# A QUALITATIVE STUDY OF CONTROLLABILITY OF A CERTAIN CLASS OF FUZZY SYSTEMS AND NONLINEAR MATRIX LYAPUNOV SYSTEMS

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# ABSTRACT

In the thesis, we study controllability of linear systems with fuzzy initial conditions and fuzzy inputs. Before establishing the controllability results for the fuzzy dynamical systems, we will first investigate the behavior of solutions of a general nonlinear system of ordinary differential equations with fuzzy initial conditions and fuzzy inputs. Although various approaches are suggested in the literature for the evolution of solution to fuzzy differential equations, we investigate controllability results by using the levelwise approach and differential inclusion approach. We also investigate controllability of nonlinearly perturbed matrix Lyapunov systems and impulsive semilinear matrix Lyapunov systems by using the tools of operator theory and nonlinear functional analysis.

The research work is mainly divided in to three parts. In the first part, we investigate the behavior of the solutions of fuzzy differential equations obtained by fuzzification of nonlinear ODEs with fuzzy initial conditions and fuzzy inputs. We consider the following $n$-dimensional nonlinear ordinary differential equations with fuzzy initial conditions and fuzzy inputs of the form

$$\dot{x}(t) = f(t, x(t), u(t)), x(t_0) = X_0, t \geq t_0 \geq 0, \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function which is measurable in $t$ and is continuous in $x$ and $u$, $X_0 \in (\mathbb{E}^1)^n$ and the fuzzy input $u(t) \in (\mathbb{E}^1)^m$. In our analysis, we employ the tools of levelwise approach of solving fuzzy differential equations along with some of the results from real analysis. We have shown that the solutions of systems of type (1) are described by a system of $2n$-ordinary differential equations with crisp initial conditions and crisp inputs corresponding to the end points of the alpha cuts of fuzzy states.

We also consider a particular case of the systems of type (1), that is, linear time-varying dynamical systems with fuzzy initial conditions and fuzzy inputs of the

form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = X_0, t \geq t_0 \geq 0, \end{cases} \tag{2}$$

where $A(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n \times n}), B(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n \times m}), X_0 \in (\mathbb{E}^1)^n = \underbrace{\mathbb{E}^1 \times \ldots \times \mathbb{E}^1}_{n-times}$ and the input $u(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1](t_1 > t_0)$. Here we use a complex number representation of the $\alpha$-level sets of the fuzzy states to characterize the solutions of such systems by a closed form formula involving the transition matrix which could be easily used in practical computations. We will use Peano-Baker type of series to obtain the transition matrix for the system (2).

We, further, consider fuzzy initial value problem of the type

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \in (\mathbb{E}^1)^n, t_1 \geq t \geq t_0, \tag{3}$$

where $f : T = [t_0, t_1] \times (\mathbb{E}^1)^n \to (\mathbb{E}^1)^n$ is continuous, $t_0 \in \mathbb{R}^+$. We study the existence and uniqueness of the solution of system (3).

The second part of the thesis deals with the problems on controllability of linear fuzzy differential dynamical systems. Here, we first consider the following linear time invariant systems with fuzzy initial condition

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = X_0, t_1 \geq t \geq t_0, \end{cases} \tag{4}$$

where $A$, $B$ are real matrices of size $n \times n$, $n \times m$, respectively and $t_0 \in \mathbb{R}^+$. The initial state $X_0 \in (\mathbb{E}^1)^n$ and the control $u(t) \in (\mathbb{E}^1)^m$.

We study controllability of the system (4) by using the levelwise approach of evolution of solutions to system (4). In controllability, one looks for a fuzzy control $u(t)$ during time-interval $[t_0, t_1]$ such that the system can be steered exactly to a desired target fuzzy state $X_1$ at time $t_1$. That is, the solution of system (4) with the appropriate fuzzy control $u(t)$ during time interval $[t_0, t_1]$ satisfies $x(t_1) = X_1$, where $X_1$ is the desired fuzzy state at time $t_1$. We establish some sufficient conditions for the controllability of the system (4). We also provide a closed form representation for the steering control when the matrices $A$ and $B$ have non-negative entries. Furthermore, we introduce the concept of 'fuzzy-controllability', a concept weaker than controllability, and establish sufficient conditions for the fuzzy dynamical systems of type (4) to be fuzzy-controllable. In fuzzy-controllability, one looks for a fuzzy-controller $u(\cdot)$ that can steer the system-state within the desired target state $X_1$ at time $t_1$. More precisely, solution of system (4) with the fuzzy control $u(t)$ during time-interval $[t_0, t_1]$ satisfies $x(t_1) \leq X_1$, where $X_1$ is the desired fuzzy state at time $t_1$. In our work, we provide a computational procedure to obtain the fuzzy-controllable initial states that can be steered to within a desired target fuzzy state $X_1$ with some suitable control.

So far in our controllability analysis, we have employed the levelwise approach of the evolution of solutions of fuzzy differential equations in order to establish controllability results. We will now establish controllability results by using the differential inclusion approach. We consider the following time-varying systems of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)U(t)\\ x(0) &= x_0 \in \mathbb{R}^n, T \geq t \geq 0,\end{aligned} \tag{5}$$

in which $A(t)$, $B(t)$ are $n \times n$, $n \times m$ continuous matrices, respectively. We assume that the control $u(t) \in (\mathbb{E}^1)^m$ and the state $x(t)$ for $t > 0$ belong to $(\mathbb{E}^1)^n$. Controllability of similar systems has been studied by other authors with an assumption of the invertibility of the matrix $B(t)$; while we obtain controllability results with a general non-invertible matrix $B(t)$. It is observed in our analysis that the system (5) may not be controllable on the whole space $(\mathbb{E}^1)^n$, instead controllability is established on a subset $\mathbb{E}_0^n$ of $(\mathbb{E}^1)^n$. This motivates us to introduce a concept of quasi-controllability, a weaker concept than controllability. We characterize the quasi-controllable subset $\mathbb{E}_0^n$ of $(\mathbb{E}^1)^n$ and establish sufficient conditions for quasi-controllability of system (5).

The third part of the thesis deals with the controllability analysis of the semilinear matrix Lyapunov systems and semilinear impulsive matrix Lyapunov systems. We use techniques from operator theory and nonlinear functional analysis to establish the complete controllability results for such systems. We study the controllability of nonlinear matrix Lyapunov systems represented by:

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), \tag{6}$$

where $X(t)$ is an $n \times n$ real matrix called state matrix, $U(t)$ is an $m \times n$ real matrix called control matrix and $G(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is a nonlinear function. $A(t)$, $B(t)$, $F(t)$ are $n \times n$, $n \times n$ and $n \times m$ real matrices, respectively. All of them are assumed to be piecewise continuous functions of $t$ in $[t_0, t_1](0 \leq t_0 < t_1 < \infty)$. Furthermore, entries in the state matrix $X(t)$ and the control matrix $U(t)$ belong to $L^2([t_0, t_1], \mathbb{R})$. The function $G$ satisfies the 'Caratheodory conditions'; that is, $G(\cdot, x)$ is measurable with respect to $t$ for all $x \in \mathbb{R}^{n \times n}$ and $G(t, \cdot)$ is continuous with respect to $x$ for almost all $t \in [t_0, t_1]$. We establish our results under the assumption that nonlinear term $G(t, X(t))$ satisfies Lipschitz condition or monotonicity condition.

We also obtain sufficient conditions for the complete controllability of the following matrix Lyapunov systems with impulse effects

$$\begin{cases} \dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), t \neq t_k, t \in [t_0, T] \\ X(t_k^+) = [I_n + D^k U(t_k)]X(t_k), k = 1, 2, \ldots, \rho \\ X(t_0) = X_0, \end{cases} \tag{7}$$

where the state $X(t)$ is an $n \times n$ real matrix, control $U(t)$ is an $m \times n$ real matrix. $A(t), B(t), F(t)$ are $n \times n$, $n \times n$, $n \times m$ real matrices with piecewise continuous entries in time interval $[t_0, T]$ and $0 \leq t_0 \leq t_1 \leq t_2 \ldots \leq t_\rho \leq T$ are the time points at which impulse control $U(t_k)$ is given to the system. For each $k = 1, 2, \ldots, \rho$, $D^k U(t_k)$ is an $n \times n$ diagonal matrix such that $D^k U(t_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^k U_{ij}(t_k) I_n$, where $I_n$ is the identity matrix on $\mathbb{R}^n$ and $d_{ij}^k \in \mathbb{R}$. $G(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is a nonlinear function and satisfies the 'Caratheodory conditions'.

Controllability of special cases of system (6) and system (7) has been studied by several authors in the literature. However, our results are more general, applicable to a much wider class of systems and extend some of the existing results on the controllability of matrix Lyapunov systems and impulsive dynamical systems.