convection diffusion reaction equation. Let N be a positive integer and $[0,T]$ be the time interval. Also, we assume that $\Delta t = T/N$ be the time-step. This scheme is unconditionally stable and second order convergent, i.e. $O(\Delta t^2)$. Moreover, we convey that one can exploit backward Euler scheme for time discretization, but the main disadvantage with the B-E scheme is that it is first order convergent. Therefore, in order to obtain optimal order of convergence in h, we essentially demand smaller values of Δt which is numerically expensive. Crank-Nicolson/VEM scheme is revealed as

$$\begin{aligned}&m_h\Big(u_h^n - u_h^{n-1}, \phi\Big) + \Delta t\, \mathcal{A}_{\text{supg},h}\Big(\frac{u_h^n + u_h^{n-1}}{2}, \phi\Big) = \Delta t\, \Big(\frac{f^n + f^{n-1}}{2}, \Pi_k^0 \phi\Big) \\ &+ \Delta t \sum_{K\in\mathcal{T}_h} \delta_K \Big(\frac{f^n + f^{n-1}}{2}, \mathbf{b}\cdot \Pi_{k-1,K}^0 \nabla\phi\Big)_K - \sum_{K\in\mathcal{T}_h} \delta_K \Big(u_h^n - u_h^{n-1}, \mathbf{b}\cdot\Pi_{k-1,K}^0\nabla\phi\Big)_K.\end{aligned} \tag{4.114}$$

We will follow standard FEM technique(mainly adopted from [59]) with minor modification in order to discuss convergence analysis for fully discrete scheme. Moreover, we exploit analogous idea as semi-discrete case for analysing fully-discrete scheme. Error involving with non stationary part can not be estimated directly. An application of stability property($stability$,[11]) along with approximation property of local L^2 projection $\Pi^0_{k,K}$ ensure that the error decreasing with optimal order in h. Additionally, temporal discretization follows same framework as[59]. In order to discuss convergence analysis precisely, we neglect some obvious estimations which are already proved in semi-discrete case.

Theorem 31. *Let u be the exact solution of (4.23) and u_h be the discrete solution of (4.27). Moreover, we assume that u_t and f belong to $L^\infty(0,T,H^k(\Omega))$,$u$ belongs to $L^\infty(0,T,H^k(\Omega))$ and $R^h u_{ttt}$ belongs to $L^2(0,T,L^2(\Omega))$. Also assume the $\theta(0) = 0$. Then there exists generic constant C such that the following estimation holds*

$$\begin{aligned}
&\|(u-u_h)(t_n)\|_0^2 + \ \Delta t \sum_{j=1}^{n} |||\frac{u(t_j)+u(t_{j-1})}{2} - \frac{u_h(t_j)+u_h(t_{j-1})}{2}|||^2 \\
&\leq \ C\, h^{2k}\left(\|u\|^2_{L^\infty(0,T,H^{k+1}(\Omega))} + \|u_t\|^2_{L^\infty(0,T,H^k(\Omega))} + \|f\|^2_{L^\infty(0,T,H^k(\Omega))}\right) \\
&\quad + C\,(\Delta t)^4\, \|R^h u_{ttt}\|^2_{L^2(0,T,L^2(\Omega))}.
\end{aligned} \tag{4.115}$$

Proof. In order to estimate fully discrete estimation, we employ the projection operator R^h at time $t = t_n$.

$$u_h(t_n) - u(t_n) = u_h(t_n) - R^h u(t_n) + R^h u(t_n) - u(t_n) = \theta^n + \rho^n. \tag{4.116}$$

Estimation of ρ^n is straightforward. However, in order to estimate $|||\theta^n|||$, we proceed as follows. Since θ^n is an element of $\mathcal{Z}_h^k$, exploiting fully-discrete variational form (4.114), we have

$$\begin{aligned}
&m_h\Big(\frac{\theta^n - \theta^{n-1}}{\Delta t}, v_h\Big) + \ \mathcal{A}_{\text{supg},h}\Big(\frac{\theta_h^n + \theta_h^{n-1}}{2}, v_h\Big) \\
&\quad + \sum_{K\in\mathcal{T}_h} \delta_K \left(\Pi^0_{k,K}\Big(\frac{\theta_h^n - \theta_h^{n-1}}{\Delta t}\Big), \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla v_h\right)_K \\
&= \Big(\frac{f^n + f^{n-1}}{2}, \Pi^0_k v_h\Big) + \sum_{K\in\mathcal{T}_h} \delta_K \left(\frac{f^n + f^{n-1}}{2}, \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla v_h\right)_K \\
&\quad - m_h\Big(\frac{R^h u(t_n) - R^h u(t_{n-1})}{\Delta t}, v_h\Big) - \mathcal{A}_{\text{supg},h}\Big(\frac{R^h u(t_n) + R^h u(t_{n-1})}{2}, v_h\Big) \\
&\quad - \sum_{K\in\mathcal{T}_h} \delta_K \left(\Pi^0_{k,K}\Big(\frac{R^h u(t_n) - R^h u(t_{n-1})}{\Delta t}\Big), \mathbf{b}\cdot \Pi^0_{k-1,K}\nabla v_h\right)_K.
\end{aligned} \tag{4.117}$$

Moreover, in order to reduce cumbersome notation, we denote $\omega_\tau^n = \frac{\omega(t_n)-\omega(t_{n-1})}{\tau}$ and $\bar{\omega}^n = \frac{\omega(t_n)+\omega(t_{n-1})}{2}$ for arbitrary function ω. Now, an application of definition of projection operator (4.32), we have

$$
\begin{aligned}
& m_h\Big(\theta^n_\tau, v_h\Big) + \ \mathcal{A}_{\mathrm{supg},h}\Big(\bar{\theta^n}, v_h\Big) + \sum_{K\in\mathcal{T}_h} \delta_K \ \Big(\Pi^0_{k,K}\theta^n_\tau, \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla v_h\Big)_K \\
& = \Big(\bar{f^n}, \Pi^0_k v_h\Big) - \Big(\bar{f^n}, v_h\Big) - m_h\Big(R^h u^n_\tau, v_h\Big) + \Big(\bar{u_t}, v_h\Big) \\
& + \sum_{K\in\mathcal{T}_h} \delta_K \ \Big(\bar{f^n}, \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla v_h\Big)_K - \sum_{K\in\mathcal{T}_h} \delta_K \ \Big(\bar{f^n}, \mathbf{b}\cdot\nabla v_h\Big)_K \\
& \quad - \sum_{K\in\mathcal{T}_h} \delta_K \ \Big(\Pi^0_{k,K}(R^h u^n_\tau), \mathbf{b}\cdot\Pi^0_{k-1,K}\nabla v_h\Big)_K + \sum_{K\in\mathcal{T}_h} \delta_K \ \Big(\bar{u_t}, \mathbf{b}\cdot\nabla v_h\Big).
\end{aligned}
\tag{4.118}
$$

An elementary manipulation yields

$$
a_h(\theta^n_\tau, \bar{\theta^n}) := \frac{1}{2\,\Delta t}\Big(a_h(\theta^n, \theta^n) - a_h(\theta^{n-1}, \theta^{n-1})\Big). \tag{4.119}
$$

Similarly other inner-products associated with $A_{1,h}(\cdot,\cdot)$ can be decomposed. Exploiting (4.119), and borrowing idea from (Theorem-5.4 [59] and Theorem-3.3 [11]) and proceeding analogously as semi-discrete case, we have

$$
\begin{aligned}
& \Big(\|\theta^n\|^2 - \|\theta^{n-1}\|^2\Big) + C_1\ \delta\ \Delta t\ \|\theta^n_\tau\|^2 + C_2\ \Delta t\ |||\bar{\theta^n}|||^2 + C_3\ \delta\ \Big(|||\theta^n|||^2 - |||\theta^{n-1}|||^2\Big) \\
& \le\ C_4\ h^{2k}\ \Big(\|u_t\|^2_{L^\infty(0,T,H^k(\Omega))} + \|f\|^2_{L^\infty(0,T,H^k(\Omega))}\Big) + C_5\ (\Delta t)^4\ \|R^h u_{ttt}\|^2_{L^2(0,T,L^2(\Omega))},
\end{aligned}
\tag{4.120}
$$

where C_1, C_2,C_3, C_4 and C_5 are generic constants. Iterating (4.120), $1 \le j \le n$, we have

$$
\begin{aligned}
& \|\theta^n\|^2 + \Delta t\ \sum_{j=1}^{n} |||\bar{\theta^j}|||^2 + \delta\ |||\theta^n|||^2 \le\ \|\theta^0\|^2 + \delta\ |||\theta^0|||^2 \\
& \quad + C\ h^{2k}\ \Big(\|u_t\|^2_{L^\infty(0,T,H^k(\Omega))} + \|f\|^2_{L^\infty(0,T,H^k(\Omega))}\Big) + \bar{C}\ (\Delta t)^4\ \|R^h u_{ttt}\|^2_{L^2(0,T,L^2(\Omega))}.
\end{aligned}
\tag{4.121}
$$

Assuming $\theta^0 = \theta(0) = 0$ and since $|||\theta^n|||^2$ is a positive number, we recast as

$$
\begin{aligned}
\|\theta^n\|^2 + \Delta t\ \sum_{j=1}^{n} |||\bar{\theta^j}|||^2 \le +C\ h^{2k}\ \Big(\|u_t\|^2_{L^\infty(0,T,H^k(\Omega))} + \|f\|^2_{L^\infty(0,T,H^k(\Omega))}\Big) \\
+ \bar{C}\ (\Delta t)^4\ \|R^h u_{ttt}\|^2_{L^2(0,T,L^2(\Omega))}.
\end{aligned}
\tag{4.122}
$$

With the help of lemma 27 at discrete level, we have

$$\|(u-u_h)(t_n)\|^2 + \Delta t \sum_{j=1}^{n} |||\bar{u^j} - \bar{u_h^j}|||^2 \leq C \; h^{2k} \Big(\|u_t\|^2_{L^\infty(0,T,H^k(\Omega))} + \|u\|^2_{L^\infty(0,T,H^{k+1}(\Omega))} + \|f\|^2_{L^\infty(0,T,H^k(\Omega))}\Big) + \bar{C} \; (\Delta t)^4 \; \|R^h u_{ttt}\|^2_{L^2(0,T,L^2(\Omega))}. \tag{4.123}$$

■

4.10 Numerical experiment

This section is dedicated to justify theoretical convergence results by robust numerical experiments. Here, we examine two examples. First one agreed with smooth solution demonstrates order of convergence in L^2 and H^1 norm and second example depicts physical behaviour of the numerical solutions.

4.10.1 Example-1

Consider the model problem (4.23) with $\mathbf{b} = (-1, 1)$, $c = 1$, $\kappa = 10^{-9}$ final time $T = 1$ and computational domain $\Omega = (0,1) \times (0,1)$. Furthermore, we consider the exact solution is $u(x, y, t) := e^{\sin(2\pi t)} \sin(2\pi x) \sin(2\pi y)$ and force function f is computed inserting exact solution u in model problem. We discretize the domain with voronoi mesh [36] where $h_K \approx h$. As we have mentioned that u_h is defined implicitly, in order to evaluate error , we borrow the innovative idea form [18]. The errors $e_{h,0}$ and $e_{h,1}$are defined as

- L^2-norm error : $e_{h,0} = \sqrt{\sum_{K\in\mathcal{T}_h} \|u - \Pi^0_{k,K} u_h\|^2_{L^2(K)}}$.
- H^1-norm error : $e_{h,1} = \sqrt{\sum_{K\in\mathcal{T}_h} |u - \Pi^\nabla_{k,K} u_h|^2_{H^1(K)}}$.

Furthermore, $r_{h,0}$ and $r_{h,1}$ denote rate of convergence in L^2-norm and H^1 semi-norm. We have employed Crank-Nicolson scheme in temporal direction and virtual element method of order $k = 1, 2$ in spatial direction. Theorem 31 illustrates that $|||u - u_h||| = O(h^k + \tau^2)$. Hence $\Delta t = O(h)$ would be ideal choice for time-step Δt in order to obtain optimal order of convergence. Stabilization parameter δ chosen as $\delta = O(\sqrt{\Delta t}\; h)$. Table 4.9 and Table 4.10 represents rate of convergence in L^2 norm and H^1 semi-norm which have good agreement with theoretical estimation revealed in Theorem 30 and Theorem 28 . All the errors are calculated at final time $T = 1$.

Table 4.9

Error table for Crank-Nicolson/VEM scheme with $k = 1$

	h	$e_{h,0}$	$r_{h,0}$	$e_{h,1}$	$r_{h,1}$
$\kappa = 10^{-9}$	1/5	3.2795e-02	-	1.3164e0	-
	1/10	7.6930e-03	2.09	6.4269e-01	1.03
	1/20	1.7670e-03	2.12	3.1392e-01	1.03
	1/40	4.4500e-04	1.98	1.5539e-01	1.01
$\kappa = 1$	1/5	5.9284e-02	-	1.2825e0	-
	1/10	1.5569e-02	1.92	6.3285e-01	1.01
	1/20	3.7310e-03	2.06	3.1230e-01	1.01
	1/40	9.1211e-04	2.03	1.5231e-01	1.03

Table 4.10

Error table for Crank-Nicolson/VEM scheme with $k = 2$

	h	$e_{h,0}$	$r_{h,0}$	$e_{h,1}$	$r_{h,1}$
$\kappa = 10^{-8}$	1/5	4.8765e-03	-	2.8329e-01	-
	1/10	5.0224e-04	3.27	6.1983e-02	2.19
	1/20	6.3432e-05	2.98	1.5903e-02	1.96
	1/40	6.9143e-06	3.19	4.0123e-03	1.98
$\kappa = 1$	1/5	3.1015e-03	-	1.9063e-01	-
	1/10	4.1342e-04	2.90	5.0124e-02	1.92
	1/20	5.1523e-05	3.00	1.1273ee-02	2.15
	1/40	6.1246e-06	3.07	2.5301e-03	2.15

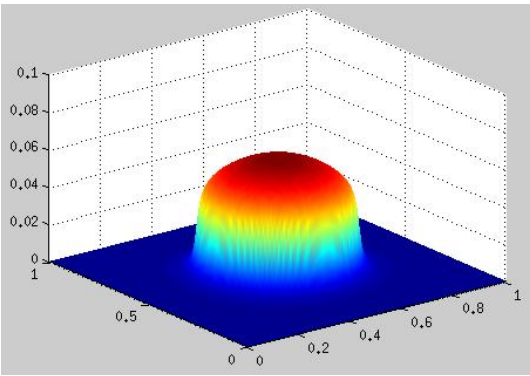
As we have mentioned that the non-polynomial stabilizer appeared in discretization of symmetric bilinear form stabilize the bilinear form, we have to pay more attention to choose stabilizers in numerical calculation. In convection dominated region, the stabilizer(non-polynomial part) appeared with reaction term and additional SUPG- stabilizer need to be balanced to obtain an optimal L^2 -error estimates. Moreover, in diffusion dominated region step-size Δt and mesh-size h need to be balanced in order to obtain optimal rate of convergence in L^2 and H^1-norm. We have chosen very small time step $\Delta t = 10^{-4}$ in order to obtain optimal rate of convergence in L^2 norm for both linear and quadratic element. In implementation, we have assumed $u_{h,0} := I_h u_0$.

4.10.2 Example-2

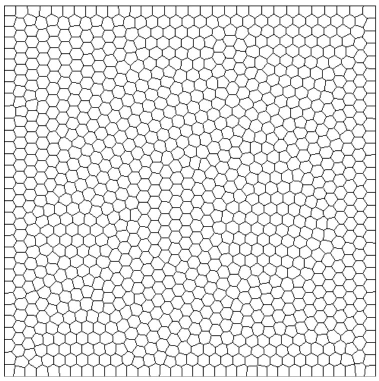
In order to depict performance of SUPG stabilizer, we examine this example. This problem is already investigated in finite element settings with SUPG stabilizer in [60]. We accomplish this numerical experiments with time-step Δt and voronoi mesh with mesh-size $h = 1/5, 1/10, 1/20, 1/40$. Since there is no appropriate quadrature rule over polygonal element, we split the polygon into triangles by joining each vertex with centroid of element and then applied Gauss-quadrature rule. Exact solution is taken as

$$u(t,x,y) := 16\, e^{\sin(\pi t)} x(1-x)y(1-y)\Big[\frac{1}{2} + \frac{\arctan\{2\,\kappa^{-1/2}(0.25^2 - (x-0.5)^2 - (y-0.5)^2)\}}{\pi}\Big] \tag{4.124}$$

In numerical experiment, we consider diffusive coefficient $\kappa = 10^{-4}$, reaction coefficient $c = 1$ and vector valued convective coefficient $\mathbf{b} = (2, 3)$. All the experiments are calculated at final time $T = 1$. Moreover, we have considered time-step $\Delta t = 10^{-2}$ and $\delta = O(\sqrt{\Delta t}\; h)$. Right-hand side f is chosen such that $u(t, x, y)$ become exact solution of model problem (4.23). In order to show the performance of SUPG stabilizer, we consider four succeeding pictures for mesh size $h = 1/5, 1/10, 1/20, 1/40$ respectively. We have observed when $h = 1/5$ oscillations are more (Figure 4.7) and reduces gradually as considering smaller mesh size. The exact solution is depicted in Figure 4.6. Also we have observed that oscillations are almost removed for $h = 1/40$ (Figure 4.8)and matching perfectly with exact solution depicted in Figure-4.6



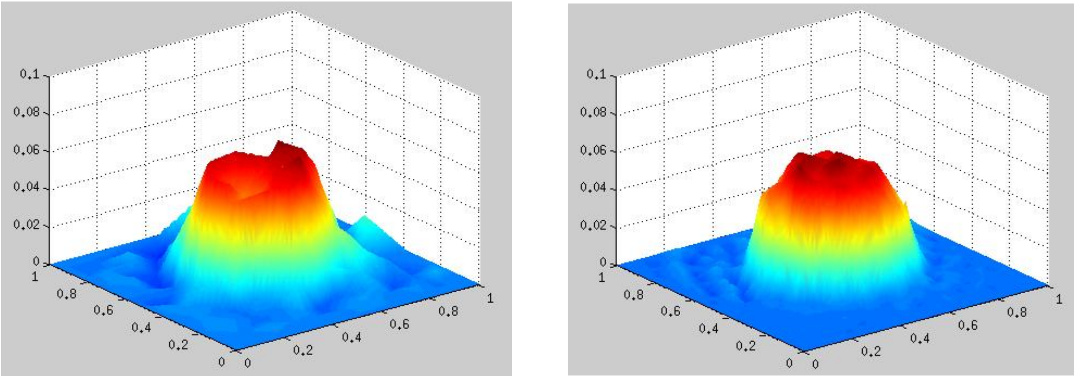
(a) Exact solution



(b) Voronoi mesh

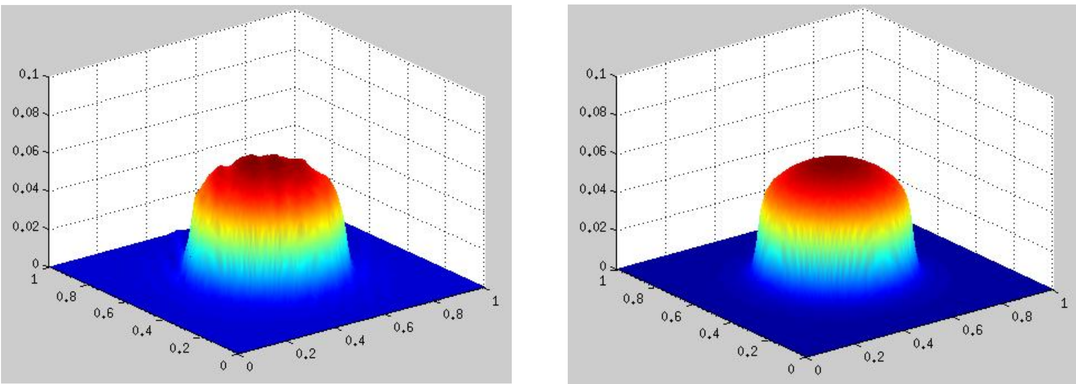
Figure 4.6

Remark 4. *In discrete formulation, we have approximated (u_t, v) and (u, v) by $m_h(u_{ht}, v_h)$ and $m_h(u_h, v_h)$ where we have employed $\Pi^0_{k,K}$ operator over VEM space of order k. One can approximate the same terms using $\Pi^0_{k-1,K}$ operator to prove semi-discrete estimations with optimal order of convergence. In such case, inverse-estimation revealed in Lemma 24 can be waived.*



(a) $h = 1/5$ **(b)** $h = 1/10$

Figure 4.7: Numerical solution at T=1.



(a) $h = 1/20$ **(b)** $h = 1/40$

Figure 4.8: Numerical solution at T=1.

4.11 Discussion

In this chapter, we have discussed convection dominated diffusion reaction equation using SUPG formulation both in the context of FEM and VEM. We have shown basic aspects of SUPG FEM considering two different bilinear forms and estimated error analysis in mesh dependent norm $|||\cdot|||$. We have also shown that the extension of SUPG stabilizers in the context of VEM is not striaghtforward and requires projection operators to suitably define the terms to be computable. We would also like to mention that the choice of stabilization parameter δ plays a vital role in determining the rate of convergence in the convection dominated regime. Since the discrete bilinear form is designed element wise, this configuration can be easily extended to higher dimension including nonconforming case. It is well known that nonconforming FEM with higher order patch-test show better rate of convergence. Moreover, we have observed that nonconforming VEM formulation has uniform framework irrespective of order of space[12, 20]. Therefore, we can easily construct nonconforming element with higher order patch-test. The extension of nonconforming VEM formulation for non-linear time dependent convection dominated diffusion reaction equation can be considered as a future work.

Chapter 5

Virtual element methods for time-dependent Stokes equation

5.1 Introduction

Non-stationary Stokes equation describes many physical phenomena of incompressible flow problems including modelling of weather predictions, ocean currents, water flow in a pipe and air flow around a wing. Therefore, it is worth to produce an efficient numerical technique for non-stationary Stokes equation. Several authors have contributed various numerical techniques in this direction. In [61], authors has studied non-stationary Stokes equation considering linear element for velocity approximation and piecewise constant function for pressure approximation ($Q_1 - P_0$ element) which is not discrete inf-sup stable. They estimate error analysis in L^2 and H^1 norm assuming the regularity assumption

$$\begin{cases} \sup\limits_{0\leq t\leq T} \Big(\|\mathbf{u}_t(t)\|_0^2 + \|\mathbf{u}(t)\|_2^2 + \|p(t)\|_1^2 \Big) \leq C, \\ \sup\limits_{0\leq t\leq T} \sigma(t)\ \|\mathbf{u}_t(t)\|_1^2 + \displaystyle\int_0^T \sigma(t) \Big(\|\mathbf{u}_{tt}(t)\|_0^2 + |\mathbf{u}_t(t)|_2^2 + \|p(t)\|_1^2 \Big)\, dt \leq C. \end{cases}$$

In this respect, we desire to convey that the above said assumptions are justified in [62, 63]. Authors also studied boundedness of $\mathbf{u}_t(t)$, $\mathbf{u}(t)$ in L^2 and H^1 norms. It is not worthy to assume exact solution $\mathbf{u}(t) \in H^{k+1}(\Omega)$ which basically contradicts natural phenomenon. Since lowest order ($Q_1 - P_0$) element does not satisfies inf-sup condition, numerical technique demands additional stabilizers. In this direction, the following contributions were studied, such as the stream upwind Petrov-Galerkin(SUPG) method [47], Brezzi-Pitkaranta method [64], the Douglas-Wang method [65], the well-known Galerkin least square(GLS) method [66], the method of bubble function enrichment [67]. Furthermore, in [68], authors examine non-stationary Navier- Stokes equation exploiting unstable velocity pressure pair of the lowest equal order finite element. Hence the existence of divergence free Fortin- operator cannot be guaranteed which make the analysis cumbersome. In view of this issue, authors have introduced discrete Stokes projection operator to estimate theoretical results. Recently, Huang et at. have reviewed equal order ($P1 - P1$) approximation for velocity pressure pair and Crank-Nicolson scheme for time-discretization. Heywood and Ranacher have proposed a fully implicit Crank-Nicolson scheme for Navier-Stokes equation in [69] and they have proved the scheme is unconditionally stable and converges optimally. An error analysis for the Crank-Nicolson extrapolation scheme of time discretization have been studied in [70], where they have utilized stabilized finite element approximation for the space variable. In the last decade, several authors paid sincere attention to study lowest equal-order finite element pair $P_1 - P_1$(linear function on triangle and tetrahedron), $Q_1 - Q_1$ (bilinear functions) or $P_1 - P_1$(linear functions on quadrilateral) using constant projection operator for pressure variable [71, 72]. The above said stabilized finite element technique does not require stabilization parameters and calculation of high order derivatives. Therefore, this technique has drawn attention of several researchers.

This chapter deals with development of virtual element methods(VEM) for time dependent Stokes equation. VEM is new technology having capability of dealing with very general type of polygonal element without explicit knowledge of basis functions. It is observed that VEM has intense connection with mimitic finite difference method(MFDM). The basic idea of MFDM has been moulded as VEM in order to generalize FEM over very general type of element. The novelty of VEM relies on simplicity in implementation, easy to extend to higher dimension (2D to 3D) without much change in mathematical foundation. Discrete bilinear form can be implemented with the help of degrees of freedom bypassing cumbersome numerical computation of basis functions. Since in virtual element discretization, discrete formulations are defined locally this method preserves material property element-wise which is desirable by scientists. Moreover, VEM deals with very distorted elements with optimal accuracy. Even non-convex elements are also allowed. These features have made VEM popular and drawn several researchers to contribute in this direction. In [25], Antonietti et al. have introduced stream virtual element formulation for Stokes problems on polygonal mesh. Discrete scheme introduced in this paper is completely computable based on the information provided by degrees of freedom. In [23], Da veiga et al. have framed a new VEM space which is divergence free and modify the stationary Stokes problem accordingly. Modified VEM space for velocity contains polynomial of order k where for pressure, space contains polynomial of order $k - 1$. The pair is inf-sup stable for $k \geq 2$. The primary drawback of this space is that vector valued L^2 projection operator is not computable on this space(optimal order). Moreover, in same paper author has designed reduced local virtual space which is computationally cheap. However, Vacca has modified the VEM space for velocity variable where L^2 projection operator is completely computable [2] and introduced VEM discretization for Darcy and Brinkman equation. Nonconforming virtual element formulation for stationary Stokes equation has been studied by Cangiani et al.[29]. Virtual element space introduced in this work is not divergent free but the projection operators introduced in this paper are easy to compute from edge moments and cell moments.

In this chapter, we mainly focus on error estimation of semi-discrete case for time dependent stokes equation In order to analyze semi-discrete case, we design Stokes projection which is compatible with discrete virtual element space. The discrete virtual element space investigated in this work is not divergence free. This drawback already has reviewed in [23]. However,the space is discrete inf-sup stable($k \geq 2$) and provide optimal order of convergence in L^2 and H^1 norms. Since the space is discrete inf-sup stable, we can construct Fortin operator which reduces the complexity of theoretical estimation. We modify the VEM space in order to compute L^2 projection operator $\Pi^0_{k,K}$ with optimal order k. The applied technique is basically an extension of idea used in [1], where $\Pi^0_{k,K}$ is computed with the help of $\Pi^\nabla_{k,K}$ operator. Furthermore, we introduce discrete Stokes projection operator and with the help of this projection operator we derive error estimation for semi-discrete case in L^2 and H^1 norm. Discrete formulation introduced in this work is computable by the informations provided by degrees of freedom. Furthermore, we have considered same set of degrees of freedom as mentioned in [14]. For pressure approximation, we consider discontinuous polynomial space of order $k-1$ same as finite element method. Since the virtual element space for pressure is only polynomial space, discrete formulation associated with pressure variable will be same as finite element approximation and also directly computable from degrees of freedom.

Rest of the Chapter is organised as follows. In Section 5.2, we recollect model problem with its continuous formulation. Section 5.3 deals with the basic construction of enhanced virtual element space and associated degrees of freedom. In this Section, we also design discrete virtual element formulation of model problem. In Section 5.5, we introduce discrete Stokes projection and utilizing this projection, we study error estimations for velocity and pressure variables. Optimal error estimations in L^2 and H^1 norm are derived in the same Section. Finally, based on theoretical estimation, we have made some conclusion in Section 5.6.

5.2 Preliminaries and governing equations

We consider the time dependent Stokes equation

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \times (0,T) \\ \text{div}\, \mathbf{u} = 0 \quad \text{in} \quad \Omega \times (0,T) \\ \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ \mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \Omega \times \{0\}, \end{cases} \tag{5.1}$$

where the vector variable $\mathbf{u}$ and scalar variable p represent velocity and pressure respectively. Moreover, we adopt standard notation for Laplacian, divergence, gradient operator as Δ, div, ∇. Additionally, we introduce some standard results which will make the representation convenient. Let us denote continuous velocity space and continuous pressure space by $\mathcal{V}$ and $\mathcal{Q}$ respectively, where

$$\mathcal{V} := [H_0^1(\Omega)]^2; \quad \mathcal{Q} := L_0^2(\Omega) = \Big\{ q \in L^2(\Omega) \; s.t. \int_\Omega q \, d\Omega = 0 \Big\}$$

equipped with natural norms

$$\|\mathbf{v}\|_1^2 := \|\mathbf{v}\|_{[H^1(\Omega)]^2}, \quad \|q\|_{\mathcal{Q}} := \|q\|_{L^2(\Omega)}.$$

We deduce that force function $\mathbf{f} \in [L^2(\Omega)]^2$ and the bilinear form $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ is defined as

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x}, \qquad b(\mathbf{v}, q) := \int_\Omega q \nabla \cdot \mathbf{v} \, d\mathbf{x},$$

where : represent tensor product of two matrices. Exploiting above two bilinear forms, we represent the continuous formulation: find $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ s.t.

$$\begin{cases} (\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u},\mathbf{v}) - b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \quad \mathbf{v} \in V \\ \qquad\qquad\qquad\qquad b(\mathbf{u}, q) = 0 \quad \forall \quad q \in Q, \end{cases} \tag{5.2}$$

where $(\cdot,\cdot)$ is denoted as L^2 inner product on Ω. It can be easily verified that $b(\cdot,\cdot)$ satisfies inf-sup condition, i.e. there exists a positive constant $C_\alpha > 0$ such that the following estimation hold

$$C_\alpha \|l\|_{\mathcal{Q}} \le \sup_{\mathbf{v}\in\mathcal{V} \; \mathbf{v}\neq 0} \frac{b(\mathbf{u}, l)}{\|\mathbf{v}\|_1}, \forall \quad l \in \mathcal{Q}. \tag{5.3}$$

The continuous bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bounded in $\mathcal{V}$ norm, i.e.,

$$\begin{aligned} |a(\mathbf{u},\mathbf{v})| \le & \; C \|\mathbf{u}\|_{\mathcal{V}} \|\mathbf{v}\|_{\mathcal{V}} \quad \forall \quad \mathbf{u},\mathbf{v} \in \mathcal{V}; \\ |b(\mathbf{u}, l)| \le & \; C \|\mathbf{u}\|_{\mathcal{V}} \|l\|_{\mathcal{Q}} \quad \forall \quad \mathbf{v} \in \mathcal{V} \text{ and } l \in \mathcal{Q}; \end{aligned} \tag{5.4}$$

where C is a generic constant. Moreover, $a(\cdot,\cdot)$ satisfies discrete coercivity on $\mathcal{V}$. That is there exists a positive constant C such that

$$a(\mathbf{v},\mathbf{v}) \ge \; C \|\mathbf{v}\|_{\mathcal{V}}^2. \tag{5.5}$$

Results revealed in (5.3),(5.4) and (5.5) ensure that Problem 5.2 has unique solution $(\mathbf{u}, p)$ ([73]) satisfying

$$\|\mathbf{u}\|_{\mathcal{V}} + \|p\|_{\mathcal{Q}} \leq \; C \; \|f\|_{[L^2(\Omega)]}. \tag{5.6}$$

Furthermore $L^p([H^s]^2)$, $1 \leq p \leq \infty$, $s \geq 0$ represent the Hilbert space of all L^p integrable functions $\psi(t) : [0,T] \to H^s(\Omega)$ with the standard norm $\|\psi\|_{L^p([H^s(\Omega)]^2)} := \Big(\int_0^T \|\psi\|^p_{[H^s(\Omega)]^2} \Big)^{1/p}$ for $p \in [1,0)$ with standard modification at $p = \infty$.

5.3 Virtual element spaces

We now demonstrate the basic construction of local and global virtual element space. The virtual element space are constructed in such a fashion that the space will be unisolvent w.r.t. a set of functionals entitled as degrees of freedom (DOF). Moreover, the space satisfies all the assumptions which we will infer in order to analyze theoretical estimations. In order to perform convergence analysis, we introduce two basic tools, L^2 orthogonal projection operator $\Pi^0_{k,K}$ and energy projection operator $\Pi^\nabla_{k,K}$.

$$\Pi^\nabla_{k,K} : [H^1(K)]^2 \to [\mathbb{P}_k(K)]^2.$$

defined by

$$\begin{cases} \displaystyle\int_K \nabla \mathbf{w}_k : \nabla(\mathbf{v}_h - \Pi^\nabla_{k,K}\mathbf{v}_h)\; dK = 0 \quad \text{for all } \mathbf{w}_k \in [\mathbb{P}_k(K)]^2, \\ \mathbf{P}_0(\mathbf{v}_h - \Pi^\nabla_{k,K}\mathbf{v}_h) = \mathbf{0}, \end{cases}$$

where $\mathbf{P}_0$ is orthogonal L^2 projection operator onto constant functions. The projection operator $\Pi^\nabla_{k,K}$ can be directly evaluated with the help of DOF. Moreover, for subsequent discussion, we demand L^2 orthogonal projection operator $\Pi^0_{k,K} : [L^2(K)]^2 \to [\mathbb{P}_k(K)]^2$ which is defined as

$$\int_K \mathbf{w}_k \cdot (\mathbf{v}_h - \Pi^0_{k,K}\mathbf{v}_h) dK = 0 \quad \text{for all } \mathbf{w}_k \in [\mathbf{P}_k(K)]^2.$$

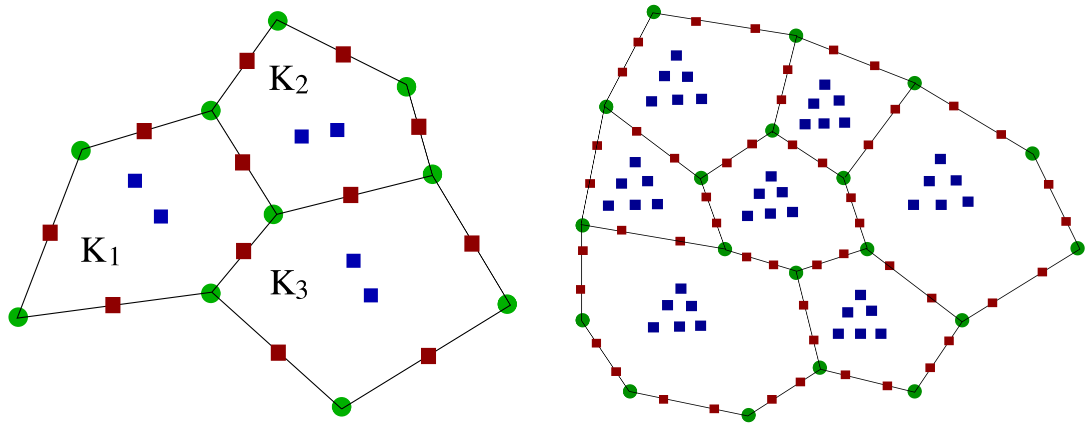
Local virtual element space: We consider the following local virtual element space $Z^k(K)$ which is already defined in [14] for elasticity problem.

$$Z^k(K) := \Big\{ \mathbf{v} \in [H^1(K)]^2 \text{ s.t. } \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2,\; \Delta\mathbf{u} \in [\mathbb{P}_{k-2}(K)]^2 \Big\},$$

where $[\mathbb{B}_k(\partial K))] := \{v \in C^0(\partial K) s.t. v|_e \in \mathbb{P}_k(e) \forall e \in \partial K\}$. Moreover, for a function $\mathbf{v} \in [H^1(K)]^2$, we defined the set of functional $\mathcal{F}_Z$ as

- $(\mathbf{D_1})$ Values of $\mathbf{v}$ at $V(K)$ vertexes of K.
- $(\mathbf{D_2})$ For $k > 1$, the values of $\mathbf{v}$ at $k-1$ uniformly spaced points on each edge $e \subset \partial K$.
- $(\mathbf{D_3})$ For $k > 1$, the moments $\frac{1}{|K|}\int_K \mathbf{w}(\mathbf{x})\,\mathbf{v}(\mathbf{x})\,d\mathbf{x} \quad \forall\, \mathbf{w}(\mathbf{x}) \in [\mathbb{P}_k(K)]^2$.

In order to depict degrees of freedom, we exhibit mesh decomposition in Figures 5.1.



(a) $k = 2$ **(b)** $k = 3$

Figure 5.1: Typical degrees of freedom of polygonal elements; (D_1) degrees of freedom are indicated by green circles; (D_2) are indicated by red squares;cell moments are indicated by blue squares.

Lemma 32. *$Z^k(K)$ is unisolvent with respect to $\mathcal{F}_Z$.*

Proof. See in details [14]. ∎

For polynomial degree k, local virtual element space $Z^k(K)$ consists of functions v which is polynomial of degree k on ∂K and Δv is polynomial of degree $k-2$ inside each polygon K. It is well known that $\dim([\mathbb{B}_k(\partial K)]^2) = 2N_K k$ and $\dim([\mathbb{P}_{k-2}(K)]^2) = k(k-1)$. Therefore, dimension of virtual element space $Z^k(K) = 2N_K k + k(k-1)$.

Computation of $\Pi^{\nabla}_{k,K}$ operator on $Z^k(K)$: Let $\boldsymbol{\phi}_i \in Z^k(K)$ be a local basis function. Then $\Pi^{\nabla}_{k,K}\boldsymbol{\phi}_i$ be an element of $(\mathbb{P}_k(K))^2$. Also assume that $\mathbf{q}_k \in (\mathbb{P}_k(K))^2$ be an arbitrary element. Then from the definition of $\Pi^{\nabla}_{k,K}$ operator, we can write as

$$\int_K \nabla \Pi^{\nabla}_{k,K}\boldsymbol{\phi}_i : \nabla \mathbf{q} = \int_K \nabla \boldsymbol{\phi}_i : \nabla \mathbf{q}$$
$$= \underbrace{-\int_K \Delta \mathbf{q} \cdot \boldsymbol{\phi}_i}_{T_1} + \underbrace{\int_{\partial K} (\nabla \mathbf{q}\mathbf{n}) \cdot \phi_i}_{T_2}.$$

Since $\Delta \mathbf{q} \in (\mathbb{P}_k(K))^2$, T_1 can be directly evaluated from internal momentum. The other term T_2 can be computed with $(\mathbf{D_1})$ and$(\mathbf{D_2})$ degrees of freedom.

It can be noticed that the projection operator $\Pi^0_{k-2,K}$ is computable on $Z^k(K)$. Right hand side load term and time dependent part are approximated with the help of $\Pi^0_{k,K}$ operator. The projection operator $\Pi^0_{k,K}$ is not computable over $Z^k(K)$ since we have internal momentum of order upto $k-2$. The same complication, we faced in scalar VEM framework where elliptic equation has been discussed. Employing analogous idea from [1], we will recast the local virtual element space $Z^k(K)$ where the projection operator $\Pi^0_{k,K}$ is fully computable. In view of this requirement, we start with introducing local classical space $\mathcal{V}^k(K)$.

On each polygon $K \in \mathcal{T}_h$, we define classical local space as

$$\mathcal{V}^k(K) := \Big\{ \mathbf{v} \in [H^1(K)]^2 \text{ s.t. } \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2,\ \Delta \mathbf{u} \in [\mathbb{P}_k(K)]^2 \Big\}.$$

Exploiting the elliptic operator $\Pi^{\nabla}_{k,K}$, we recast local modified virtual element space which is basically restriction of $\mathcal{V}^k(K)$

$$\mathcal{W}^k(K) := \Big\{ \mathbf{v} \in \mathcal{V}^k(K)\ s.t. \int_K \mathbf{q}_k \cdot (\mathbf{v} - \Pi^{\nabla}_{k,K}\mathbf{v}) = 0 \quad \forall\, \mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2 \Big\}.$$

Global virtual element space is defined as

$$\mathcal{W}^k_h := \Big\{ \mathbf{v} \in [H^1_0(\Omega)]^2\ s.t.\ \mathbf{v}|_K \in \mathcal{W}^k(K) \Big\}.$$

In classical virtual element space $\mathcal{V}^k(K)$, we consider Laplacian of $\mathbf{w} \in \mathcal{V}^k(K)$ is an element of $(\mathbb{P}_k(K))^2$ which is extra requirement. Modified VEM space is designed as restriction of functions of the space $\mathcal{V}^k(K)$. It seems that the dimension of $\mathcal{W}^k(K)$ is more than dimension of $Z^k(K)$. However, this prediction is no longer true and we will show that $\mathcal{F}_Z$ forms degrees of freedom for $\mathcal{W}^k(K)$. In view of this, we continue our discussion with the following lemma.

Lemma 33. *The dimension of* $\mathcal{V}^k(K)$ *is*

$$dim(\mathcal{V}^k(K)) = 2N_K k + (k+1)(k+2).$$

Furthermore, the set of functional $\mathcal{F}_Z$ *along with the moments* $\mathcal{F}_V$ *which is defined as*

$$\mathcal{I}(\mathbf{v}) := \int_K \mathbf{v} \cdot \mathbf{q}_k \quad \textit{for all} \quad \mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2.$$

form degrees of freedom for $\mathcal{V}^k(K)$.

Proof. The framework of the proof is analogous to scalar valued function mentioned in [1]. Hence, briefly we exhibit the abstract framework of proof. Let $\mathbf{w}_h$ be an element of $\mathcal{V}^k(K)$. We show that

$$\mathbf{w}_h|_{\partial K} = 0 \quad \text{and} \quad \Pi^0_{k,K}\mathbf{w}_h = 0$$

imply $\mathbf{w}_h = 0$. Basically these two conditions indicate $\mathbf{A}\mathbf{w}_h = 0$, where the operator $\mathbf{A}$ is defined as

$$\mathbf{A}\mathbf{w} := \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix},$$

and

$$\mathbf{w} := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Proceeding same as [9, 14], we can construct an one-to-one mapping $\mathbf{R} : [\mathbb{P}_k(K)]^2 \to [\mathbb{P}_k(K)]^2$ which implies isomorphism between internal moments of function $\mathbf{w} \in [H^1_0(K)]^2$ and their Laplacian which is a polynomial of order k. The mapping $\mathbf{R}$ is defined as

$$\mathbf{R}\mathbf{q} := \Pi^0_{k,K}\mathbf{A}^{-1}\mathbf{q}. \quad \text{for all} \quad \mathbf{q} \in [\mathbb{P}_k(K)]^2.$$

Hence the dimension of $\mathcal{V}^k(K)$ is

$$\begin{aligned} \dim(\mathcal{V}^k(K)) &= \dim([\mathbb{B}_k(\partial K)]^2) + \dim([\mathbb{P}_k(K)]^2). \\ &= 2N_K k + (k+1)(k+2). \end{aligned}$$

■

Remark 5. *On $\mathcal{W}^k(K)$, the L^2 projection operator $\Pi^0_{k,K}$ is fully computable. We evaluate $\Pi^0_{k,K}$ operator employing $\Pi^\nabla_{k,K}$ operator. Moreover, for each $\mathbf{v} \in [H^1(K)]^2$, $\Pi^0_{k,K}\mathbf{v}$ and $\Pi^\nabla_{k,K}\mathbf{v}$ are polynomial approximation in $[\mathbb{P}_k(K)]^2$. Hence we can roughly assume that $\Pi^0_{k,K}\mathbf{v} \approx \Pi^\nabla_{k,K}\mathbf{v}$. Furthermore, from the construction of VEM space $\mathcal{W}^k(K)$, it can be deduced that $(\mathbf{q}_k, \mathbf{v}_h - \Pi^\nabla_{k,K}\mathbf{v}_h) = 0$ for $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2$. This condition reduces the dimension of the space $\mathcal{W}^k(K)$.*

Lemma 34. *$\mathcal{W}^k(K)$ is unisolvent with respect to set of functions $\mathcal{F}_Z$.*

Proof. In the modified VEM space $\mathcal{W}^k(K)$, we have additional conditions that are $\dim([\mathbb{P}_k(K)]^2) - \dim([\mathbb{P}_{k-2}(K)]^2) = 4k + 2$. Hence, removing these additional conditions, we obtain

$$\dim(\mathcal{W}^k(K)) \geq \dim[\mathcal{V}^k(K)] - (4k+2) = \dim(\mathcal{Z}^k(K)).$$

In order to show that $\mathcal{W}^k(K)$ is unisolvent with respect to $\mathcal{F}_Z$, we show that an element $\mathbf{w}_h \in \mathcal{W}^k(K)$ with $\mathcal{F}_Z(\mathbf{w}_h) = 0$, is identically zero. Since $\mathcal{F}_Z(\mathbf{w}_h) = 0$ implies $\Pi^{\nabla}_{k,K}\mathbf{w}_h = 0$. Furthermore, from the definition of the VEM space $\mathcal{W}^k(K)$, we have

$$\int_K \mathbf{w}_h \mathbf{q}_k = \int_K \Pi^{\nabla}_{k,K}\mathbf{w}_h \mathbf{q}_k \quad \forall \quad \mathbf{q}_k \in [\mathbb{P}_k(K)]^2)/[\mathbb{P}_{k-2}(K)]^2.$$

Hence we deduce that $\mathcal{F}_V(\mathbf{w}_h) = 0$. Hence from Lemma 33, we can conclude that $\mathbf{w}_h = 0$. ∎

Computation of L^2 projection operator $\Pi^0_{k,K}$ on modified VEM space $\mathcal{W}^k(K)$:

Lemma 35. *The orthogonal L^2 projection operator $\Pi^0_{k,K}$ is computable on $\mathcal{W}^k(K)$.*

Proof. Let $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2$ be an arbitrary function. $\mathbf{q}_k$ can be decomposed as

$$\mathbf{q}_k = \mathbf{q_1} + \mathbf{q_2},$$

where $\mathbf{q_1} \in [\mathbb{P}_{k-2}(K)]^2$ and $\mathbf{q_2} \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2$. Let $\boldsymbol{\phi}_i \in \mathcal{W}^k(K)$ be a local basis function. Hence, we write the following estimation

$$\begin{aligned}
\int_K \Pi^0_{k,K}\boldsymbol{\phi}_i \cdot \mathbf{q}_k &= \underbrace{\int_K \boldsymbol{\phi}_i \cdot \mathbf{q}_k}_{\text{Def. of } \Pi^0_{k,K}} \\
&= \int_K \boldsymbol{\phi}_i \cdot \mathbf{q_1} + \int_K \boldsymbol{\phi}_i \cdot \mathbf{q_2} \\
&= \int_K \boldsymbol{\phi}_i \cdot \mathbf{q_1} + \underbrace{\int_K \Pi^{\nabla}_{k,K}\, \boldsymbol{\phi}_i \cdot \mathbf{q_2}}_{\text{Def. of } \mathcal{W}^k(K)}.
\end{aligned}$$

It is well know that on the space $\mathcal{W}^k(K)$, the energy projection operator $\Pi^{\nabla}_{k,K}$ is computable. Hence, we can enumerate $\Pi^{\nabla}_{k,K}\, \boldsymbol{\phi}_i$. First term involving polynomial of degree $k-2$ is computable and the second term involving the integration $\int_K \Pi^{\nabla}_{k,K}\, \boldsymbol{\phi}_i \cdot \mathbf{q_2}$ is entirely computable. Therefore, we can evaluate $\Pi^0_{k,K}\phi$ for $\phi \in \mathcal{W}^k(K)$. ∎

For pressure approximation, we consider the standard finite dimensional space

$$\mathcal{Q}_k(K) := \mathbb{P}_{k-1}(K).$$

Globally, the pressure space is defined as

$$\mathcal{Q}_h^k := \{l \in L_0^2(\Omega) \text{ s.t. } l|_K \in \mathcal{Q}_k(K) \quad \text{for all } K \in \mathcal{T}_h\}.$$

Moreover, for the local pressure $q \in \mathcal{Q}_k(K)$, we consider the linear operator D_Q

$D_Q :=$ The moment upto order $k-1$ of q, i.e.

$$\int_K q\, p_{k-1} dK \quad \text{for all} \quad p_{k-1} \in \mathbb{P}_{k-1}(K).$$

It can be easily observed that $\mathcal{Q}_k(K)$ is unisolvent w.r.t. D_Q.

5.4 Discrete virtual element formulation

In this section, we construct discrete virtual element formulation of the weak formulation (5.2). Employing L^2 projection operator $\Pi^0_{k,K}$, we approximate non-stationary part (u_t, v). Local discrete formulations consist of two parts such as polynomial part and non-polynomial part. Global formulation is obtained by summing the local formulation on each polygon. For $\mathbf{u}, \mathbf{v} \in [L_0^2(K)]^2$, we define the inner-product

$$m^K(\mathbf{u}, \mathbf{v}) := \int_K \mathbf{u} \cdot \mathbf{v},$$

where K denotes polygon.

Employing inner-product $m^K(\cdot,\cdot)$, we define local discrete formulation for time derivative part as

$$m_h^K(\mathbf{u}_h, \mathbf{v}_h) := m^K(\Pi^0_{k,K}\mathbf{u}_h,\ \Pi^0_{k,K}\mathbf{v}_h) + \mathcal{S}_m^K\Big((I - \Pi^0_{k,K})\mathbf{u}_h,\ (I - \Pi^0_{k,K})\mathbf{v}_h\Big),$$

where $\mathcal{S}_m^K\Big(\cdot,\cdot\Big) : \mathcal{W}^k(K) \times \mathcal{W}^k(K) \to \mathbb{R}$ be a symmetric bilinear form that stabilizes the bilinear form $m_h^K(\cdot,\cdot)$ satisfying

$$\beta_*\, m^K(\mathbf{v}_h, \mathbf{v}_h) \le \mathcal{S}_m^K(\mathbf{v}_h, \mathbf{v}_h) \le \beta^* m^K(\mathbf{v}_h, \mathbf{v}_h) \quad \text{for all} \quad \mathbf{v}_h \in \ker(\Pi^0_{k,K}), \tag{5.7}$$

where $\ker(\Pi^0_{k,K}) \subset \mathcal{W}^k(K)$ denotes kernel of $\Pi^0_{k,K}$. Furthermore, employing energy projection operator $\Pi^\nabla_{k,K}$, we discretize the bilinear form $a(\mathbf{u}, \mathbf{v})$ as

$$a_h^K(\mathbf{u}_h, \mathbf{v}_h) := a^K(\Pi^\nabla_{k,K}\mathbf{u}_h, \Pi^\nabla_{k,K}\mathbf{v}_h) + \mathcal{S}_a^K\Big((I - \Pi^\nabla_{k,K})\mathbf{u}_h, (I - \Pi^\nabla_{k,K})\mathbf{v}_h\Big),$$

where $\mathbf{u}_h, \mathbf{v}_h \in \mathcal{W}^k(K)$ and the stabilizer $\mathcal{S}_a^K\Big(\cdot,\cdot\Big) : \mathcal{W}^k(K) \times \mathcal{W}^k(K) \to \mathbb{R}$ satisfies

$$\alpha_* \, a^K(\mathbf{v}_h, \mathbf{v}_h) \leq \mathcal{S}_a^K(\mathbf{v}_h, \mathbf{v}_h) \leq \alpha^* a^K(\mathbf{v}_h, \mathbf{v}_h) \quad \text{for all} \quad \mathbf{v}_h \in \ker(\Pi^{\nabla}_{k,K}). \tag{5.8}$$

α_*, α^*, β_* and β^* are positive constants independent of K and h. The global formulation is defined as adding local bilinear form over each polygon K.

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_K a_h^K(\mathbf{u}_h, \mathbf{v}_h), \\ m_h(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_K m_h^K(\mathbf{u}_h, \mathbf{v}_h). \end{aligned}$$

Stability An application of equations (5.7) and (5.8) yields four positive constants independent of h and K such that for $\mathbf{v}_h \in \mathcal{W}^k(K)$, it holds

$$\begin{aligned} \min\{\alpha_*, 1\} \, a^K(v,v) \ &\leq \ a_h^K(v,v) \leq \ \max\{\alpha^*, 1\} \, a^K(v,v); \\ \min\{\beta_*, 1\} \, m^K(v,v)_K \ &\leq \ m_h^K(v,v) \leq \ \max\{\beta^*, 1\} \, m^K(v,v), \end{aligned} \tag{5.9}$$

In order to derive error estimation for semi-discrete case in L^2 and H^1 norm, we desire polynomial consistency property of discrete bilinear forms $a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$ which can be stated as

Lemma 36. *For all polygonal element $K \in \mathcal{T}_h$, the bilinear form $m_h^K(\cdot,\cdot)$ and $a_h^K(\cdot,\cdot)$ satisfy the following consistency property*

$$\begin{aligned} a_h^K(\mathbf{q}_k, \mathbf{v}_h) &= a^K(\mathbf{q}_k, \mathbf{v}_h), \\ m_h^K(\mathbf{q}_k, \mathbf{v}_h) &= m^K(\mathbf{q}_k, \mathbf{v}_h), \end{aligned}$$

for all $\mathbf{q}_k \in \mathbb{P}_k(K)$ and $\mathbf{v}_h \in \mathcal{W}_h^k(K)$.

Proof. Since $\int_K (\nabla \Pi^{\nabla}_{k,K} \mathbf{v}_h - \nabla \mathbf{v}_h) : \nabla \mathbf{q}_k = 0$ for all $\mathbf{q}_k \in \mathbb{P}_k(K)$, we have $\mathcal{S}_a^K(\mathbf{q}_k, \mathbf{v}_h) = 0$. Similarly, applying definition of L^2 projection operator $\Pi^0_{k,K}$, we deduce that $\mathcal{S}_m^K(\mathbf{q}_k, \mathbf{v}_h) = 0$. Moreover, both the operators are identity on polynomial space $(\mathbb{P}_k(K))^2$, i.e., $\Pi^{\nabla}_{k,K}(\mathbb{P}_k(K))^2 = (\mathbb{P}_k(K))^2$ and $\Pi^0_{k,K}(\mathbb{P}_k(K))^2 = (\mathbb{P}_k(K))^2$ which gives the final thesis. ■

Approximation of right-hand side load term $(\mathbf{f}_h, \mathbf{v}_h)$**:** Now, we frame discretization of right-hand side load term $(\mathbf{f}, \mathbf{v})$ where $\mathbf{f}$ denotes force function. For all $K \in \mathcal{T}_h$, exploiting $\Pi^0_{k,K}$ operator, we approximate load term $\mathbf{f}_h$ as

$$\mathbf{f}_h|_K := \Pi^0_{k,K} \mathbf{f}, \quad \text{for all } K \in \mathcal{T}_h.$$

Hence, utilizing orthogonality property of $\Pi^0_{k,K}$ operator, we can recast as

$$(\mathbf{f}_h, \mathbf{v}_h) := \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f}_h \cdot \mathbf{v}_h \, dK = \sum_{K \in \mathcal{T}_h} \int_K \Pi^0_{k,K} \mathbf{f} \cdot \mathbf{v}_h \, dK = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f} \cdot \Pi^0_{k,K} \mathbf{v}_h \, dK.$$

In contrast with FEM, convergence analysis depends on regularity of force function $\mathbf{f}$. An approximation property of the projection operator $\Pi^0_{k,K}$ ensures optimal order of convergence. On $\mathcal{W}_h$, the projection operator $\Pi^0_{k,K}$ is fully computable. Moreover, $\Pi^0_{k,K}\mathbf{v}_h$ can be written in terms of polynomial. Hence, the right-hand side reduces to integration of known function, which can be evaluated by applying appropriate quadrature rule.

Semidiscrete variational formulation of (5.2) is defined as follows: find $(\mathbf{u}_h, p_h) \in \mathcal{W}^k_h \times \mathcal{Q}_h$ s.t.

$$\begin{cases} m_h(\partial_t \mathbf{u}_h, \mathbf{v}_h) + a_h(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h), \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad b(\mathbf{u}_h, q_h) = 0. \end{cases} \tag{5.10}$$

Moreover, the pair $(\mathcal{W}^k_h, \mathcal{Q}_h)$ satisfies discrete inf-sup condition

Lemma 37. *The family of virtual element spaces $\{(\mathcal{W}^k_h, \mathcal{Q}_h)\}_{h>0}$ satisfies the discrete inf-sup condition, i.e., there exists a positive constant $\mathcal{B} > 0$ such that the following holds*

$$\mathcal{B} \, \|\mathbf{q}_h\|_{\mathcal{Q}} \leq \; C \sup_{\substack{\mathbf{w}_h \in \mathcal{W}^k_h \\ \mathbf{w}_h \neq 0}} \frac{b(\mathbf{w}_h, \mathbf{q}_h)}{\|\mathbf{w}_h\|_{\mathcal{V}}} \quad \forall \quad \mathbf{q}_h \in \mathcal{Q}^k_h. \tag{5.11}$$

Proof. see in detail in [14]. ∎

5.5 Convergence analysis

In this section, we perform error estimations in L^2 and H^1 norm employing discrete Stokes projection operator (Π^s_h, Π^p_h). Furthermore, on virtual element space $\mathcal{W}^k_h$, we have the following approximation property.

Lemma 38. *For all h, let $K \in \mathcal{T}_h$ and k be a natural number. Then for all $\mathbf{w} \in [H^{m+1}(K)]^2$ where $0 \leq m \leq k$, there exists a polynomial function $\mathbf{u}_\pi \in [\mathbb{P}_k(K)]^2$, such that*

$$\|\mathbf{u} - \mathbf{u}_\pi\|_{0,K} + h_K \, |\mathbf{u} - \mathbf{u}_\pi|_{1,K} \leq \; C \, h_K^{m+1} \, |\mathbf{u}|_{m+1,K}.$$

Proof. The result follows from classical result by Scott-Dupont. ∎

Moreover, since the pair $(\mathcal{W}_h^k, \mathcal{Q}_h^k)$ satisfies discrete inf-sup condition, proceeding analogously as [14], we can prove the the following result.

Lemma 39. *For each enough regular* $\mathbf{u} \in V$*, there exists* $\mathbf{u}_I \in V_h$*, s.t. the following condition holds*

$$\begin{aligned} \Pi^0_{k-1,K}(\nabla \cdot \mathbf{u}_I) &= \Pi^0_{k-1,K}(\nabla \cdot \mathbf{u}) \,\forall\, K \in \mathcal{T}_h, \\ \|\mathbf{u} - \mathbf{u}_I\|_V &\leq C \inf_{v_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_V. \end{aligned}$$

Proof. See in detail in [14]. ■

Next, we move to define discrete Stokes projection operator which will be utilized to derive error estimation for velocity and pressure variable in L^2 and H^1 norms.

Discrete Stokes Projection: Let (u, p) be the solution of model problem (5.1), then discrete stokes projection is defined as $(\Pi_h^s u, \Pi_h^p p) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$

$$\begin{aligned} a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, \Pi_h^p p) = a(\mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, p) \,\forall\, \mathbf{v}_h \in \mathcal{W}_h^k \\ b(\Pi_h^s \mathbf{u}, q_h) = b(\mathbf{u}, q_h) \,\forall\, q_h \in \mathcal{Q}_h^k. \end{aligned} \tag{5.12}$$

For each $\mathbf{u}$, the approximation $\Pi_h^s \mathbf{u} \in \mathcal{W}_h^k$ converges with optimal order in L^2 and H^1 norm. An application of Lemma 39, and discrete inf-sup condition yields the following result.

Lemma 40. *Let* $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ *satisfies equation (5.1) and* $(\Pi_h^s \mathbf{u},\ \Pi_h^p p)$ *be the discrete Stokes projection. Then the following estimations hold*

$$\begin{aligned} \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0 + h\, |u - \Pi_h^s u|_1 &\leq C\, h^{k+1} \Big(|\mathbf{u}|_{k+1} + |p|_k\Big), \\ \|p - \Pi_h^p p\|_0 &\leq C\, h^k \Big(|\mathbf{u}|_{k+1} + |p|_k\Big). \end{aligned}$$

Proof. We split the error $\mathbf{u} - \Pi_h^s \mathbf{u}$ as follows

$$\mathbf{u} - \Pi_h^s \mathbf{u} = \mathbf{u} - \mathbf{u}_I + \mathbf{u}_I - \Pi_h^s \mathbf{u},$$

where $\mathbf{u}_I$ be the interpolation of $\mathbf{u}$, defined in lemma 39. Let $\eta = \Pi_h^s \mathbf{u} - \mathbf{u}_I$ be an element of $\mathcal{W}_h^k$. Hence, employing discrete coercivity of $a_h(\cdot,\cdot)$, we have

$$\begin{aligned}
|\eta|_1^2 &\leq a_h(\Pi_h^s \mathbf{u} - \mathbf{u}_I, \eta) \\
&= a_h(\Pi_h^s \mathbf{u}, \eta) - a_h(\mathbf{u}_I, \eta) \\
&= a(\mathbf{u}, \eta) + b(\eta, \Pi_h^p p - p) - a_h(\mathbf{u}_I, \eta) \\
&= a(\mathbf{u}, \eta) - a_h(\mathbf{u}_I, \eta) + b(\eta, \Pi_h^p p - p_\pi + p_\pi - p) \\
&= a(\mathbf{u}, \eta) - a_h(\mathbf{u}_I, \eta) + b(\eta, p_\pi - p) \\
&\leq \ C\ h^k\ |\mathbf{u}|_{k+1}\ |\eta|_1 + C\ h^k\ |p|_k\ |\eta|_1.
\end{aligned}$$

Since $|\eta|_1 \neq 0$, we have

$$|\eta|_1 \leq C\ h^k\ (|\mathbf{u}|_{k+1} + |p|_k).$$

Together with the estimation $|\mathbf{u} - \mathbf{u}_I|_1 \leq \ C\ h^k\ |\mathbf{u}|_{k+1}$, we deduce

$$|\mathbf{u} - \Pi_h^s \mathbf{u}|_1 \leq C\ h^k \Big(|\mathbf{u}|_{k+1} + |p|_k \Big). \tag{5.13}$$

Now, we proceed to estimate $\|p - \Pi_h^p p\|_0$. Let q_h be an arbitrary element of $\mathcal{Q}_h^k$, then from discrete inf-sup condition (Lemma 37), we have

$$\begin{aligned}
\beta\ \|\Pi_h^p p - q_h\|_0 &\leq \sup_{\mathbf{v}_h \in V_h \setminus \{0\}} \frac{b(\mathbf{v}_h, \Pi_h^p p - q_h)}{\|\mathbf{v}_h\|_1} \\
&= \sup_{\mathbf{v}_h \in V_h \setminus \{0\}} \frac{b(\mathbf{v}_h, \Pi_h^p p - p) + b(\mathbf{v}_h, p - q_h)}{\|\mathbf{v}_h\|_1}.
\end{aligned}$$

An application of definition of Stokes projection(5.12) helps to derive

$$\begin{aligned}
b(\mathbf{v}_h, \Pi_h^p p - p) &= a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h) \\
&= \sum_{K \in \mathcal{T}_h} \Big(a_h^K(\Pi_h^s \mathbf{u} - \mathbf{u}_\pi, \mathbf{v}_h) + a^K(\mathbf{u}_\pi - \mathbf{u}, \mathbf{v}_h) \Big).
\end{aligned}$$

With the help of estimation (5.13) and Lemma 38, we can write

$$b(\mathbf{v}_h, \Pi_h^p p - p) \leq C\ h^k\ (|\mathbf{u}|_{k+1} + |p|_k)\ |\mathbf{v}_h|_1. \tag{5.14}$$

Therefore, the error $\|p - \Pi_h^p p\|$ can be rewritten as

$$\|p - \Pi_h^p p\|_0 \leq C\ h^k\ (|\mathbf{u}|_{k+1} + |p|_k) + \inf_{q_h \in \mathcal{Q}_h} \|p - q_h\|, \tag{5.15}$$

which gives the required thesis.

In order to estimate $\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0$, we apply the duality argument. We consider the following model problem

$$
\begin{aligned}
-\Delta \boldsymbol{\Psi} + \nabla r &= \mathbf{u} - \Pi_h^s \mathbf{u} \\
\nabla \cdot \boldsymbol{\Psi} &= 0.
\end{aligned}
$$

Moreover, we have the following regularity result

$$
\|\boldsymbol{\Psi}\|_2 + |r|_1 \leq C\, \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0.
$$

We have

$$
\begin{aligned}
\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0^2 &= a(\boldsymbol{\Psi}, \mathbf{u} - \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r) \\
&= a(\boldsymbol{\Psi} - \boldsymbol{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) + a(\boldsymbol{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r).
\end{aligned} \tag{5.16}
$$

The first term of estimation (5.16), can be estimated as

$$
a(\boldsymbol{\Psi} - \boldsymbol{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) \leq \; C\, h^{k+1}\, \|\boldsymbol{\Psi}\|_2 \Big(|\mathbf{u}|_{k+1} + |p|_k \Big).
$$

The other two terms of estimation (5.16) can be bounded as

$$
\begin{aligned}
a(\boldsymbol{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r) = a(\boldsymbol{\Psi}_I, \mathbf{u}) - a(\boldsymbol{\Psi}_I, \Pi_h^s \mathbf{u}) + a_h(\boldsymbol{\Psi}_I, \Pi_h^s \mathbf{u}) \\
- a_h(\boldsymbol{\Psi}_I, \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r).
\end{aligned}
$$

We denote

$$
H1 = a_h(\boldsymbol{\Psi}_I, \Pi_h^s \mathbf{u}) - a(\boldsymbol{\Psi}_I, \Pi_h^s \mathbf{u}),
$$

and

$$
H2 = a(\boldsymbol{\Psi}_I, \mathbf{u}) - a_h(\boldsymbol{\Psi}_I, \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r).
$$

An application of definition of projection operators implies

$$
\begin{aligned}
H2 &= b(\boldsymbol{\Psi}_I, p) - b(\boldsymbol{\Psi}_I, \Pi_h^p p) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r) \\
&= b(\boldsymbol{\Psi}_I - \boldsymbol{\Psi}, p - \Pi_h^p p) + \underbrace{b(\boldsymbol{\Psi}, p - \Pi_h^p p)}_{0} - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r - r_\pi).
\end{aligned}
$$

Employing the Lemma 39 and estimation 5.15, we have

$$\begin{aligned} |H2| &\leq |\mathbf{\Psi}_I - \mathbf{\Psi}|_1 \, \|p - \Pi_h^p p\|_0 + |\mathbf{u} - \Pi_h^s \mathbf{u}|_1 \, \|r - r_\pi\|_0 \\ &\leq \; C \, h^{k+1} \Big(|\mathbf{u}|_{k+1} + |p|_k \Big) \, \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0. \end{aligned} \tag{5.17}$$

An application of polynomial consistency property of $a_h(\cdot,\cdot)$ and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} a_h(\Pi_h^s \mathbf{u}, \mathbf{\Psi}_I) - a(\Pi_h^s \mathbf{u}, \mathbf{\Psi}_I) &= \sum_{K \in \mathcal{T}_h} \Big(a_h^K(\Pi_h^s \mathbf{u} - \mathbf{u}_\pi, \mathbf{\Psi}_I - \mathbf{\Psi}_\pi) - a(\Pi_h^s \mathbf{u} - \mathbf{u}_\pi, \mathbf{\Psi}_I - \mathbf{\Psi}_\pi) \Big) \\ &\leq \; C \, |\Pi_h^s \mathbf{u} - \mathbf{u}_\pi|_1 \, |\mathbf{\Psi}_I - \mathbf{\Psi}_\pi| \\ &\leq \; C \, h^{k+1} \Big(|\mathbf{u}|_{k+1} + |p|_k \Big) \, \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0. \end{aligned} \tag{5.18}$$

Inserting the estimations (5.17) and (5.18) in (5.16), we obtain

$$\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0 \leq \; C \, h^{k+1} \Big(|\mathbf{u}|_{k+1} + |p|_k \Big). \tag{5.19}$$

An application of (5.19) and (5.13) yield the desired thesis. ■

5.5.1 Error estimation for the velocity

Optimal L^2 error estimation

Theorem 41. *Let $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies (5.1) and $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ satisfies (5.10). Then the following estimation holds*

$$\begin{aligned} \|(\mathbf{u} - \mathbf{u}_h)(t)\|_0 \leq \|(\mathbf{u} - \mathbf{u}_h)(0)\|_0 + C \, h^{k+1} \Big(& |\mathbf{u}_0|_{k+1} + |\mathbf{u}|_{k+1} + |p|_k \\ & + \|\mathbf{f}\|_{L^2(0,T,H^{k+1}(\Omega))} + \|\mathbf{u}_t\|_{L^2(0,T,H^{k+1}(\Omega))} \Big). \end{aligned}$$

Proof. In order to estimate $\|\mathbf{u} - \mathbf{u}_h\|_0$, we split the term with the help of Stokes projection $\Pi_h^s \mathbf{u}$.

$$\mathbf{u} - \mathbf{u}_h = \mathbf{u} - \Pi_h^s \mathbf{u} + \Pi_h^s \mathbf{u} - \mathbf{u}_h.$$

Estimation of $\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0$ is known. Now, we proceed to estimate $\boldsymbol{\zeta} = \mathbf{u}_h - \Pi_h^s \mathbf{u}$. Putting $\boldsymbol{\zeta}$ in the semi-discrete approximation (5.10), we get

$$m_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_t, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h). \tag{5.20}$$

We choose $\mathbf{v}_h = \boldsymbol{\zeta}$ in estimation (5.20) which reduces to

$$m_h(\boldsymbol{\zeta}_t, \boldsymbol{\zeta}) + a_h(\boldsymbol{\zeta}, \boldsymbol{\zeta}) - b(\boldsymbol{\zeta}, p_h) = (\mathbf{f}_h, \boldsymbol{\zeta}_h) - m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_h) - a_h(\Pi_h^s \mathbf{u}, \boldsymbol{\zeta}_h). \quad (5.21)$$

Since $b(\boldsymbol{\zeta}, q_h) = 0$ for all $q_h \in Q_h$ and the discrete bilinear form $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ satisfies stability property (5.9), estimation (5.21)reduces to

$$\frac{d}{dt}\|\boldsymbol{\zeta}\|_0^2 + |\boldsymbol{\zeta}|_1^2 \leq C_1 \underbrace{\Big((\mathbf{f}_h, \boldsymbol{\zeta}) - (\mathbf{f}, \boldsymbol{\zeta})\Big)}_{T_1} + C_2 \underbrace{\Big(-m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}) + (\mathbf{u}_t, \boldsymbol{\zeta})\Big)}_{T_2}. \quad (5.22)$$

Exploiting approximation property of L^2 projection operator $\Pi_{k,K}^0$ and Cauchy-Schwartz inequality, we have

$$\begin{aligned} |T_1| &= |(\mathbf{f}_h, \boldsymbol{\zeta}) - (\mathbf{f}, \boldsymbol{\zeta})| \\ &\leq \sum_K |(\mathbf{f}_h - \mathbf{f}, \boldsymbol{\zeta})_K| \\ &\leq \sum_K \|\Pi_{k,K}^0 \mathbf{f} - \mathbf{f}\|_K \, \|\boldsymbol{\zeta}\|_K \\ &\leq C \sum_K h_K^{k+1} |\mathbf{f}|_{k+1,K} \, \|\boldsymbol{\zeta}\|_{0,K} \\ &\leq C \, h^{k+1} \, |\mathbf{f}|_{k+1} \, \|\boldsymbol{\zeta}\|. \end{aligned} \quad (5.23)$$

Non-stationary part can be estimated with the help of polynomial consistency property of bilinear form $m_h(\cdot,\cdot)$

$$\begin{aligned} |T_2| &= |-m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}) + (\mathbf{u}_t, \boldsymbol{\zeta})| \\ &\leq \sum_K |m_h^K(\Pi_h^s \mathbf{u}_t - \Pi_{k,K}^0 \mathbf{u}_t, \boldsymbol{\zeta})| + |(\Pi_{k,K}^0 \mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})_K| \\ &\leq C \sum_K \Big(|m_h^K(\Pi_h^s \mathbf{u}_t - \mathbf{u}_t + \mathbf{u}_t - \Pi_{k,K}^0 \mathbf{u}_t, \boldsymbol{\zeta})| + |(\Pi_{k,K}^0 \mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})_K|\Big) \\ &\leq C \sum_K \Big(|m_h^K(\Pi_h^s \mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})| + |m_h^K(\mathbf{u}_t - \Pi_{k,K}^0 \mathbf{u}_t, \boldsymbol{\zeta})| + |(\Pi_{k,K}^0 \mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})_K|\Big) \\ &\leq C \sum_K \|\Pi_{k,K}^0 \mathbf{u}_t - \mathbf{u}_t\|_{0,K} \, \|\boldsymbol{\zeta}\|_{0,K} \\ &\leq C \, h^{k+1} \, |\mathbf{u}_t|_{k+1} \, \|\boldsymbol{\zeta}\|_{0,K}. \end{aligned} \quad (5.24)$$

Inserting (5.23) and (5.24) into (5.22), we have

$$\|\boldsymbol{\zeta}\|_0 \, \frac{d}{dt}\|\boldsymbol{\zeta}\|_0 + |\boldsymbol{\zeta}|_1^2 \leq C \, h^{k+1}\Big(|\mathbf{f}|_{k+1} + |\mathbf{u}_t|_{k+1}\Big) \, \|\boldsymbol{\zeta}\|_0.$$

Since the term $|\boldsymbol{\zeta}|_1^2$ is positive, we can estimate as

$$\frac{d}{dt}\|\boldsymbol{\zeta}\|_0 \leq \ C\ h^{k+1}\Big(|\mathbf{f}|_{k+1} + |\mathbf{u}_t|_{k+1}\Big). \tag{5.25}$$

Taking integration from 0 to t, we have

$$\|\boldsymbol{\zeta}(t)\|_0 \leq \ \|\boldsymbol{\zeta}(0)\|_0 + C\ h^{k+1}\ \Big(|\mathbf{f}|_{L^2(0,T,H^{k+1}(\Omega))} + |\mathbf{u}_t|_{L^2(0,T,H^{k+1}(\Omega)}\Big). \tag{5.26}$$

Moreover, we have

$$\|\boldsymbol{\zeta}(0)\| \leq \|\mathbf{u}(0) - \mathbf{u}_h(0)\|_0 + \ C\ h^{k+1}\ |\mathbf{u}(0)|_{H^{k+1}(\Omega)}. \tag{5.27}$$

Inserting (5.27) into (5.26) and with the help of Lemma 40, we obtain final estimation

$$\begin{aligned}\|(\mathbf{u} - \mathbf{u}_h)(t)\| \leq \|(\mathbf{u} - \mathbf{u}_h)(0)\| + \ C\ h^{k+1}\ \Big(&|\mathbf{u}(0)|_{H^{k+1}(\Omega)} + |\mathbf{u}|_{k+1} + |p|_k \\ &+ |\mathbf{f}|_{L^2(0,T,H^{k+1}(\Omega))} + |\mathbf{u}_t|_{L^2(0,T,H^{k+1}(\Omega)}\Big).\end{aligned}$$

Next, we move to estimate convergence analysis in H^1-norm.

Theorem 42. *Let $(\mathbf{u}, p) \in V \times Q$ satisfies (5.1) and $(\mathbf{u}_h, p_h) \in W_h^k \times Q_h$ satisfies (5.10). Moreover, we assume that $\mathbf{f} \in L^2(0,T;(H^{k+1}(\Omega))^2)$, $\mathbf{u} \in L^2(0,T;(H^{k+1}(\Omega))^2)$ and $\mathbf{u} \in L^2(0,T;(H^{k+1}(\Omega))^2)$. Then there exists a generic constant C such that the following estimation holds*

$$\begin{aligned}|\mathbf{u}(t) - \mathbf{u}_h(t)|_1 \leq \ &|\mathbf{u}(0) - \mathbf{u}_h(0)|_1 + C_1\ h^k\Big(|\mathbf{u}(0)|_{k+1} + |\mathbf{u}|_{k+1} + |p|_k\Big) \\ &+ \ C_2\ h^{k+1}\ \Big(|\mathbf{f}|_{L^2(0,T;H^{k+1}(\Omega))} + |\mathbf{u}_t|_{L^2(0,T;H^{k+1}(\Omega))}\Big).\end{aligned}$$

We review the following equation

$$m_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_t, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h). \tag{5.28}$$

Discrete Stokes projection operator Π_h^s define in (5.12) commute with time-derivative. Hence, we have the result

$$\boldsymbol{\zeta}_t = \frac{d}{dt}(\mathbf{u}_h - \Pi_h^s \mathbf{u}) = \mathbf{u}_{ht} - \Pi_h^s\ \mathbf{u}_t.$$

Choosing $\mathbf{v}_h = \boldsymbol{\zeta}_t$ in (5.28) and since $b(\boldsymbol{\zeta}_t, q_h) = 0$ for all $q_h \in Q_h$, we have

$$m_h(\boldsymbol{\zeta}_t, \boldsymbol{\zeta}_t) + a_h(\boldsymbol{\zeta}, \boldsymbol{\zeta}_t) = (\mathbf{f}_h, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_t) - a_h(\Pi_h^s \mathbf{u}, \boldsymbol{\zeta}_t).$$

Exploiting stability property of discrete bilinear form $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ revealed in (5.9) and making use that time derivative which commutes with discrete bilinear form $a_h(\cdot,\cdot)$, we have

$$\|\boldsymbol{\zeta}_t\|_0^2 + \frac{1}{2}\frac{d}{dt}|\boldsymbol{\zeta}|_1^2 \leq C\,(\mathbf{f}_h - \mathbf{f}, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_t) + (\mathbf{u}_t, \boldsymbol{\zeta}_t). \tag{5.29}$$

An analogous estimation as (5.23) yields the following result

$$\|(\mathbf{f}_h - \mathbf{f}, \boldsymbol{\zeta}_t)\| \leq C\, h^{k+1}\, |\mathbf{f}|_{k+1}\, \|\boldsymbol{\zeta}_t\|_0. \tag{5.30}$$

Exploiting polynomial consistency property, continuity of discrete bilinear form $m_h(\cdot,\cdot)$ and standard approximation property of projection operator Π_h^s and Cauchy-Schwartz inequality, we have

$$|-m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_t) + (\mathbf{u}_t, \boldsymbol{\zeta}_t)| \leq C\, h^{k+1}\, |\mathbf{u}_t|_{k+1}\, \|\boldsymbol{\zeta}_t\|_0. \tag{5.31}$$

Inserting (5.30) and (5.31) into (5.29), we obtain

$$\|\boldsymbol{\zeta}_t\|_0^2 + \frac{1}{2}\frac{d}{dt}|\boldsymbol{\zeta}|_1^2 \leq C\, h^{2k+2}\Big(|\mathbf{f}|_{k+1}^2 + |\mathbf{u}_t|_{k+1}^2\Big).$$

Since $\|\boldsymbol{\zeta}_t\|_0^2$ is positive quantity, we have

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{\zeta}|_1^2 \leq C\, h^{2k+2}\,\Big(|\mathbf{f}|_{k+1}^2 + |\mathbf{u}_t|_{k+1}^2\Big). \tag{5.32}$$

Integrating the above equation (5.32) form 0 to t, we have

$$|\boldsymbol{\zeta}(t)|_1 \leq |\boldsymbol{\zeta}(0)|_1 + C\, h^{k+1}\Big(|\mathbf{f}|_{L^2(0,T;H^{k+1}(\Omega))} + |\mathbf{u}_t|_{L^2(0,T;H^{k+1}(\Omega))}\Big). \tag{5.33}$$

Employing approximation property of Π_h^s operator, we have

$$|\boldsymbol{\zeta}(0)|_1 \leq |\mathbf{u}_h(0) - \mathbf{u}(0)|_1 + C\, h^k\, |\mathbf{u}(0)|_{k+1}. \tag{5.34}$$

Inserting (5.34) into (5.33) and exploiting Lemma 40, we have

$$\begin{aligned}|\mathbf{u}(t) - \mathbf{u}_h(t)|_1 \leq\ & |\mathbf{u}(0) - \mathbf{u}_h(0)|_1 + C_1\, h^k\Big(|\mathbf{u}(0)|_{k+1} + |\mathbf{u}(t)|_{k+1} + |p|_k\Big)\\ & + C_2\, h^{k+1}\,\Big(|\mathbf{f}|_{L^2(0,T;H^{k+1}(\Omega))} + |\mathbf{u}_t|_{L^2(0,T;H^{k+1}(\Omega))}\Big).\end{aligned}$$

Now, we proceed to estimate $\|(\mathbf{u}_t - \mathbf{u}_{ht})(t)\|_0$. ■

Theorem 43. *Let $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies (5.1) and $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ be the* *corresponding discrete solution satisfying (5.10). Moreover, we assume that* $\mathbf{f}_t(t) \in (H^{k+1}(\Omega))^2$, $\mathbf{u}_{tt}(t) \in (H^{k+1}(\Omega))^2$ *and* $\mathbf{u}_t(t) \in (H^{k+1}(\Omega))^2$ *for all* $t \in [0,T]$. *Then there exists a generic constant* C *such that the following estimation holds*

$$\|(\mathbf{u}_t - \mathbf{u}_{ht})(t)\| \leq \|\mathbf{u}_t(0) - \mathbf{u}_{ht}(0)\| + C\, h^{k+1} \Big(|\mathbf{u}_t(0)|_{k+1} + |\mathbf{u}_t(t)|_{k+1} + |p_t|_k\Big) \\ + C\, h^{k+1} \Big(|\mathbf{f}_t|_{L^2(0,t;H^{k+1}(\Omega))} + |\mathbf{u}_{tt}|_{L^2(0,t;H^{k+1}(\Omega))}\Big).$$

Proof. We first consider the equation

$$m_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_t, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h). \quad (5.35)$$

Differentiating equation (5.35) with respect to t and since $\mathbf{v}_h \in \mathcal{W}_h^k$ is independent of temporal variable t and Stokes projection Π_h^s commutes with time variable t, we obtain

$$m_h(\boldsymbol{\zeta}_{tt}, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) - b(\mathbf{v}_h, p_{ht}) = (\mathbf{f}_{ht}, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_{tt}, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u_t}, \mathbf{v}_h). \quad (5.36)$$

Replacing $\mathbf{v}_h$ by $\boldsymbol{\zeta}_t$ in Estimation 5.36 and since $b(\boldsymbol{\zeta}_t, p_{ht}) = 0$, we acquire

$$\begin{aligned} m_h(\boldsymbol{\zeta}_{tt}, \boldsymbol{\zeta}_t) + a_h(\boldsymbol{\zeta}_t, \boldsymbol{\zeta}_t) &= (\mathbf{f}_{ht}, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_{tt}, \boldsymbol{\zeta}_t) - a_h(\Pi_h^s \mathbf{u_t}, \boldsymbol{\zeta}_t). \\ &= (\mathbf{f}_{ht}, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_{tt}, \boldsymbol{\zeta}_t) - (\mathbf{f}, \boldsymbol{\zeta}_t) + (\mathbf{u}_{tt}, \boldsymbol{\zeta}_t). \end{aligned} \quad (5.37)$$

Employing approximation property of Stokes projection Π_h^s and Cauchy-Schwarz inequality, we have

$$|(\mathbf{f}_{ht}, \boldsymbol{\zeta}_t) - (\mathbf{f}, \boldsymbol{\zeta}_t)| \leq C\, h^{k+1}\, |\mathbf{f}_t|_{k+1}\, \|\boldsymbol{\zeta}_t\|_0. \quad (5.38)$$

In view of polynomial consistency property of $m_h(\cdot,\cdot)$, standard approximation property of Stokes projection Π_h^s and L^2 projection $\Pi_{k,K}^0$, Cauchy-Schwarz inequality, we have

$$|m_h(\Pi_h^s u_{tt}, \boldsymbol{\zeta}_t) - (\mathbf{u}_{tt}, \boldsymbol{\zeta}_t)| \leq C\, h^{k+1}\, |\mathbf{u}_{tt}|_{k+1}\, \|\zeta_t\|. \quad (5.39)$$

Inserting (5.38) and (5.39) into (5.37), we get

$$\frac{d}{dt}\, \|\boldsymbol{\zeta}_t\|^2 + |\boldsymbol{\zeta}_t|_1^2 \leq C\, h^{k+1} \Big(|\mathbf{f}_t|_{k+1} + |\mathbf{u}_{tt}|_{k+1}\Big)\, \|\zeta_t\|.$$

Furthermore, we utilize stability property of bilinear forms $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ in order to derive the above estimation. Since $|\boldsymbol{\zeta}_t|_1^2$ is positive quantity, hence we can neglect this term and we obtain

$$\|\boldsymbol{\zeta}_t\|\, \frac{d}{dt}\, \|\boldsymbol{\zeta}_t\| \leq C\, h^{k+1} \Big(|\mathbf{f}_t|_{k+1} + |\mathbf{u}_{tt}|_{k+1}\Big) \|\boldsymbol{\zeta}_t\|.$$

Without loss of generality, we assume that $\|\boldsymbol{\zeta}_t\| \neq 0$. Therefore, the above equation reduces to

$$\frac{d}{dt}\,\|\boldsymbol{\zeta}_t\| \leq \; C\, h^{k+1} \Big(|\mathbf{f}_t|_{k+1} + |\mathbf{u}_{tt}|_{k+1}\Big).$$

A straightforward integration of the above equation from 0 to t implies

$$\|\boldsymbol{\zeta}_t(t)\| \leq \; \|\boldsymbol{\zeta}_t(0)\| + \; C\, h^{k+1} \Big(|\mathbf{f}_t|_{L^2(0,t;H^{k+1}(\Omega))} + |\mathbf{u}_{tt}|_{L^2(0,t;H^{k+1}(\Omega))}\Big). \tag{5.40}$$

Furthermore, an application of approximation property of Stokes projection Π_h^s mentioned in Lemma 40 reduces the estimation as

$$\|\mathbf{u}_{ht}(t) - \Pi_h^s \mathbf{u}_t\| \leq \; C\, \|\mathbf{u}_{ht}(0) - \mathbf{u}_t(0)\| + \; C\, h^{k+1}\, |\mathbf{u}_t(0)|_{k+1}. \tag{5.41}$$

Utilizing (5.41), Lemma 40 and the Estimation (5.40), we obtain the final result

$$\begin{aligned}\|(\mathbf{u}_t - \mathbf{u}_{ht})(t)\| \leq \; &\|\mathbf{u}_t(0) - \mathbf{u}_{ht}(0)\| + \; C\, h^{k+1} \Big(|\mathbf{u}_t(0)|_{k+1} + |\mathbf{u}_t(t)|_{k+1} + |p_t|_k\Big) \\ &+ \; C\, h^{k+1} \Big(|\mathbf{f}_t|_{L^2(0,t;H^{k+1}(\Omega))} + |\mathbf{u}_{tt}|_{L^2(0,t;H^{k+1}(\Omega))}\Big).\end{aligned}$$

■

5.5.2 Error estimate for the pressure variable

Exploiting the operator Π_h^p and discrete inf-sup condition, we exhibit that discrete solution p_h converges optimally in L^2 norm.

Theorem 44. *Let* $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ *satisfies (5.1) and* $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ *satisfies (5.10). Moreover, we deduce that all assumptions of Theorem 41, Theorem 42, Theorem 43 hold. Then there exists a positive constant* C *depending on regularity of* $\mathbf{u}$, $\mathbf{u}_t$, $\mathbf{u}_{tt}$, p, p_t, $\mathbf{f}$ *and* $\mathbf{f}_t$ *such that the following estimation holds*

$$\|p - p_h\| \leq \; C\, h^k.$$

Proof. Let $q_h \in \mathcal{Q}_h^k$ be an arbitrary element. Then $(p_h(t) - q_h) \in \mathcal{Q}_h^k$. An application of discrete inf-sup condition (Lemma 37) implies that

$$\begin{aligned}\mathcal{B}\, \|p_h(t) - q_h\|_0 \leq \; &\sup_{\substack{\mathbf{v}_h \in \mathcal{W}_h^k \\ \mathbf{v}_h \neq 0}} \frac{b(\mathbf{v}_h,\; p_h(t) - q_h)}{\|\mathbf{v}_h\|_1} \\ = \; &\sup_{\substack{\mathbf{v}_h \in \mathcal{W}_h^k \\ \mathbf{v}_h \neq 0}} \frac{b(\mathbf{v}_h,\; p_h(t) - p(t)) + b(\mathbf{v}_h, p(t) - q_h)}{\|\mathbf{v}_h\|_1}.\end{aligned} \tag{5.42}$$

Since $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies non-stationary Stokes equation (5.1), we have

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) - (\mathbf{u}_t, \mathbf{v}). \tag{5.43}$$

Again, since $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ satisfies the discrete equation (5.10), we obtain

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\mathbf{u}_{ht}, \mathbf{v}_h). \tag{5.44}$$

Replacing $\mathbf{v}$ by $\mathbf{v}_h$ in (5.43) and then subtracting (5.44) form (5.43), we have

$$b(\mathbf{v}_h, p - p_h) = \underbrace{(\mathbf{f} - \mathbf{f}_h, \mathbf{v}_h)}_{H_1} + \underbrace{m_h(\mathbf{u}_{ht}, \mathbf{v}_h) - (\mathbf{u}_t, \mathbf{v}_h)}_{H_2} + \underbrace{a_h(\mathbf{u}_h, \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h)}_{H_3}. \tag{5.45}$$

In order to reduce the cumbersome notation, we split the load term into three parts H_1, H_2 and H_3. Exploiting approximation property of $\Pi^0_{k,K}$ operator and Cauchy-Schwartz inequality, we estimate that

$$\begin{aligned} |H_1| &= |\sum_K (f - \Pi^0_{k,K} f, \mathbf{v}_h)_K| \\ &\leq \sum_K \|f - \Pi^0_{k,K} f\|_{0,K} \, \|\mathbf{v}_h\|_{0,K} \\ &\leq C\, h^k \, |\mathbf{f}|_k \, \|\mathbf{v}_h\|. \end{aligned} \tag{5.46}$$

In order to estimate second term H_2, we proceed as follows

$$\begin{aligned} |H_2| &\leq \sum_K \left| m_h^K(\mathbf{u}_{ht}, \mathbf{v}_h) - (\mathbf{u}_t, \mathbf{v}_h)_K \right| \\ &\leq \sum_K \left| m_h^K(\mathbf{u}_{ht}, \mathbf{v}_h) - m_h^K(\Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) + m_h^K(\Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) - (\mathbf{u}_t, \mathbf{v}_h)_K \right| \\ &\leq \sum_K \left(\left| m_h^K(\mathbf{u}_{ht} - \Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) \right| + \left| (\Pi^0_{k,K}\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}_h)_K \right| \right) \\ &\leq \sum_K \left(\left| m_h^K(\mathbf{u}_{ht} - \mathbf{u}_t + \mathbf{u}_t - \Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) \right| + \left| (\Pi^0_{k,K}\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}_h)_K \right| \right) \\ &\leq C \sum_K \left(\|\mathbf{u}_{ht} - \mathbf{u}_t\|_{0,K} \, \|\mathbf{v}_h\|_{0,K} + \|\Pi^0_{k,K}\mathbf{u}_t - \mathbf{u}_t\|_{0,K} \, \|\mathbf{v}_h\|_{0,K} \right) \\ &\leq C \, \|\mathbf{u}_{ht} - \mathbf{u}_t\| \, \|\mathbf{v}_h\| + C\, h^k \, |\mathbf{u}_t|_k \, \|\mathbf{v}_h\|. \end{aligned} \tag{5.47}$$

H_3 can be bounded as

$$
\begin{aligned}
|H_3| &= |a_h(\mathbf{u}_h, \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h)| \\
&\le \sum_K \Big(|a_h^K(\mathbf{u}_h - \mathbf{u}_\pi, \mathbf{v}_h)| + |a_h^K(\mathbf{u}_\pi, \mathbf{v}_h) - a^K(\mathbf{u}, \mathbf{v}_h)| \Big) \\
&\le \sum_K \Big(|a_h^K(\mathbf{u}_h - \mathbf{u} + \mathbf{u} - \mathbf{u}_\pi, \mathbf{v}_h)| + |a^K(\mathbf{u} - \mathbf{u}_\pi, \mathbf{v}_h)| \Big) \\
&\le \sum_K \Big(|\mathbf{u}_h - \mathbf{u}|_{1,K} \, |\mathbf{v}_h|_{1,K} + |\mathbf{u}_\pi - \mathbf{u}|_{1,K} \, |\mathbf{v}_h|_{1,K} \Big) \\
&\le C \Big(|\mathbf{u}_h - \mathbf{u}|_1 + h^k \, |\mathbf{u}|_{k+1} \Big) \, |\mathbf{v}_h|_1.
\end{aligned}
\tag{5.48}
$$

The estimation of $|\mathbf{u}_h - \mathbf{u}|_1$ can be evaluated from Theorem 42. Inserting results (5.46), (5.47), and (5.48) into (5.45), we deduce that

$$
\begin{aligned}
\frac{b(\mathbf{v}_h, p - p_h)}{\|\mathbf{v}_h\|_1} &\le C \Big(|\mathbf{u}(0) - \mathbf{u}_h(0)|_1 + \|\mathbf{u}_{ht}(0) - \mathbf{u}_t(0)\| \Big) + C \, h^k \Big(|\mathbf{u}(0)|_{k+1} \\
&+ |\mathbf{u}_t(0)|_{k+1} + |\mathbf{u}|_{k+1} + |p|_k + |p_t|_k + |\mathbf{f}|_{L^2(0,t;H^{k+1}(\Omega))} \\
&+ |\mathbf{u}_t|_{L^2(0,t;H^{k+1}(\Omega))} + |\mathbf{f}_t|_{L^2(0,t;H^{k+1}(\Omega))} + |\mathbf{u}_{tt}|_{L^2(0,t;H^{k+1}(\Omega))} \Big).
\end{aligned}
$$

Considering $\mathbf{u}_h(0) := I_h \mathbf{u}(0)$ and $\mathbf{u}_{ht}(0) := I_h \mathbf{u}_t(0)$, we get

$$
\frac{b(\mathbf{v}_h, p - p_h)}{\|\mathbf{v}_h\|_1} \le C \, h^k, \tag{5.49}
$$

where C is positive generic constant. Inserting (5.49) into (5.42), we obtain

$$
\mathcal{B} \, \|p_h(t) - q_h\| \le C \, h^k + \|p - q_h\|.
$$

Again, we have

$$
\begin{aligned}
\|p - p_h\| &\le C \, \|p - q_h\| + \|p_h - q_h\| \\
&\le C \Big(h^k + \|p - q_h\| \Big),
\end{aligned}
\tag{5.50}
$$

where q_h is an arbitrary element of $\mathcal{Q}_h^k$. As a consequence, the estimation (5.50) can be recast as

$$
\|p - p_h\| \le C \, h^k + \inf_{q_h \in \mathcal{Q}_h^k} \|p - q_h\|.
$$

Choosing $q_h = \Pi_h^p p$, and exploiting Lemma 40, we have

$$
\|p - p_h\| \le C \, h^k.
$$

■

5.6 Discussion

In this work, we have initiated the investigation of virtual element method for time dependent Stokes equations. The VEM space under our discussion is discrete inf-sup stable, H^1 conforming space but not divergence free. The main advantage of this space is that computation of orthogonal projection operators $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ on $\mathcal{W}^k(K)$ is straightforward like scalar valued operators. Moreover, the construction of the space is natural extension of framework used for Laplace equation. Hence, implementation of discrete bilinear form is effortless and can be easily extended for nonconforming virtual element method. Furthermore, we have modified the virtual element space as demanded by theory, where orthogonal L^2 projection operator $\Pi^0_{k,K}$ is fully computable. The approximation of pressure space is same as finite element method where piecewise discontinuous polynomials are employed. We focus on to construct discrete Stokes projection like FEM and the standard theory for FEM can be incorporated in VEM framework and provide an unified VEM framework for arbitrary order $k \geq 2$. Since our VEM space $(\mathcal{W}^k_h, \mathcal{Q}_h)$ is discrete inf-sup stable, we can construct Fortin operator that reformulates the analysis in simple form. However, the primary issue related with the space is that this pair is inf-sup stable which motivates us to study Navier-Stokes equation on this space. Moreover, we have justified the computation of projection operators $\Pi^0_{k,K}$ and $\Pi^\nabla_{k,K}$ on $\mathcal{W}^k(K)$ but we did not verify theoretical results by conducting numerical experiments. We have considered computation of vector valued operator $\Pi^0_{k,K}$, $\Pi^\nabla_{k,K}$ and implementation of proposed scheme as future work.

Chapter 6

Conclusion and Future Work

In this work, we have presented virtual element methods for time dependent partial differential equations(PDEs) on polygonal meshes. Our main contribution lies in the computation of nonlinear load term. We stress that there are several contributions on FEM dealing with nonlinear problems; however, the same idea can not be extended in the context of VEM. Therefore, in this work, we have given a new idea to compute nonlinear term with the help of degrees of freedoms (DoF). We point out that classical VEM space revealed in [9] does not support to compute L^2 projection operator $\Pi^0_{k,K}$. But our theoretical framework for convergence analysis related to time dependent PDE demands evaluation of $\Pi^0_{k,K}$. In view of this difficulty, we reformulate the VEM space that allows to compute $\Pi^0_{k,K}$ without changing DoF. In addition to this with the help of L^2 projection operator $\Pi^0_{k,K}$, required results concerning error estimates are established in different norms. Numerical experiments have been conducted at the end of each chapter (except the last Chapter) in order to justify our theoretical findings.

In Chapter-2, we have attempted to solve semilinear parabolic equation by introducing a graceful idea consists of an application of L^2- projection operator $\Pi^0_{k,K}$. It is well-known that the discrete function u_h consists of polynomial part and non-polynomial part. In order to evaluate the nonlinear part, we have considered only polynomial part of u_h. It seems that there will be a loss of order of convergence. However, we have represented a robust theoretical analysis in order to ensure optimal order of convergence. After time discretization, discrete bilinear form reduces to system of nonlinear equations, which can be solved using well-known Newton method. In Chapter-2 we have employed implicit Euler method for temporal discretization. In light of the above discussion, we assert that our method discretize non-linear model problem which can be computed from degrees of freedom. Therefore, an extension of VEM formulation for nonlinear problems can be considered as future works.

Moreover, the discrete bilinear form is defined element-wise locally on each polygon K. Finally, we obtain global form by adding local bilinear forms. This aspect has a great similarity with non-conforming finite element settings. Therefore, naturally one question arises that whether it is possible to review the analysis in the context of nonconforming VEM. We have experienced that nonconforming FEM has more advantage than conforming FEM from the engineering point of view. Moreover, higher order patch-test satisfied by nonconforming element ensure optimal order of convergence with better accuracy. Following idea from [12, 20], nonconforming VEM can be examined for linear, quasi-linear, nonlinear parabolic equations. This can be considered as future work.

In Chapter-3, we have studied semi-linear second order hyperbolic equation. Right hand side non-linear load term is discretized exploiting analogous technique revealed in Chapter-2. We have employed Newmark scheme for time discretization. Practical importance and utility of Newmark schemes are revealed in the corresponding Chapter. Future study includes further development of VEM technique for second order non-linear wave equation. In order to describe practical utility of semilinear hyperbolic equation, we have examined Sine-Gordon equation. Moreover, first order hyperbolic equation is not yet studied in the context of VEM. A straight forward extension our method for semi-linear hyperbolic conservation laws can be considered as future work. In addition to the previous discussion, we notice that standard Galerkin method produces unstable numerical solution for first order hyperbolic equation. Consequently, in order to diminish non-physical oscillations, we add additional stabilizer. Therefore, VEM approximation for first order hyperbolic equation adding additional stabilizer can be considered as future work.

In Chapter-4, we have initiated our investigation for stationary convection dominated diffusion reaction equation in the context of FEM and later we extend our discussion for time dependent convection-diffusion-reaction equation (convection dominated) in the context of VEM. It is well-known that standard Galerkin method produces non-physical oscillations in convection dominated region. Indeed, there are plenty of stabilizing techniques studied by researchers in order to reduce oscillations in the context of FEM. Among them, SUPG method can be directly incorporated in VEM. The stabilizing idea for VEM used in this chapter is mainly followed from [18].

Furthermore, one can recast the analysis considering symmetric stabilizers. Moreover, the technique employed to solve semi-linear parabolic and hyperbolic PDE revealed in Chapter-2 and Chapter-3 respectively can be recast to solve semi-linear non-stationary convection diffusion reaction (CDR)problem. The extension of VEM analysis for system of time-dependent CDR equation may require deeper analysis. Especially, the formulation of stabilizers that ensure stability of discrete bilinear form need careful attention. Moreover, nonconforming VEM has several advantages. Therefore, it is worthy to extend nonconforming VEM formulation for convection dominated diffusion reaction equation which can be considered as future work.

Finally, in Chapter-5, we explore time dependent Stokes equation which is extensively used by engineers, physicist, mathematician in order to model incompressible fluid flow problems. We exploit modified virtual element space having similarity with the space discussed in [14] used for linear elasticity problem. Basically, velocity space contains continuous polynomial of order k and pressure space contains discontinuous polynomial of order $k-1$. Our velocity-pressure space satisfies discrete inf-sup conditions for $k \geq 2$ that ensures well-posedness of discrete solution. The key point of our method is constructing discrete Stokes projection operator which represents the analysis in a simple way. Additionally, the computation of vector valued L^2 projection operator $\Pi^0_{k,K}$ and energy projection operator $\Pi^{\nabla}_{k,K}$ follow analogous approach like scalar valued projection operator and hence it is straightforward to implement. Stationary Navier-Stokes equation is approximated exploiting divergence-free space in [23]. Extension of the method for non-stationary Navier-Stokes equation could be considered as future work. Moreover, we have observed that lowest order $P_1 - P_0$ element and $Q_1 - P_0$ element do not satisfy discrete in-sup condition but these elements are cheap to implement. Hence, possible future work could be development of VEM technique for equal order velocity-pressure element adding additional residual based stabilizers. Furthermore, non-stationary Navier-Stokes equation can be studied over nonconforming space developed in [29].

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- Dibyendu Adak, E. Natarajan and Sarvesh Kumar, "Convergence analysis of virtual element methods for semilinear parabolic problems on polygonal meshes" Accepted, Numerical Methods for Partial Differential Equation, 2018.
- Dibyendu Adak, E. Natarajan, "SUPG virtual element formulation for evolutionary convection diffusion reaction equation"(submitted).
- Dibyendu Adak, E. Natarajan and Sarvesh Kumar, An H1-conforming virtual element method for time dependent Stokes equation on polygonal meshes (submitted).
- Dibyendu Adak, E. Natarajan, Conforming virtual element method for semi-linear elliptic equations on polygonal meshes.(Preprint)